Chapter **2**

Polynomial Equations and Inequalities

Curriculum Expectations

Polynomial and Rational Functions

Connecting Graphs and Equations of Polynomial Functions

C1.7 determine an equation of a polynomial function that satisfies a given set of conditions (e.g., degree of the polynomial, intercepts, points on the function), using methods appropriate to the situation (e.g., using the x-intercepts of the function; using a trial-and-error process with a graphing calculator or graphing software; using finite differences), and recognize that there may be more than one polynomial function that can satisfy a given set of conditions (e.g., an infinite number of polynomial functions satisfy the condition that they have three given x-intercepts)

C1.8 determine the equation of the family of polynomial functions with a given set of zeros and of the member of the family that passes through another given point [e.g., a family of polynomial functions of degree three with zeros 5, -3, and -2 is defined by the equation $f(x) = k(x - 5)(x + 3)(x + 2)$, where k is a real number, $k \neq 0$; the member of the family that passes through $(-1, 24)$ is $f(x) = -2(x - 5)(x + 3)(x + 2)$]

Solving Polynomial and Rational Equations

C3.1 make connections, through investigation using technology (e.g., computer algebra systems), between the polynomial function $f(x)$, the divisor $x - a$, the remainder from the division $\dfrac{f(x)}{x - a}$, and $f(a)$ to verify the remainder theorem and the factor theorem

C3.2 factor polynomial expressions in one variable, of degree no higher than four, by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring, factoring by grouping, remainder theorem, factor theorem)

C3.3 determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), the connection between the real roots of a polynomial equation and the x-intercepts of the graph of the corresponding polynomial function, and describe this connection [e.g., the real roots of the equation $x^4 - 13x^2 + 36 = 0$ are the x-intercepts of the graph of $f(x) = x^4 - 13x^2 + 36$]

C3.4 solve polynomial equations in one variable, of degree no higher than four (e.g., $2x^3 - 3x^2 + 8x - 12 = 0$), by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring, factoring by grouping, remainder theorem, factor theorem), and verify solutions using technology (e.g., using computer algebra systems to determine the roots; using graphing technology to determine the x-intercepts of the graph of the corresponding polynomial function)

C3.7 solve problems involving applications of polynomial and simple rational functions and equations [e.g., problems involving the factor theorem or remainder theorem, such as determining the values of k for which the function $f(x) = x^3 + 6x^2 + kx - 4$ gives the same remainder when divided by $x - 1$ and $x + 2$]

Solving Inequalities

C4.1 explain, for polynomial and simple rational functions, the difference between the solution to an equation in one variable and the solution to an inequality in one variable, and demonstrate that given solutions satisfy an inequality (e.g., demonstrate numerically and graphically that the solution to $\dfrac{1}{x + 1} < 5$ is $x < -1$ or $x > -\dfrac{4}{5}$)

C4.2 determine solutions to polynomial inequalities in one variable [e.g., solve $f(x) \geq 0$, where $f(x) = x^3 - x^2 + 3x - 9$] and to simple rational inequalities in one variable by graphing the corresponding functions, using graphing technology, and identifying intervals for which x satisfies the inequalities

Technology Notes
The technology used in this chapter is primarily graphing calculators, specifically the TI-83 Plus/TI-84 Plus series, and a computer algebra system (CAS), specifically the TI-89/89T series.

C4.3 solve linear inequalities and factorable polynomial inequalities in one variable (e.g., $x^3 + x^2 > 0$) in a variety of ways (e.g., by determining intervals using x-intercepts and evaluating the corresponding function for a single x-value within each interval; by factoring the polynomial and identifying the conditions for which the product satisfies the inequality), and represent the solutions on a number line or algebraically (e.g., for the inequality $x^4 - 5x^2 + 4 < 0$, the solution represented algebraically is $-2 < x < -1$ or $1 < x < 2$)

