Chapter 4

Calculus of Variations

4.1 INTRODUCTION

Calculus of variations deals with certain kinds of "external problems" in which expressions involving integrals are optimized (maximized or minimized). Euler and Lagrange in the 18th century laid the foundations, with the classical problems of determining a closed curve in the plane enclosing maximum area subject to fixed length and the brachistochrone problem of determining the path between two points in minimum time. The present day problems include the maximization of the entropy integral in third law of thermodynamics, minimization of potential and kinetic energies integral in Hamilton's principle in mechanics, the minimization of energy integral in the problems in elastic behaviour of beams, plates and shells. Thus calculus of variations deals with the study of extrema of "functionals".

Functional: A real valued function f whose domain is the set of real functions $\{y(x)\}$ is known as a functional (or functional of a single independent variable). Thus the domain of definition of a functional is a set of **admissible functions**. In ordinary functions the values of the independent variables are numbers. Whereas with functionals, the values of the independent variables are functions.

Example: The length L of a curve, c whose equation is $y = f(x)$, passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx$$

where y' denotes derivative of y w.r.t. x.

Now the length L of the curve passing through A and B depends on $y(x)$ (the curve). Than L is a function of the independent variable $y(x)$, which is a function. Thus

$$L\{y(x)\} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx$$

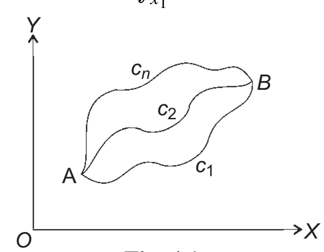

Fig. 4.1

defines a functional which associates a real number L uniquely to each $y(x)$ (the independent variable). Further suppose we wish to determine the curve having shortest (least) distance between the two given points A and B, i.e., curve with minimum length L. This is a classical example of a variational problem in which we wish to determine, the particular curve $y = y(x)$ which minimizes the functional $L\{y(x)\}$ given by (1). Here the two conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, which are imposed on the curve $y(x)$ are known as *end conditions* of the problem. Thus variational problems involves determination of maximum or minimum or stationary values of a functional. The term extremum is used to include maximum or minimum or stationary values.

4.2 VARIATIONAL PROBLEM

Consider the general integral (a functional)

$$I\{y(x)\} = \int_{x_1}^{x_2} f(x, y, y')dx \tag{1}$$

Extremal: A function $y = y(x)$ which extremizes (1) and satisfies the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ is known as an extremal or extremizing function of the functional I (given by (1)). A variational problem is to find such an extremal function $y(x)$.

Variation of a Function and a Functional

When the independent variable x changes to $x + \Delta x$ then the dependent variable y of the function $y = f(x)$ changes to $y + \Delta y$. Thus Δy is the change of the function, the differential dy provides the variation in y. Consider a function $f(x, y, y')$ which for a fixed x, becomes a functional defined on a set of functions $\{y(x)\}$.

For a fixed value of x, if $y(x)$ is changed to $y(x) + \epsilon\eta(x)$, where ϵ is independent of x, then $\epsilon\eta(x)$ is known as the *variation* of y and is denotd by δy. Similarly, variation of y' is $\epsilon\eta'(x)$ and is denoted by $\delta y'$. Now the change in f is given by

$$\Delta f = f(x, y + \epsilon\eta, y' + \epsilon\eta') - f(x, y, y')$$

Expanding the first term on R.H.S. by Maclaurins series in powers of ϵ, we get

$$\Delta f = f(x, y, y') + \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)\epsilon + \\ + \left(\frac{\partial^2 F}{\partial y^2}\eta^2 + \frac{2\partial^2 F}{\partial y y'}\eta\eta' + \frac{\partial^2 F}{\partial y'^2}\eta'^2\right)\frac{\epsilon^2}{2!} + \\ + \cdots - F(x, y, y')$$

or approximately, neglecting higher powers of ϵ.

$$\Delta f = \frac{\partial f}{\partial y}\eta\epsilon + \frac{\partial f}{\partial y'}\eta'\epsilon\epsilon = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

Thus the *variation of a functional f* is denoted by δf and is given by

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

which is analogous to the differential of a function.

Result:
(a) $\delta(f_1 \pm f_2) = \delta f_1 \pm \delta f_2$
(b) $\delta(f_1 f_2) = f_1\delta f_2 + f_2\delta f_1$
(c) $\delta(f^\eta) = \eta f^{\eta-1}\delta f$
(d) $\delta\left(\frac{f_1}{f_2}\right) = \frac{f_2\delta f_1 - f_1\delta f_2}{f_2^2}$
(e) $\frac{d}{dx}(\delta y) = \frac{d}{dx}(\epsilon\eta) = \epsilon\frac{d\eta}{dx} = \epsilon\eta' = \delta y' = \delta\left(\frac{dy}{dx}\right)$.

Thus taking the variation of a functional and differentiating w.r.t. the independent variable x are commutative operations.

Result: The necessary condition for the functional I to attain an extremum is that its variation vanish i.e., $\delta I = 0$.

4.3 EULER'S EQUATION

A necessary condition for the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx \tag{1}$$

to attain an extreme value is that the extremizing function $y(x)$ should satisfy

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \tag{2}$$

for $x_1 \le x \le x_2$.

Note 1: The second order differential equation (2) is known as Euler-Lagrange or simply Euler's equation for the integral (1).

Note 2: The solutions (integral curves) of Euler's equation are known as *extremals* (or stationary functions) of the functional. Extremum for a functional can occur only on extremals.

Proof: Assume that the function $y = y(x)$, is twice-differentiable on $[x_1, x_2]$, satisfies the end (boundary) conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ and extremizes (maximizes or minimizes) the integral I given by (1). To determine such a function $y(x)$, construct the class of *comparison functions* $Y(x)$ defined by

$$Y(x) = y(x) + \epsilon\eta(x) \tag{2}$$

on the interval $[x_1, x_2]$. For any function $\eta(x)$, $y(x)$ is a member of this class of functions $\{Y(x)\}$ for $\epsilon = 0$. Assume that

$$\eta(x_1) = \eta(x_2) = 0 \tag{3}$$

Differentiating (2),

$$Y'(x) = y'(x) + \epsilon\eta'(x) \tag{4}$$

Replacing y and y' in (1) Y and Y' from (2) and (4), we obtain the integral

$$I(\epsilon) = \int_{x_1}^{x_2} f(x, Y, Y')dx \tag{5}$$

which is a function of the parameter ϵ. Thus the problem of determining $y(x)$ reduces to finding the extremum of $I(\epsilon)$ at $\epsilon = 0$ which is obtained by solving $I'(\epsilon = 0) = 0$. For this, differentiate (5) w.r.t. ϵ, we get

$$\frac{dI}{d\epsilon} = I'(\epsilon) = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \epsilon}\right)dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta'\right)dx$$

putting $\epsilon = 0$,

$$I'(0) = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx \tag{6}$$

because for $\epsilon = 0$, we have from (2) $Y = y$ and $Y' = y'$. Integrating the second integral in R.H.S. of (6) by parts, we have

$$I'(0) = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\eta + \left[\frac{\partial f}{\partial y'}\eta\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx\right]$$

