Expected rate of return

In the example given in the text, a \$1,000 machine that yields a one-time added profit of \$1,100 one year later has an expected rate of return equal to 10%: (\$1,100 – \$1,000)/\$1,000 = 0.10, or 10%. But what if the added profit is spread out over many years? Is there a method for determining the "rate of return" under these circumstances? As you might suspect, the answer is "yes," but the solution is not as simple as in this one-period example. To begin, we will need to investigate how the value of money varies with the time at which it is received.

Suppose you have the opportunity to obtain an annual return of r on an investment of \$$X$. At the end of the year, your investment would be worth the principal, X, plus interest, rX, or $X(1 + r)$. Call this amount Z_1, so that $Z_1 = X(1 + r)$. Alternatively, we might ask: How much money would have to be invested now at an interest rate of r, so that at the end of one year, you would have accumulated a total of Z_1? Clearly, this amount is $X = \frac{Z_1}{1+r}$. The value X is known as the "present value" of Z_1, and tells us the current "worth" of Z_1 to be received one year from now.

Instead of withdrawing your investment at the end of the first year, however, suppose you "let it ride" and accumulate an additional year's interest. At the end of the second year, your initial investment of X would be worth Z_1 (its value at the end of year 1) plus the second year's interest, rZ_1. Call this amount Z_2: $Z_2 = Z_1(1 + r)$. Substituting $Z_1 = X(1 + r)$ from before, we find $Z_2 = X(1 + r)^2$. Again, we might ask: What is the current value of an amount Z_2 to be received two years from now? Working backwards, we see this is $X = \frac{Z_2}{(1+r)^2}$, the present value of Z_2. In like manner, we find that the present value of Z_t received t years in the future is $\frac{Z_t}{(1+r)^t}$.

More generally, suppose we expect to receive a stream of payments equal to $Z_0, Z_1, Z_2, \ldots Z_n$, where the subscript references the year the payment is to be received. The present value of this stream of payments is $X = Z_0 + \frac{Z_1}{1+r} + \frac{Z_2}{(1+r)^2} + \ldots + \frac{Z_n}{(1+r)^n}$.

The "expected rate of return" on a given investment is defined as the value of r that equates the present value of its stream of expected benefits to its stream of expected costs. That is, r is the solution to the equation:

$$B_0 + \frac{B_1}{1+r} + \frac{B_2}{(1+r)^2} + \ldots + \frac{B_n}{(1+r)^n} = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}$$

where B_i and C_i represent expected benefits and costs, respectively, in year i.

If we let Z_t be the difference between benefits and costs of the investment in year t, $Z_t = B_t - C_t$, then we see that the rate of return on the investment is the value of r that equates the present value of net benefits to zero: $0 = (B_0 - C_0) + \frac{(B_1 - C_1)}{1+r} + \frac{(B_2 - C_2)}{(1+r)^2} + \ldots + \frac{(B_n - C_n)}{(1+r)^n} = Z_0 + \frac{Z_1}{1+r} + \frac{Z_2}{(1+r)^2} + \ldots + \frac{Z_n}{(1+r)^n}$. If we multiply each term on both sides of this equation by $(1 + r)^n$, we get an n^{th}-degree polynomial in $1 + r$: $0 = Z_0(1 + r)^n + Z_1(1 + r)^{n-1} + Z_2(1 + r)^{n-2} + \ldots + Z_n$, which can be solved for r.

The one-year example in the text is easily seen, then, as a special case of this formula. In that case, $n = 1$; $Z_0 = B_0 - C_0 = 0 - 1{,}000$; $Z_1 = B_1 - C_1 = 1{,}100 - 0$, and we have r as the solution to:

$0 = -1{,}000(1 + r) + 1{,}100$ or, $r = \frac{1{,}100 - 1{,}000}{1{,}000} = 0.10$.

To address the question first raised in this note, what if the $1,100 is instead received over two years—say, $550 at the end of the first and second years? Under these circumstances the rate of return is the solution to: $(0 - 1{,}000)(1 + r)^2 + (550 - 0)(1 + r) + (550 - 0) = 0$, whose solution is $r = .0659646$, or approximately 6.6%.

Why is this rate of return lower than our previous result of 10%? Intuitively, it takes another year to obtain the same number of dollars, $1,000. Funds received two years from now are of lower value than funds received only one year from now.