Chapter 2 Planning Chart

Section Suggested Timing	Student Text Page(s)	Teacher's Resource Blackline Masters	Assessment	Tools
Chapter 2 Opener • 10 min	81			
Prerequisite Skills • 45–60 min	82–83	• BLM 2–1 Prerequisite Skills		
2.1 The Remainder Theorem • 45–65 min	84–93	• T–4 The Computer Algebra System (CAS) on the TI-89 Calculator • BLM 2–2 Section 2.1 Practice		• computer algebra system
2.2 The Factor Theorem • 45–60 min	94–103	• T–4 The Computer Algebra System (CAS) on the TI-89 Calculator • BLM 2–3 Section 2.2 Practice	• BLM 2–4 Section 2.2 Achievement Check Rubric	• computer algebra system • graphing calculator
2.3 Polynomial Equations • 40–55 min	104–112	• BLM 2–5 Section 2.3 Practice		• graphing calculator
2.4 Families of Polynomial Functions • 40–60 min	113–122	• G–3 Four Quadrant Grids • BLM 2–6 Section 2.4 Practice		• grid paper • graphing calculator
2.5 Solve Inequalities Using Technology • 40–60 min	123–131	• G–1 Grid Paper • T–4 The Computer Algebra System (CAS) on the TI-89 Calculator • BLM 2–7 Section 2.5 Practice		• grid paper • graphing calculator • computer algebra system
2.6 Solve Factorable Polynomial Inequalities Algebraically • 45–60 min	132–139	• G–5 Number Lines • BLM 2–8 Section 2.6 Practice		
Chapter 2 Review • 45–50 min	140–141	• G–5 Number Lines • T–4 The Computer Algebra System (CAS) on the TI-89 Calculator • BLM 2–9 Chapter 2 Review		• graphing calculator • computer algebra system
Chapter 2 Problem Wrap-Up • 30–50 min	141	• G–1 Grid Paper	• BLM 2–10 Chapter 2 Problem Wrap-Up Rubric	• grid paper • graphing calculator • computer with *The Geometer's Sketchpad*®
Chapter 2 Practice Test • 30–45 min	142–143	• G–1 Grid Paper • T–4 The Computer Algebra System (CAS) on the TI-89 Calculator	• BLM 2–11 Chapter 2 Test	• grid paper • graphing calculator • computer algebra system
Chapter 2 Task: Can You Tell Just by Looking? • 60–75 min	144	• T–2 *The Geometer's Sketchpad*® 4 • BLM 2–13 BLM Answers	• BLM 2–12 Task: Can You Tell Just by Looking? Rubric	• graphing calculator

Chapter 2 Blackline Masters Checklist

	BLM	Title	Purpose
Prerequisite Skills			
	BLM 2–1	Prerequisite Skills	Practice
2.1 The Remainder Theorem			
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 2–2	Section 2.1 Practice	Practice
2.2 The Factor Theorem			
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 2–3	Section 2.2 Practice	Practice
	BLM 2–4	Section 2.2 Achievement Check Rubric	Assessment
2.3 Polynomial Equations			
	BLM 2–5	Section 2.3 Practice	Practice
2.4 Families of Polynomials Functions			
	G–3	Four Quadrant Grids	Student Support
	BLM 2–6	Section 2.4 Practice	Practice
2.5 Solve Inequalities Using Technology			
	G–1	Grid Paper	Student Support
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 2–7	Section 2.5 Practice	Practice
2.6 Solve Factorable Polynomial Inequalities Algebraically			
	G–5	Number Lines	Student Support
	BLM 2–8	Section 2.6 Practice	Practice
Chapter 2 Review			
	G–5	Number Lines	Student Support
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 2–9	Chapter 2 Review	Practice
Chapter 2 Problem Wrap-Up			
	G–1	Grid Paper	Student Support
	BLM 2–10	Chapter 2 Problem Wrap-Up Rubric	Assessment
Chapter 2 Practice Test			
	G–1	Grid Paper	Student Support
	T–4	The Computer Algebra System (CAS) on the TI-89 Calculator	Technology
	BLM 2–11	Chapter 2 Test	Summative Assessment
Chapter 2 Task: Can You Tell Just by Looking?			
	BLM 2–12	Task: Can You Tell Just by Looking? Rubric	Assessment
	BLM 2–13	BLM Answers	Answers

Prerequisite Skills

Differentiated Instruction
Use an **anticipation guide** to get students thinking about the concepts in this chapter.

Assessment
You may wish to use **BLM 2–1 Prerequisite Skills** as a diagnostic assessment. Refer students to the Skills Appendix for examples and further practice of topics.