Since by (3), $\eta(x_1) = \eta(x_2) = 0$, the second term vanishes and using $I'(0) = 0$, we get

$$I'(0) = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\, dx = 0 \tag{7}$$

Since $\eta(x)$ is arbitrary, equation (7) holds good only when the integrand is zero

i.e.,
$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \tag{2}$$

Note: Equation (2) is not sufficient condition. Solution of (2) may be maximum or minimum or a horizontal inflexion. Thus $y(x)$ is known as extremizing function or extremal and the term extremum includes maximum or minimum or stationary value.

EQUIVALENT FORMS OF EULER'S EQUATION:

(**I**) Differentiating f, which is a function of x, y, y', w.r.t. x, we get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} \tag{8}$$

Consider

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}y'' \tag{9}$$

Subtracting (9) from (8), we have

$$\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Rewriting this

$$\frac{d}{dx}\left\{f - y'\frac{\partial f}{\partial y'}\right\} - \frac{\partial f}{\partial x} = y'\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\} \tag{10}$$

Since by Euler's Equation (2), the R.H.S. of (10) is zero, we get another form of Euler's equtaion

$$\boxed{\frac{d}{dx}\left\{f - y'\frac{\partial f}{\partial y'}\right\} - \frac{\partial f}{\partial x} = 0} \tag{11}$$

(**II**) Since $\frac{\partial f}{\partial y'}$ is also function ϕ of x, y, y' say $\frac{\partial f}{\partial y'} = \phi(x, y, y')$. Differentiating w.r.t. x

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} + \frac{\partial \phi}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + y'\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right) + y''\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial^2 f}{\partial x \partial y'} + y'\frac{\partial^2 f}{\partial y \partial y'} + y''\frac{\partial^2 f}{\partial y'^2} \tag{12}$$

Substituting (12) in the Euler's equation (2), we have

$$\boxed{\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0} \tag{13}$$

General case: the necessary condition for the occurrence of extremum of the general integral

$$\int_{x_1}^{x_2} f(x, y_1, y_2, \ldots, y_\eta, y_1', y_2', \ldots, y_\eta')dx$$

involving η functions $y_1, y_2, \ldots, y_\eta$, is given by the set of η Euler's equations

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

for $i = 1, 2, 3, \ldots, \eta$.

First integrals of the Euler-Lagrang's equation:

Degenerate cases: Euler's equation is readily integrable in the following cases:

Case (a): If f is independent of x, then $\frac{\partial f}{\partial x} = 0$ and equivalent form of Euler's Equation (11) reduces to

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Integrating, we get the first integral of Euler's equation

$$\boxed{f - y'\frac{\partial f}{\partial y'} = \text{constant}} \qquad (14)$$

Thus the extremizing function y is obtained as the solution of a first-order differential equation (14) involving y and y' only.

Case (b): If f is independent of y, then $\frac{\partial f}{\partial y} = 0$, and the Euler's Equation (2) reduces to

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Integrating, we get the first integral of the Euler's equation as,

$$\boxed{\frac{\partial f}{\partial y'} = \text{constant}} \qquad (15)$$

which is a first order differential equation involving y' and x only.

Case (c): If f is independent of x and y then the partial derivative $\frac{\partial f}{\partial y'}$ is independent of x and y and is therefore function of y' alone. Now (15) of case (b) $\frac{\partial f}{\partial y'} = \text{constant}$ has the solution.

$$y' = \text{constant} = c_1$$

Integrating, the extremizing function is a linear function of x given by

$$y = c_1 x + c_2$$

Case (d): If f is independent of y', then $\frac{\partial f}{\partial y'} = 0$ and the Euler's Equation (2) reduces to

$$\frac{\partial f}{\partial y} = 0$$

Integrating, we get $\boxed{f = f(x)}$, i.e., function of x alone.

Geodesics: A geodesic on a surface is a curve on the surface along which the distance between any two points of the surface is a minimum.

4.4 STANDARD VARIATIONAL PROBLEMS

Shortest distance

Example 1: Find the shortest smooth plane curve joining two distinct points in the plane.

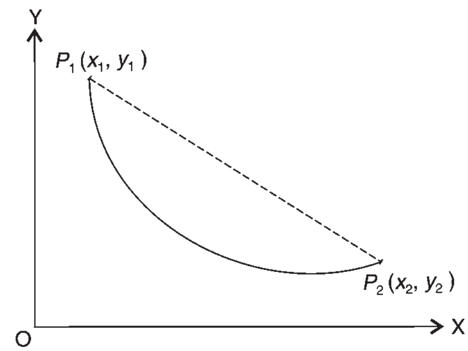

Fig. 4.2

Solution: Assume that the two distinct points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in the XY-Plane. If $y = f(x)$ is the equation of any plane curve c in XY-Plane and passing through the points P_1 and P_2, then the length L of curve c is given by

$$L[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\,dx \qquad (1)$$

The variational problem is to find the plane curve whose length is shortest i.e., to determine the function $y(x)$ which minimizes the functional (1). The condition for extrema is the Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Here $f = \sqrt{1 + y'^2}$ so $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = \frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}}$

Then

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

or $\quad y' = k\sqrt{1 + y'^2} \quad$ where $k = \text{constant}$

Squaring $\quad y'^2 = k^2(1 + y'^2)$

i.e., $$y' = \sqrt{\frac{k^2}{1-k^2}} = m = \text{constant}.$$

Integrating, $y = mx + c$, where c is the constant of integration. Thus the straight line joining the two points P_1 and P_2 is the curve with shortest length (distance).

Brachistochrone (shortest time) problem

Example 2: Determine the plane curve down which a particle will slide without friction from the point $A(x_1, y_1)$ to $B(x_2, y_2)$ in the shortest time.