Chapter Problem

The Chapter Problem is introduced on page 93. Have students discuss their understanding of the problem, and how mathematics is relevant to running a business. The Chapter Problem is revisited in Sections 2.1 (question 17), 2.2 (question 16), 2.3 (question 16), 2.4 (question 18), and 2.5 (question 13). These questions are designed to help students move toward the Chapter 2 Problem Wrap-Up on page 141. The Chapter Problem questions may be assigned in each section where they appear. Alternatively, you may wish to assign them all with the Chapter Problem Wrap-Up when students have completed the chapter, as part of a summative assessment.

2.1 The Remainder Theorem

Teaching Suggestions

- Have students work in pairs on **Investigate 1**. The intent of **Investigate 1** is to help students remember the steps of long division and reinforce the related vocabulary. **Prerequisite Skills** question 1 supports this Investigate.
- When presenting **Example 1**, it is helpful to use different colours of chalk to correspond to each step of the Method 1 solution. Point out the importance of writing the polynomial in descending order of powers.
- Technology tip for **Example 1**, Method 2:
  - Another way to divide a polynomial by a binomial using CAS is to press $\boxed{\text{F2}}$ and select **3: expand**. Enter the division expression, and then press $\boxed{\text{ENTER}}$.
- In **Example 2**, the placeholder is helpful to subtract like terms.
- In **Example 3**, the remainder is 0 and the trinomial quotient can be factored further.
- Have students work in pairs to complete **Investigate 2**. The purpose of this Investigate is to have students discover the remainder theorem by making the connection between $P(b)$ and the remainder when $P(x)$ is divided by $x - b$. If needed, use **T–4 The Computer Algebra System (CAS) on the TI-89 Calculator** to support Method 2.
- In **Example 4**, stress that even though long division may be used to verify the remainder, the remainder theorem is the preferred method to determine the remainder.
- In **Example 5**, students should understand that long division should not be used to solve for k.
- For question C1 of **Communicate Your Understanding,** students should understand that division with polynomials represents a rational expression where the numerator is the dividend and the denominator is the divisor. Since division by zero is undefined the divisor cannot equal zero hence the restriction. For question C2, students should understand that a placeholder is used when aligning the terms in descending powers. Placeholders help to simplify the subtraction steps in the division. Question C4 points toward the factor theorem, which is dealt with in the next section.
- **Question 5** allows students to solve a problem by reflecting on the strategy needed to determine possible dimensions for the box and by using algebraic reasoning to make conjectures about how to do this. They will also need to select tools and make connections among different strands of mathematics to solve this problem.
- In **questions 12** and **13**, it is not necessary to know the remainder.
- **Question 18** helps students make the connection to the slope formula.
- **Question 18** requires students to reflect and interpret part of the question geometrically and part algebraically. It will be necessary to select tools in order to produce a geometric interpretation and make a representation in the form of a diagram for part d). Reasoning strategies and connecting skills will allow students to ascertain certain information that they will then communicate throughout the question.
- **Questions 20** and **21** lead students to solve a system of linear equations.
- Use **BLM 2–2 Section 2.1 Practice** for remediation or extra practice.

Investigate Answers (pages 84, 87–88)

Investigate 1

1. a) i) 753 **ii)** 22 **iii)** 34 **iv)** 5

  b) i) $x^2 + 5x + 7$ **ii)** $x + 2$ **iii)** $x + 3$ **iv)** 1

2. Answers may vary. Sample answers:

  a) The divisor 22 divides into 75, three times. Write 3 above the 5 found under the division sign. Multiply 22 by 3 to get 66. Write 66 below 75. Subtract to get 9. Bring down the 3. 22 divides into 93 four times. Write 4 above the 3. Multiply 22 by 4 to get 88. Write this under 93 and subtract to get the remainder 5.

  b) The x in the divisor divides into x^2 of the dividend x times. Write x above the term $5x$. Multiply $x + 2$ with x to get $x^2 + 2x$. Write these terms below $x^2 + 5x$. Subtract to get $3x$. Bring down the 7. The x in the divisor divides into $3x$, three times. Write 3 above the 7. Multiply $x + 2$ with 3 to get $3x + 6$. Write these terms below $3x + 7$ and subtract to get the remainder 1.

  c) Similarities: In both long divisions we multiply and subtract to obtain the result. For division of numbers, the digits are lined up by place value. For trinomial division, the divisor terms are lined up with like terms in the dividend.

  Differences: When dividing the trinomial, we only use the x in the divisor to determine how many times it divides into x^2, then the other terms as the subtraction occurs.