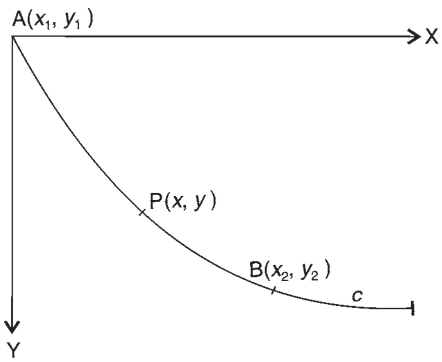

Fig. 4.3

Solution: Assume the positive direction of the y-axis is vertically downward and let $x_1 < x_2$. Let $P(x, y)$ be the position of the particle at any time t, on the curve c. Since energy is conserved, the speed v of the particle sliding along any curve is given by

$$v = \sqrt{2g(y - y^*)}$$

where $y^* = y_1 - \left(\frac{v_1^2}{2g}\right)$. Here g is acceleration due to gravity, v_1 is the initial speed. Choose the origin at A so that $x_1 = 0$, $y_1 = 0$ and assume that $v_1 = 0$. Then

$$\frac{ds}{dt} = v = \sqrt{2gy}$$

Integrating this, we get the time taken by the particle moving under gravity (and neglecting friction along the curve and neglecting resistance of the medium) from $A(0, 0)$ to $B(x_2, y_2)$ is

$$t[y(x)] = \int \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_{x=0}^{x=x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx \quad (1)$$

subject to the boundary conditions $y(0) = 0$ and $y(x_2) = y_2$. Integral (1) is convergent although it is improper. Here

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

which is independent of x. Now

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{1+y'^2}} \cdot \frac{1}{2} \cdot 2y'$$

The Euler's equation

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$$

reduces to

$$\frac{d}{dx}\left[\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}}\right] = 0$$

Integrating

$$\frac{\sqrt{1+y'^2}\sqrt{1+y'^2} - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = k_1 = \text{constant}$$

or $$y(1 + y'^2) = k_2 \quad (1)$$

where $k_2 = \left(\frac{1}{k_1}\right)^2$, put $y' = \cot\theta$ where θ is a parameter. Then from (1)

$$y = \frac{k_2}{1+y'^2} = \frac{k_2}{1+\cot^2\theta} = k_2 \sin^2\theta = \frac{k_2}{2}(1 - \cos 2\theta) \quad (2)$$

Now

$$dx = \frac{dy}{y'} = \frac{\frac{k_2}{2}(+2 \cdot \sin 2\theta)d\theta}{\cot\theta}$$
$$= \frac{k_2 2 \cdot \sin\theta \cdot \cos\theta d\theta}{\cot\theta} = 2k_2 \sin^2\theta d\theta$$
$$dx = k_2 \cdot (1 - \cos 2\theta)d\theta.$$

Integrating, $x = k_2\left(\theta - \frac{\sin 2\theta}{2}\right) + k_3$, where k_3 is constant of integration. So

$$x - k_3 = \frac{k_2}{2}(2\theta - \sin 2\theta) \quad (2)$$

Since $y = 0$ at $x = 0$, we have $k_3 = 0$. Put $2\theta = \phi$ in (1) and (2), then

$$x = \frac{k_2}{2}(\phi - \sin\phi),\ y = \frac{k_2}{2}(1 - \cos 2\phi) \quad (3)$$

Equation (3) represents a one parameter family of cycloids with $\frac{k_2}{2}$ as the radius of the rolling circle. Using the condition that the curve (cycloid) passes through $B(x_2, y_2)$, the value of the constant k_2 can be determined.

Note: A curve having this property of shortest time is known as "brachistochrone" with Greek words 'brachistos' meaning shortest and 'chronos' meaning time. In 1696 John Bernoulli advanced this 'brachistochrone' problem, although it was also studied by Leibnitz, Newton and L'Hospital.

Minimal surface area

Example 3: Find the curve c passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ such that the rotation of the curve c about x-axis generates a surface of revolution having minimum surface area.

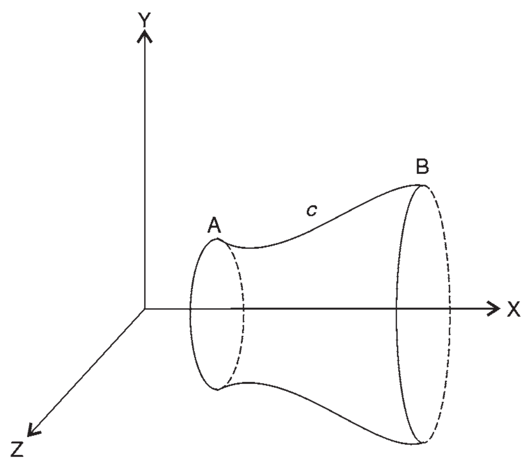

Fig. 4.4

Solution: The surface area S generated by revolving the curve c defined by $y(x)$ about x-axis is

$$S[y(x)] = \int_A^B 2\pi y\, ds = \int_{x=x_1}^{x_2} 2\pi y\sqrt{1+y'^2}dx \quad (1)$$

To find the extremal $y(x)$ which minimizes (1). Here $f = y\sqrt{1+y'^2}$ which is independent of x. The Euler's equation is

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \quad \text{or} \quad f - y'\frac{\partial f}{\partial y'} = \text{constant} = c_1$$

Substituting f and $\frac{\partial f}{\partial y'}$, we have

$$y\sqrt{1+y'^2} - y'\frac{y}{2}\frac{1}{\sqrt{1+y'^2}} \cdot 2y' = c_1$$

$$\frac{y\{(1+y'^2) - y'^2\}}{\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}} = c_1 \quad (2)$$

Put $y' = \sinh t$, then from (2)

$$\frac{y}{\sqrt{1+\sin^2 \mathrm{h}t}} = \frac{y}{\cosh t} = c_1 \quad \text{or} \quad y = c_1 \cosh t \quad (3)$$

$$\text{So} \quad dx = \frac{dy}{y'} = \frac{c_1 \sinh t\, dt}{\sinh t} = c_1\, dt$$

$$\text{Integrating} \quad x = c_1 t + c_2 \quad (4)$$

where c_2 is the constant of integration. Eliminating 't' between (3) and (4)

$$t = \frac{x - c_2}{c_1}$$

$$\text{therefore} \quad y = c_1 \cosh t = c_1 \cosh\left(\frac{x - c_2}{c_1}\right) \quad (5)$$

Equation (5) represents a two parameter family of catenaries. The two constants C_1 and C_2 are determined using the end (boundary) conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Solid of revolution with least resistance

Example 4: Determine the shape of solid of revolution moving in a flow of gas with least resistance.

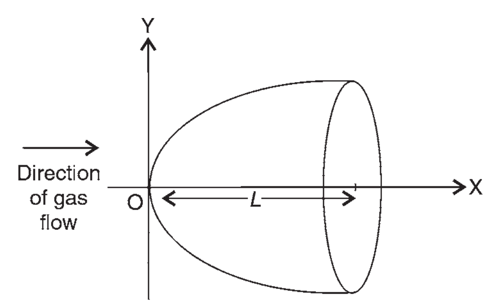

Fig. 4.5

Solution: The total resistance experienced by the body is

$$F[y(x)] = 4\pi\rho v^2 \int_0^L yy'^3\, dx$$

with boundary conditions $y(0) = 0$, $y(L) = R$. Here ρ is the density, v is the velocity of gas relative to solid. Here $f = yy'^3$ is independent of x. The Euler's equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = y'^3 - \frac{d}{dx}(3yy'^2) = 0 \qquad (1)$$

Multiplying (1) by y', we get

$$\frac{d}{dx}(yy'^3) = 0$$

Integrating

$$yy'^3 = c_1^3 \quad \text{or} \quad y' = \frac{c_1}{y^{\frac{1}{3}}}$$

Integrating $y^{\frac{1}{3}}dy = c_1 dx$ yields

$$\frac{y^{\frac{4}{3}}}{\frac{4}{3}} = c_1 x + c_2$$

$$\text{or} \qquad y(x) = (c_3 x + c_4)^{\frac{3}{4}} \qquad (2)$$

Using boundary conditions

$$0 = y(0) = 0 + c_4 \qquad \therefore c_4 = 0$$

$$R = y(L) = (c_3 L)^{\frac{3}{4}} \qquad \therefore c_3 = \frac{R^{\frac{4}{3}}}{L}$$

The the required function $y(x)$ is given by

$$y(x) = R\left(\frac{x}{L}\right)^{\frac{3}{4}}.$$

Geodesics

Example 5: Find the geodesics on a sphere of radius 'a'.

Solution: In spherical coordinates r, θ, ϕ, the differential of arc length on a sphere is given by

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$

Since $r = a = \text{constant}$, $dr = 0$. So

$$\left(\frac{ds}{d\theta}\right)^2 = a^2 + a^2\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2$$

Integrating w.r.t. θ between θ_1 and θ_2,

$$s = \int_{\theta_1}^{\theta_2} a\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

Here $f = a\sqrt{1 + \sin^2\theta\cdot\left(\frac{d\phi}{d\theta}\right)^2}$ is independent of ϕ, but is a function of θ and $\frac{d\phi}{d\theta}$. Denoting $\frac{d\phi}{d\theta} = \phi'$, the Euler's equation reduces to

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial\phi'}\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial\phi'} = \text{constant}.$$

$$\text{i.e., } a\cdot\frac{1}{\sqrt{1 + \sin^2\theta\phi'^2}}\cdot\frac{1}{2}2\cdot\sin^2\theta\cdot\phi' = k = \text{constant}$$

Squaring $a^2\sin^4\theta\cdot\phi'^2 = k^2(1 + \sin^2\theta\cdot\phi'^2)$

$$\text{or } \frac{d\phi}{d\theta} = \phi' = \frac{k}{\sin\theta\cdot\sqrt{\sin^2\theta - k^2}} = \frac{k\text{cosec}^2\theta}{\sqrt{1 - c^2\text{cosec}^2\theta}}$$

Integrating, we get

$$\phi(\theta) = \int\frac{k\text{cosec}^2\theta\, d\theta}{\sqrt{(1-k^2) - (k\cot\theta)^2}} + c_2$$

$$\phi(\theta) = \cos^{-1}\left\{\frac{k\cot\theta}{\sqrt{1-k^2}}\right\} + c_2$$

where c_2 is constant of integration. Rewriting

$$\frac{k\cot\theta}{\sqrt{1-k^2}} = \cos(\phi - c_2) = \cos\phi\cdot\cos c_2 + \sin\phi\cdot\sin c_2$$

$$\text{or} \quad \cot\theta = A\cos\phi + B\sin\phi$$

$$\text{where} \quad A = \frac{(\cos c_2)(\sqrt{1-k^2})}{k},$$

$$B = (\sin c_2)\frac{(\sqrt{1-k^2})}{k}$$

Multiplying by $a\sin\theta$, we have

$$a\cos\theta = A\cdot a\cdot\sin\theta\cdot\cos\phi + B\cdot a\cdot\sin\theta\cdot\sin\phi$$

Since $r = a$, the spherical coordinates are $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$, so

$$z = Ax + By$$

which is the equation of plane, passing through origin (0, 0, 0) (since no constant term) the centre of sphere. This plane cuts the sphere along a great circle. Hence the great circle is the geodesic on the sphere.

WORKED OUT EXAMPLES

Variational problems.
f is dependent on x, y, y'

Example 1: Find a complete solution of the Euler-Lagrange equation for

$$\int_{x_1}^{x_2}\left[y^2-(y')^2-2y\cosh x\right]dx \qquad (1)$$

Solution: Here $f(x, y, y') = y^2 - (y')^2 - 2y\cosh x$, which is a function of x, y, y'. The Euler-Lagrange equation is

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \qquad (2)$$

Differentiating (1) partially w.r.t. y and y', we get

$$\frac{\partial f}{\partial y}=2y-2\cosh x \qquad (3)$$

$$\frac{\partial f}{\partial y'}=-2y' \qquad (4)$$

Substituting (3) and (4) in (2), we have

$$2y-2\cosh x-\frac{d}{dx}(-2y')=0$$

$$y''+y=\cosh x \qquad (5)$$

The complimentary function of (5) is

$$y_c=c_1\cos x+c_2\sin x$$

and particular integral of (5) is

$$y=\frac{1}{2}\cosh x.$$

Thus the complete solution Euler-Lagrange Equation (5) is

$$y(x)=c_1\cos x+c_2\sin x+\frac{1}{2}\cosh x.$$

f is independent of x

Example 1: Find the extremals of the functional

$$I[y(x)]=\int_{x_1}^{x_2}\frac{(1+y^2)}{y'^2}dx$$

Solution: Here $f=\frac{1+y^2}{y'^2}$ which is independent of x. So the Euler's equation becomes

$$\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=0 \qquad (1)$$

Here $$\frac{\partial f}{\partial y'}=\frac{\partial}{\partial y'}\left(\frac{1+y^2}{y'^2}\right)=-\frac{2(1+y^2)}{y'^3} \qquad (2)$$

Substituting (2) in (1), we have

$$\frac{d}{dx}\left(\frac{1+y^2}{y'^2}-y'\frac{(-2)(1+y^2)}{y'^3}\right)=3\frac{d}{dx}\left(\frac{1+y^2}{y'^2}\right)=0$$

$$\frac{y'^2(2yy')-(1+y^2)2y'y''}{y'^4}=0$$

or $$(1+y^2)y''-yy'^2=0 \qquad (3)$$

Put $y'=p$, then $y''=\frac{d}{dx}y'=\frac{d}{dx}p=\frac{dp}{dy}\frac{dy}{dx}=y'\frac{dp}{dy}=p\frac{dp}{dy}$. Putting these values in (3),

$$(1+y^2)p\frac{dp}{dy}-yp^2=0 \quad \text{or} \quad \frac{dp}{dy}=\frac{py}{1+y^2}$$