3. a) Multiply the quotient by the divisor and add the remainder to get the dividend.

  b) $753 = (22)(34) + 5$; $x^2 + 5x + 7 = (x + 2)(x + 3) + 1$

Investigate 2

METHOD 1

1. a) i) 1 **ii)** -1 **iii)** 2 **iv)** -2

  b) i) 1 **ii)** -1 **iii)** 2 **iv)** -2

2. a) the remainders   **i)** 5 **ii)** -1 **iii)** 32 **iv)** 8

  b) i) 5 **ii)** -1 **iii)** 32 **iv)** 8

  c) They are the same.

  d) Substitute the value b, from the binomial $x - b$, into the polynomial. The value $P(b)$ is the remainder when a polynomial function $P(x)$ is divided by $(x - b)$.

3. a) i) 7 **ii)** 15 **iii)** 55 **iv)** 19

  b) Long division gives the same remainders.

4. The remainder is equal to $P(b)$. The name *remainder theorem* is appropriate because it identifies that the resulting value is the remainder when a long division is performed.

METHOD 2

2. 5

3. 5

4. a) Yes, the remainder of $P(x) \div (x - 1)$ is $P(1)$.

  b) $-1, 32, 8$

  c) $-1, 32, 8$

  d) The answers are the same.

  e) Substitute the value b, from the binomial $x - b$, into the polynomial. The value $P(b)$ is the remainder when a polynomial function $P(x)$ is divided by $(x - b)$.

5. a) i) 7 **ii)** 15 **iii)** 55 **iv)** 19

  b) The CAS gives the same remainders.

6. The remainder is equal to $P(b)$. The name remainder theorem is appropriate because it identifies that the resulting value is the remainder when a long division is performed.

Communicate Your Understanding Responses (page 90)

C1. Division by zero is not defined. The value that makes the divisor zero is the restriction. If the divisor is $x - b$, then $x - b \neq 0$, so $x \neq b$.

C2. A placeholder might be necessary to replace a missing x^n-term, where n is less than the degree of the polynomial. The placeholder is helpful when subtracting like terms in the process of long division.

C3. The remainder is incorrect; it should be $R = \dfrac{15}{x - 2}$.

C4. The divisor is $(x + 3)$. The remainder is 0. The divisor is a factor of $P(x)$.

	Dividend	Divisor	Quotient	Remainder
a)	$6x^2 + 5x - 7$	$3x + 1$	$2x + 1$	-8
b)	$12x^3 + 2x^2 + 11x + 14$	$3x + 2$	$4x^2 - 2x + 5$	4
c)	$5x^3 - 7x^2 - x + 6$	$x - 1$	$5x^2 - 2x - 3$	3

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	5, 6
Reasoning and Proving	4–6, 10–22
Reflecting	5, 6, 16, 18
Selecting Tools and Computational Strategies	1–6, 10–22
Connecting	5–22
Representing	18
Communicating	16–19

COMMON ERRORS

- Students use long division to determine a remainder or solve for an unknown coefficient.

R_x Remind students that the remainder theorem is an efficient and simple means of determining a remainder and solving for unknown values, as shown in Examples 4 and 5.

- Students have difficulty with the steps of long division with polynomials.

R_x Have students practise long division with numbers, and then transfer those steps to polynomial division.

- Students write inaccurate division statements.

R_x Have students practise writing division statements to represent long divisions with numbers. Stress the importance of knowing the vocabulary (*quotient*, *divisor*, *dividend*, and *remainder*) and understanding the relationship between these. It helps to verify that the division statement results in the given polynomial.