Integrating $$\frac{dp}{p}=\frac{y\,dy}{1+y^2}=\frac{1}{2}\frac{d(1+y^2)}{(1+y^2)}$$

$$p^2=c_1^2(1+y^2).$$

so $$p=c_1\sqrt{(1+y^2)} \quad \text{or} \quad \frac{dy}{dx}=c\sqrt{1+y^2}$$

Integrating $$\frac{dy}{\sqrt{1+y^2}}=c_1\,dx \quad \text{we get}$$

$$\sinh^{-1}y=c_1x+c_2$$

Thus the required extremal function is

$$y(x)=\sinh(c_1x+c_2)$$

where c_1 and c_2 are two arbitrary constant.

f is independent of y

Example 3: If the rate of motion $v=\frac{ds}{dt}$ is equal to x then the time t spent on translation along the curve $y=y(x)$ from one point $P_1(x_1, y_1)$ to another point $P_2(x_2, y_2)$ is a functional. Find the extremal of this functional, when $P(1, 0)$ and $P_2(2, 1)$.

Solution: Given $\frac{ds}{dt}=x$ or $\frac{ds}{x}=dt$.

But $ds=\sqrt{1+y'^2}dx$ so $\sqrt{1+y'^2}\frac{dx}{x}=dt$.

Integrating from P_1 to P_2

$$\int_{x_1}^{x_2} dt = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx. \text{ The functional is}$$

$$t[y(x)] = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx$$

Here $f = \frac{\sqrt{1+y'^2}}{x}$ which is independent of y. Euler's equation is $\frac{d}{dx}\left\{\frac{\partial f}{\partial y'}\right\} = 0$

$$\frac{d}{dx}\left\{\frac{1}{x}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1+y'^2}}\cdot 2y'\right\} = 0$$

$$\frac{x\sqrt{(1+y'^2)}y'' - y'\left\{(1+y'^2) + xy'y''\right\}}{x^2(1+y'^2)^{\frac{3}{2}}}$$

$$= \frac{xy'' - y'(1+y'^2)}{x^2(1+y'^2)^{\frac{3}{2}}} = 0$$

or $\quad xy'' - y'(1+y'^2) = 0.$

Put $y' = u$, then $x\frac{du}{dx} - u(1+u^2) = 0$

$$\frac{du}{u(1+u^2)} = \frac{du}{u} - \frac{udu}{1+u^2} = \frac{dx}{x}$$

Integrating $\left(\frac{u}{x}\right)^2 = c_1^2(1+u^2)$

$$y'^2 = c_1^2x^2(1+y'^2)$$

or $\quad y' = c_1x\sqrt{(1+y'^2)}.$

Put $y' = \tan v$, then $\sqrt{1+y'^2} = \sqrt{1+\tan^2 v} = \sqrt{\sec^2 v}$

so $\quad x = \dfrac{y'}{c_1(1+y'^2)} = \dfrac{1}{c_1}\dfrac{\tan v}{\sec v} = \dfrac{1}{c_1}\sin v \quad (1)$

and $\quad dx = \dfrac{1}{c_1}\cos v\, dv$

Now $\quad \dfrac{dy}{dx} = y' = \tan v$

$$dy = \tan v\, dx = \tan v \cdot \frac{1}{c_1}\cdot \cos v\, dv =$$

$$= \frac{1}{c_1}\sin v\, dv$$

Integrating $\quad y = -c_2\cos v + c_3 \quad (2)$

where $\quad c_2 = \dfrac{1}{c_1}$ or $\quad y - c_3 = -c_2 \cos v \quad (3)$

Squaring (1) and (3) and adding

$$x^2 + (y-c_3)^2 = (c_2\sin v)^2 + (-c_2\cos v)^2$$
$$= c_2^2 = c_4 \quad (4)$$

Equation (4) represents a two-parameter family of circles. If (4) passes through $P_1(1, 0)$ Then $y(0) = 1$. Then (4) becomes

$$1 + (0-c_3)^2 = c_4 \quad \text{or} \quad 1 + c_3^2 = c_4$$

If (4) passes through $P_2(2, 1)$ then $y(2) = 1$. So from (4),

$$4 + (1-c_3)^2 = c_4 = 1 + c_3^2 \quad \therefore c_3 = -2$$

and $c_4 = 5$. Thus the required extremal satisfying the end points P_1 and P_2 is

$$x^2 + (y+2)^2 = 5.$$

Invalid variational problem

Example 4: Test for an extremum of the functional

$$I[y(x)] = \int_0^1 (xy + y^2 - 2y^2y')dx, \text{ with } y(0)=1,\ y(1)=2.$$

Solution: Here $f = xy + y^2 - 2y^2y'$. Differentiating partially w.r.t. y and y', we have

$$\frac{\partial f}{\partial y} = x + 2y - 4yy' \quad \text{and} \quad \frac{\partial f}{\partial y'} = -2y^2.$$

Substituting these in the Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = (x+2y-4yy') - \frac{d}{dx}(-2y^2) = 0$$

$$= x + 2y - 4yy' + 4yy' = 0$$

or $\quad x + 2y = 0 \quad$ i.e., $y = -\dfrac{x}{2}$.

However, this function $y = f(x)$ does not satisfy the given boundary conditions $y(0) = 1$ and $y(1) = 2$ i.e., $1 = y(0) \neq 0$ and $2 = y(1) \neq= -\frac{1}{2}$. Thus an extremum can not be achieved on the class of continuous functions.

Geodesics

Example 5: Determine the equation of the geodesics on a right circular cylinder of radius 'a'.

Solution: In cylindrical coordinates (r, θ, z), the differential of arc ds on a cylinder is given by

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2$$

Since radius $r = a =$ constant, $dr = 0$. Then

$$\left(\frac{ds}{d\theta}\right)^2 = a^2 + \left(\frac{dz}{d\theta}\right)^2 \quad \text{or} \quad \frac{ds}{d\theta} = \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}$$

Integrating

$$s = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta.$$

Since geodesic is curve with minimum length, we have to find minimum of s. Here $f = \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}$ which is independent of the variable z. Now the Euler's equation is

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial z'} = \text{constant} = k$$

$$\text{so} \quad \frac{\partial f}{\partial z'} = \left\{\sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\right\} = \frac{1}{2}\frac{2 \cdot z'}{\sqrt{a^2 + z'^2}} = k$$

$$\text{or} \quad z'^2 = k^2(a^2 + z'^2)$$

$$z'^2 = \frac{k^2a^2}{1 - k^2}$$

$$\text{i.e.,} \quad z' = \frac{dz}{d\theta} = \frac{ka}{\sqrt{1 - k^2}}$$

Integrating $z(\theta) = \frac{ka\theta}{\sqrt{1-k^2}} + c_1$. Thus the equation of the geodesics which is a circular helix is

$$z = k^*\theta + c_1 \quad \text{and} \quad r = a$$

$$\text{where} \quad k^* = \frac{ka}{\sqrt{1 - k^2}}.$$

Example 6: Find the geodesics on a right circular cone of semivertical angle α.

Solution: In spherical coordinates (r, θ, ϕ) the differential of an arc ds on a right circular cone is given by

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (r \sin\alpha\, d\phi)^2.$$

With vertex of the cone at the origin and z-axis as the axis of the cone, the polar equation of cone is $\theta = \alpha =$ constant so $d\theta = 0$.

Then

$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dr}{d\phi}\right)^2 + r^2 \sin^2\alpha$$

Integrating w.r.t., ϕ

$$s = \int_{\phi_1}^{\phi_2} \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2 \sin^2\alpha} \cdot d\phi$$

The arc length s of the curve is to be minimized. Here $f = \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2 \sin^2\alpha}$ is independent of ϕ. Then the integral of Euler's equation is

$$f - r'\frac{\partial f}{\partial r'} = \text{constant} = k$$

$$\text{or} \quad \sqrt{r'^2 + r^2\sin^2\alpha} - r' \cdot \frac{1}{2}\frac{2r'}{\sqrt{r'^2 + r^2\sin^2\alpha}} = k$$

$$r'^2 + r^2\sin^2\alpha - r'^2 = k\sqrt{r'^2 + r^2\sin^2\alpha}$$

$$\text{squaring,} \quad r^4\sin^4\alpha = k^2(r'^2 + r^2\sin^2\alpha)$$

$$r'^2 = \frac{r^2\sin^2\alpha(r^2\sin^2\alpha - k^2)}{k^2}$$

$$\text{or} \quad \frac{dr}{d\phi} = \frac{r\sin\alpha}{k} \cdot \sqrt{r^2\sin^2\alpha - k^2}$$

$$\text{i.e.,} \quad \frac{kdr}{r\sqrt{r^2\sin^2\alpha - k^2}} = \sin\alpha \cdot d\phi.$$

$$\text{Integrating} \quad k \cdot \int \frac{dr}{r\sqrt{r^2\sin^2\alpha - k^2}} = \sin\alpha \cdot \phi + c_1$$

where c_1 is the constant of integration. Introducing $r = \frac{1}{t}$, $dr = -\frac{1}{t^2}dt$, $t = \frac{1}{r}$, the L.H.S. integral transforms to

$$k \cdot \int \cdot t\frac{1}{\sqrt{\frac{\sin^2\alpha}{t^2} - k^2}} \cdot \left(-\frac{dt}{t^2}\right) = -k\int \frac{dt}{\sqrt{\sin^2\alpha - k^2t^2}}$$

$$= \cos^{-1}\left(\frac{kt}{\sin\alpha}\right).$$

$$\text{Then} \quad \cos^{-1}\left(\frac{kt}{\sin\alpha}\right) = \phi\sin\alpha + c_1$$

$$\frac{kt}{\sin\alpha} = \cos(\phi\sin\alpha + c_1)$$

$$\text{Thus} \quad \frac{k}{r\sin\alpha} = \cos(\phi\sin\alpha + c_1)$$

and $\theta = \alpha$ are the equations of the geodesics.

EXERCISE

Variational problems

1. Test for extremum of the functional

$$I[y(x)] = \int_0^{\frac{\pi}{2}} (y'^2 - y^2)dx, \ y(0) = 0, \ y\left(\frac{\pi}{2}\right) = 1.$$

Hint: Euler's Equation (EE): $y'' + y = 0$, $y = c_1 \cos x + c_2 \sin x$ using B.C, $c_1 = 0$, $c_2 = 1$

Ans. $y = \sin x$

Find the extremal of the following functionals

2. $\int_{x_1}^{x_2} (y^2 + y'^2 - 2y \sin x)dx$

Hint: EE: $2y - 2\sin x - 2y'' = 0$

Ans. $y = c_1 e^x + c_2 e^{-x} + \frac{\sin x}{2}$

3. $\int_0^1 (y'^{12} + 12xy)dx$, $y(0) = 0$, $y(1) = 1$.

Hint: EE: $y'' = 6x$, $y = x^3 + c_1 x + C_2$, $C = 0$, $c_2 = 0$

Ans. $y = x^3$

4. $\int_0^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy)dy$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 0$

Hint: EE: $y'' + y = x$, $y = c_1 \cos x + c_2 \sin x + x$

Ans. $y = x - \frac{\pi}{2} \sin x$

5. $\int_{x_1}^{x_2} (y^2 + 2xyy')dx$, $y(x_1) = y_1$, $y(x_2) = y_2$

Hint: EE: $2y + 2xy' - 2(xy' + y) = 0$ i.e., $0 = 0$

Ans. Invalid problem

6. $\int_1^2 \frac{x^3}{y'^2} dx$, $y(1) = 1$, $y(2) = 4$

Ans. $y = x^2$

7. $\int_2^3 \frac{y'^2}{x^3} dx$, $y(2) = 1$, $y(3) = 16$

Hint: EE: $\frac{y''}{y'} = \frac{3}{x}$, $y' = cx^3$, $y = c_1 x^4 + c_2$

Ans. $y = \frac{3}{13}x^4 - \frac{35}{13}$

8. $\int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x)\, dx$

Ans. $y = Ae^x + Be^{-x} + \frac{1}{2}xe^x$

9. $\int_0^{\pi} (4y \cos x - y^2 + y'^2)dx$, $y(0) = 0$, $y(\pi) = 0$

Hint: EE: $y'' + y = 2\cos x$, $y = c_1 \cos x + c_2 \sin x + x \sin x$, $c_1 = 0$, $c_2 =$ arbitrary

Ans. $y = (C + x) \sin x$.

4.5 ISOPERIMETRIC PROBLEMS

In calculus, in problems of extrema with constraints it is required to find the maximum or minimum of a function of several variably $g(x_1, x_2, \ldots, x_\eta)$ where the variables $x_1, x_2, \ldots, x_\eta$ are connected by some given relation or condition known as a constraint.