2.2 The Factor Theorem

Teaching Suggestions

• In the **Investigate**, students discover the factor theorem. This can be completed individually or in pairs.
• Discuss the meaning of "if and only if" in the margin feature on page 95. To support this definition you may wish to provide an example from geometry, such as "A triangle is equilateral if and only if all three angles are equal."
• Discuss with students that up to this point they have learned how to factor quadratics, or polynomials of degree two. The methods in this section will allow them to factor polynomials of degree higher than two.
• Refer to **Prerequisite Skills** questions 2, 4, and 5 for this lesson.
• In **Example 1**, students should understand that the sign of the substituted value b is opposite to the sign of b in $x - b$. In part v), point out how to find the value $\frac{b}{a} = \frac{1}{2}$. In Method 1, students should be careful when evaluating the expression. Remind them of BEDMAS; that is, they must first evaluate exponents, such as $(-2)^3$, then multiply, and then add or subtract. In particular they should be careful when evaluating with negative integers. Fraction evaluations—as in part v)—may need to be explained in detail. The graphing calculator illustrated in Method 2 may be new for many students, and is a very efficient means of finding the y-value of a function at a given x-value. For integral values of x, the corresponding y-values may also be verified using the table values by pressing [2nd] [TABLE], though this would not work as easily for rational values.
• Point out that in **Example 1** the factors to be tested were given. This is not the case in **Example 2**, where you are required to find the factors. Trial and error is used to find the first factor. This is where the integral zero theorem and factor theorem are used. Once one factor is determined, use division to draw out the other factors. Division allows you to break up the given polynomial into linear and quadratic factors. Once a quadratic factor is found, previous factoring methods are applied and division is no longer necessary.
• Two options for division (by hand) are long division and synthetic division. The latter is very efficient, in particular when two or more divisions are performed (as in the case for polynomials of degree higher than three).
• Point out that synthetic division also works with the value b if addition is used instead of subtraction. For instance, in **Example 2**, Method 2, you could use 2 instead of -2 in the very first column, and then replace the minus sign with a plus sign. Then, you would *add* instead of subtracting to determine the values in the bottom row.
• The polynomial in **Example 3** is a quartic. Division may be applied twice but in this case it is also possible to factor by grouping, a method that students may not be familiar with.
• Students should understand that the polynomials in **Examples 2** and **3** have a leading coefficient of 1. **Example 4** illustrates how to apply the rational zero theorem when the leading coefficient is not 1. Remind them that a rational number is any number that can be expressed as a fraction.
• For question C2 of **Communicate Your Understanding**, provide an example (such as $6x^2 - x - 2$) that students could use to discuss the methods for factoring trinomials of this form (decomposition, product sum method, fast method, etc.).
• Remind students to use division when completing **questions 6** and **11**.

Use **concept attainment** to teach the factor theorem.

Use a **Frayer model** to summarize the factor theorem.

COMMON ERRORS

- When evaluating $P(b)$, students use the sign of the factor for b. For instance, given $x + 2$, students evaluate $P(2)$ and not $P(-2)$.

R_x Remind them to use the value that makes the factor zero; that is $x = -2$ makes $x + 2 = 0$.

- Students have difficulty evaluating polynomial expressions involving fractions.

R_x Practise more questions such as Prerequisite Skills question 2.

- Students have difficulty factoring quadratics.

R_x Review factoring methods involving quadratics. Practise more questions such as Prerequisite Skills question 4.

ONGOING ASSESSMENT

Achievement Check, question 16, on student text page 103.

- **Question 8** gives students the opportunity to make connections with material they have learned previously, to select the necessary tools to do this, and to reason through strategies that will help them to solve the problem of finding the possible dimensions of the block.
- In **questions 9** and **10**, students need to remember what it means to be a factor.
- **Questions 12** and **13** focus on patterns for factoring sums and differences of cubes.
- **Question 12** allows students to use reasoning and reflecting skills to solve the problem of pattern prediction. They will have to select tools to do this and make connections with concepts learned in their past. Once they feel that they can predict a pattern, they will use their communicating skills to describe the prediction.
- For **question 14**, students need only test the values found using the integral root theorem.
- **Question 15** illustrates the value of synthetic division for repeated division.
- The method in **question 17** changes the quartic expression to a quadratic expression. Remind students to give the final answer in terms of x and not x^2.
- **Question 18** involves solving a system of linear equations.
- The point of **question 19** is to have students work backwards from the zeros to the factors. Members of a family of polynomial functions share the same factors. The given point is needed to determine a particular family member.
- **Question 21** develops a pattern to factor difference of powers.
- Use **BLM 2–3 Section 2.2 Practice** for remediation or extra practice.

Investigate Answers (page 94)

1. a) 0

  b) $x^3 + 2x^2 - x - 2 = (x - 1)(x^2 + 3x + 2)$

  c) $(x - 1)(x + 1)(x + 2)$

  d) If $P(b) = 0$, then $x - b$ is a factor of $P(x)$.

2. a) B, C, E

  b) If $(x - b)$ is a factor of $P(x)$, then $P(b) = 0$.