The variational problems considered so far find the extremum of a functional in which the argument functions could be chosen arbitrarily except for possible end (boundary) conditions. However, the class of variational problems with subsidiary conditions or constraints imposed on the argument functions, apart from the end conditions, are branded as isoperimetric problems. In the original isoperimetric ("iso" for same, "perimetric" for perimeter) problem it is required to find a closed curve of given length which enclose maximum area. It is known even in ancient Greece that the solution to this problem is circle. This is an example of the extrema of integrals under constraint consists of maximumizing the area subject to the constraint (condition) that the length of the curve is fixed.

The simplest isoperimetric problem consists of finding a function $f(x)$ which extremizes the functional

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')dx \qquad (1)$$

subject to the constraint (condition) that the second integral

$$J[y(x)] = \int_{x_1}^{x_2} g(x, y, y')dx \qquad (2)$$

assumes a given prescribed value and satisfying the prescribed end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. To solve this problem, use the method of Lagrange's multipliers and form a new function

$$H(x, y, y') = f(x, y, y') + \lambda g(x, y, y') \qquad (3)$$

where λ is an arbitrary constant known as the Lagrange multiplier. Now the problem is to find the extremal of the new functional,

$I^*[y(x)] = \int_{x_1}^{x_2} H(x, y, y')dx$, subject to no constraints (except the boundary conditions). Then the modified Euler's equation is given by

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \tag{4}$$

The complete solution of the second order Equation (4) contains, in general, two constants of integration say c_1, c_2 and the unknown Lagrange multiplier λ. These 3 constants c_1, c_2, λ will be determined using the two end conditions $y(x_1) = y_1, y(x_2) = y_2$ and given constraint (2).

Corollary: Parametric form: To find the extremal of the functional

$$I = \int_{t_1}^{t_2} f(x, y, \dot{x}, \dot{y}, t)dt$$

subject to the constraint

$$J = \int_{t_1}^{t_2} g(x, y, \dot{x}, \dot{y}, t)dt = \text{constant}$$

solve the system of two Euler equations given by

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0 \quad \text{and} \quad \frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$$

resulting in the solution $x = x(t)$, $y = y(t)$, which is the parametric representation of the required function $y = f(x)$ which is obtained by elimination of t. Here $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$ and

$$H(x, y, \dot{x}, \dot{y}, t) = f(x, y, \dot{x}, \dot{y}, t) + \lambda\, g(x, y, \dot{x}, \dot{y}, t)$$

The two arbitrary constants c_1, c_2 and λ are determined using the end conditions and the constraint.

4.6 STANDARD ISOPERIMETRIC PROBLEMS

Circle

Example 1: Isoperimetric problem is to determine a closed curve C of given (fixed) length (perimeter) which encloses maximum area.

Solution: Let the parametric equation of the curve C be

$$x = x(t), \quad y = y(t) \tag{1}$$

where t is the parameter. The area enclosed by curve C is given by the integral

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - \dot{x}y)dt \tag{2}$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$. We have $x(t_1) = x(t_2) = x_0$ and $y(t_1) = y(t_2) = y_0$, since the curve is closed. Now the total length of the curce C is given by

$$J = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}dt \tag{3}$$

Form $$H = \frac{1}{2}(x\dot{y} - \dot{x}y) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2} \tag{4}$$

Here λ is the unknown Lagrangian multiplier. Problem is to find a curve with given perimeter for which area (2) is maximum. Euler equations are

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0 \tag{5}$$

and $$\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0 \tag{6}$$

Differentiating H in (4) w.r.t. $x, \dot{x}, y, \dot{y}$ and substituting them in (5) and (6), we get

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{7}$$

$$-\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{8}$$

Integrating (7) and (8) w.r.t. 't', we get

$$y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_1 \tag{9}$$

and $$x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_2 \tag{10}$$

where c_1 and c_2 are arbitrary constants. From (9) and (10) squaring $(y - c_1)$ and $(x - c_2)$ and adding, we get

$$(x - c_2)^2 + (y - c_1)^2 = \left(\frac{-\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)^2 + \left(\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)^2$$

$$= \lambda^2\frac{(\dot{x}^2 + \dot{y}^2)}{(\dot{x}^2 + \dot{y}^2)} = \lambda^2$$

i.e., $(x - c_2)^2 + (y - c_1)^2 = \lambda^2$

which is the equation of circle. Thus we have obtained the well-known result that the closed curve of given perimeter for which the enclosed area is a maximum is a circle.

Catenary

Example 2: Determine the shape an absolutely flexible, inextensible homogeneous and heavy rope of given length L suspended at the points A and B

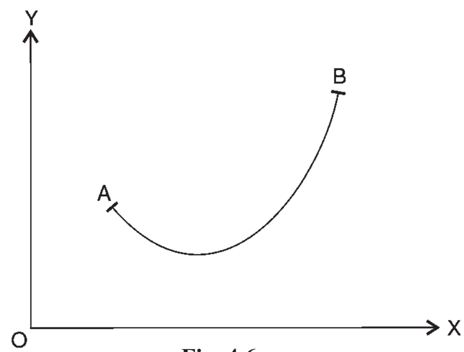

Fig. 4.6

Solution: The rope in equilibrium take a shape such that its centre of gravity occupies the lowest position. Thus to find minimum of y-coordinate of the centre of gravity of the string given by

$$I[y(x)] = \frac{\int_{x_1}^{x_2} y\sqrt{1+y'^2}dx}{\int_{x_1}^{x_2}\sqrt{1+y'^2}dx} \tag{1}$$

subject to the constraint

$$J[y(x)] = \int_{x_1}^{x_2}\sqrt{1+y'^2}dx = L = \text{constant} \tag{2}$$

Thus to minimize the numerator in R.H.S. of (1) subject to (2). Form

$$H = y\sqrt{(1+y'^2)} + \lambda\sqrt{1+y'^2} = (y+\lambda)\sqrt{1+y'^2} \tag{3}$$

where λ is Lagrangian multiplier. Here H is independent of x. So the Euler equation is

$$H - y'\frac{\partial H}{\partial y'} = \text{constant} = k_1$$

i.e., $$(y+\lambda)(\sqrt{1+y'^2}) - y'(y+\lambda)\cdot\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}} = k_1$$

$$(y+\lambda)\left\{(1+y'^2) - y'^2\right\} = k_1(\sqrt{1+y'^2})$$

or $$y + \lambda = k_1\sqrt{1+y'^2} \tag{4}$$

Put $y' = \sinh t$, where t is a parameter, in (4)

Then $$y+\lambda = k_1\sqrt{1+\sin^2 \mathrm{h}t} = k_1\cosh t \tag{5}$$

Now $$dx = \frac{dy}{y'} = \frac{k_1\sinh t\ dt}{\sinh t} = k_1 dt$$

Integrating $$x = k_1 t + k_2 \tag{6}$$

Eliminating 't' between (5) and (6), we have

$$y+\lambda = k_1\cosh t = k_1\cosh\left(\frac{x-k_2}{k_1}\right) \tag{7}$$

Equation (7) is the desired curve which is a catenary.

Note: The three unknowns λ, k_1, k_2 will be determined from the two boundary conditions (curve passing through A and B) and the constraint (2).

Worked Out Examples

Example 1: Find the extremal of the function $I[y(x)] = \int_0^\pi (y'^2 - y^2)dx$ with boundary conditions $y(0) = 0$, $y(\pi) = 1$ and subject to the constraint $\int_0^\pi y\ dx = 1$.

Solution: Here $f = y'^2 - y^2$ and $g = y$. So choose $H = f + \lambda g = (y'^2 - y^2) + \lambda y$ where λ is the unknown Lagrange's multiplier. The Euler's equation for H is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

Using derivatives of H w.r.t. y and y', we get

$$(-2y+\lambda) - \frac{d}{dx}(2y') = 0$$

or $$y'' + y = \lambda$$

whose general solution is

$$y(x) = CF + PI = (c_1\cos x + c_2\sin x) + (\lambda) \tag{1}$$

The three unknowns c_1, c_2, λ in (1) will be determined using the two boundary conditions and the given constraint. From (1)

$$0 = y(0) = c_1 + c_2 \cdot 0 + \lambda \quad \text{or} \quad c_1 + \lambda = 0$$
$$1 = y(\pi) = -c_1 + c_2 \cdot 0 + \lambda \quad \text{or} \quad -c_1 + \lambda = 1$$

Solving $\lambda = \frac{1}{2}, c_1 = -\lambda = -\frac{1}{2}$
Now from the given constraint

$$\int_0^{\pi} y\, dx = 1, \quad \text{we have}$$

$$\int_0^{\pi} (c_1 \cos x + c_2 \sin x + \lambda) dx = 1$$

$$c_1 \sin x - c_2 \cos x + \lambda x \Big|_0^{\pi} = 1$$

$$(0 + c_2 + \lambda\pi) - (0 - c_2 + 0) = 1$$

$$\text{or} \qquad 2c_2 = 1 - \pi\lambda = \left(1 - \frac{\pi}{2}\right)$$

Thus the required extremal function $y(x)$ is

$$y(x) = -\frac{1}{2}\cos x + \left(\frac{1}{2} - \frac{\pi}{4}\right)\sin x + \frac{1}{2}.$$

Example 2: Show that the extremal of the isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2} y'^2\, dx$ subject to the condition $J[y(x)] = \int_{x_1}^{x_2} y\, dx = \text{constant} = k$ is a parabola. Determine the equation of the parabola passing through the points $P_1(1, 3)$ and $P_2(4, 24)$ and $k = 36$.

Solution: Here $f = y'^2$ and $g = y$. So form

$$H = f + \lambda g = y'^2 + \lambda y.$$

The Euler equation for H is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$\lambda - \frac{d}{dx}(2y') = 0$$

$$\text{or} \quad y'' - \frac{\lambda}{2} = 0$$

Integrating twice,

$$y(x) = \frac{\lambda}{2}\frac{x^2}{2} + c_1 x + c_2 \tag{1}$$

which is a parabola. Here c_1 and c_2 are constants of integration. To determine the particular parabola, use B.C's $y(1) = 3$ and $y(4) = 24$ (i.e., passing through points P_1 and P_2) and the given constraint. From (1)

$$3 = y(1) = \frac{\lambda}{4} + c_1 + c_2 \tag{2}$$

Again from (1)

$$24 = y(4) = 4\lambda + 4c_1 + c_2 \tag{3}$$

Now from the constraint

$$\int_{x_1=1}^{x_2=4} y(x)dx = 36$$

$$\text{or} \quad \int_1^4 \left(\frac{\lambda}{4}x^2 + c_1 x + c_2\right) dx = 36$$

$$\text{i.e.,} \quad \frac{\lambda}{4}\cdot\frac{x^3}{3} + c_1\frac{x^2}{2} + c_2 x\Big|_1^4 = 36$$

$$\text{or} \quad 42\lambda + 60c_1 + 24c_2 = 288 \tag{4}$$

From (2) & (3):

$$\lambda - c_2 = 12$$

and from (3) & (4)

$$2\lambda - c_2 = 8$$

Solving $\lambda = -4, c_2 = -16, c_1 = 20$. Thus the specific parabola satisfying the given B.C.'s (passing through P_1 and P_2) is

$$y = -\frac{4}{4}x^2 + 20x - 16$$

$$\text{i.e.,} \quad y = -x^2 + 20x - 16.$$

EXERCISE

1. Find the curve of given length L which joins the points $(x_1, 0)$ and $(x_2, 0)$ and cuts off from the first quadrant the maximum area.

Ans. $(x - c)^2 + (y - d)^2 = \lambda^2, c = \frac{x_1+x_2}{2}$, $a = \frac{(x_2-x_1)}{2}, \lambda^2 = d^2 + a^2, \sqrt{d^2 + a^2}$ $\cot^{-1}\left(\frac{d}{a}\right) = \frac{L}{2}$.

2. Determine the curve of given length L which joins the points $(-a, b)$ and (a, b) and generates the minimum surface area when it is revolved about the x-axis.

Ans. $y = c\cosh\frac{x}{c} - \lambda$, where $c = \frac{a}{\sin h^{-1}\left(\frac{L}{2}\right)}$, $\lambda = \frac{c}{2}\sqrt{4 + L^2} - b$

3. Find the extremal of $I = \int_0^\pi y'^2 dx$ subject to $\int_0^\pi y^2 dx = 1$ and satisfying $y(0) = y(\pi) = 0$

   Hint: EE: $y'' - \lambda y = 0$

Ans. $y_\eta(x) = \pm\sqrt{\frac{2}{\pi}} \sin \eta x$, $\eta = 1, 2, 3 \ldots$

4. Show that sphere is the solid of revolution which has maximum volume for a given surface area.

   Hint: $H = \pi y^2 + \lambda[(2\pi y)\sqrt{(1 + y'^2)}]$, EE: $y' = \frac{\sqrt{4\lambda^2 - y^2}}{y}$, $(x - 2\lambda)^2 + y^2 = (2\lambda)^2$; circle, solid of revolution sphere.

5. Find the curve of given length L which minimizes the curved surface area of the solid generated by the revolution of the curve about the x-axis.

Ans. Catenary

6. Determine $y(x)$ for which $\int_0^1 (x^2 + y'^2) dx$ is stationary subject to $\int_0^1 y^2 dx = 2$, $y(0) = 0$, $y(1) = 0$.

Ans. $y = \pm 2 \sin m\pi x$, where m is an integer.