3. a) If $P(b) = 0$, then $(x - b)$ is a factor of $P(x)$.

  b) $(x - 2)(x - 1)(x + 1)$

Communicate Your Understanding Responses (page 101)

C1. a) B, D, F

  b) $(x - 2)(x + 1)(2x + 3)$

C2. When factoring trinomials, using either the product and sum method or the method of decomposition, we determine two numbers whose product is ac. Both methods take into account the leading coefficient, as does the rational zero theorem.

C3. Find a value $x = b$ such that $P(b) = 0$. Then, use division to determine the other factors.

C4. None of the test values—± 1, ± 2, or ± 4—results in 0 when evaluated, so the polynomial does not have integral factors.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	8, 12–15, 19–21
Reasoning and Proving	8–10, 12–16, 17–21
Reflecting	12, 13, 19–21
Selecting Tools and Computational Strategies	1–8, 11–16, 18–21
Connecting	3–6, 8–21
Representing	16
Communicating	12, 13, 16, 20, 21

Achievement Check, question 16, student text page 103

This performance task is designed to assess the specific expectations covered in Section 2.2.
   The following Math Process Expectations can be assessed.

- Connecting
- Selecting Tools
- Communicating
- Reflecting

Achievement Chart Category	Related Math Processes
Knowledge and Understanding	Selecting tools and computational strategies
Thinking	Problem solving Reasoning and proving Reflecting
Communication	Communicating, Representing
Application	Selecting tools and computational strategies Connecting

Sample Solution

Provide students with **BLM 2–4 Section 2.2 Achievement Check Rubric** to help them understand what is expected.

a) $P(x) = x^3 - 6x^2 + 9x$
   $P(1) = 1^3 - 6(1)^2 + 9(1)$
   $\quad\;\; = 1 - 6 + 9$
   $\quad\;\; = 4$
   Since $P(1) \neq 0$, $x - 1$ is not a factor of $P(x)$.

b) Possible values of $\dfrac{b}{a}$ are ± 1, ± 3, and ± 9.

c)
$$
\begin{array}{r}
x^2 - 3x \\
x - 3 \overline{)\, x^3 - 6x^2 + 9x} \\
\underline{x^3 - 3x^2} \\
-3x^2 + 9x \\
\underline{-3x^2 + 9x} \\
0
\end{array}
$$

d) $P(x) = 4x^3 + 12x^2 - 16x$
   $\quad\;\;\; = 4x(x^2 + 3x - 4x)$
   $\quad\;\;\; = 4x(x + 4)(x - 1)$

2.3 Polynomial Equations

Teaching Suggestions

- Allow students to work individually or in pairs on the **Investigate**.
- Refer to **Prerequisite Skills** questions 6 and 7 for this lesson.
- The polynomials in **Example 1** may be factored without using division. Emphasize the importance of using the most efficient strategy when factoring a polynomial. Reinforce that the word *solve* means to find all x-values that satisfy the equation. The first part of the solution involves factoring. Once the factors are found, the roots are determined by setting each factor to zero.
- The factor theorem and division are needed to factor the polynomial in **Example 2**. Discuss whether you want students to show all the test values, as in the solution to part a). Perhaps it is more efficient for them to verify only the test value that they use to find the first factor. Remind them that depending on the test value selected, it is possible to have a different first factor. For instance, in **Example 2**, if $x = -3$ was tested first, you would divide by $x + 3$.
- Before presenting **Example 3**, point out that a polynomial equation of degree n will always have n roots, however, not all the roots may be real numbers. The x-intercepts of the graph correspond to the real roots, also demonstrated in **Example 3**. Students should only use the quadratic formula when the corresponding trinomial cannot be factored, that is, when the discriminant $b^2 - 4ac$ is not a perfect square. Once again, it should not be necessary for students to list all the test values indicated in **Example 3**.
- Students should understand that many polynomial equations do not have any roots that are integers, and so factoring cannot be used. **Example 4** demonstrates how technology is used to find the roots of these polynomials. Have students refer to the Technology Appendix on page 506 for instructions on how to use the **Zero** operation. Discuss why the solution may be found by graphing the two functions (as suggested in the margin feature) and then using the **Intersect** operation.
- **Example 4** demonstrates that the test values may not provide the solutions to the given equation.
- In **Communicate Your Understanding** questions C1 and C2, draw out the difference between a root and an x-intercept. A root could also be a complex number, but an x-intercept is only a real number.
- The x-intercepts of the graphs in **question 2** are all integer values. Students should pay attention to the given scale for each axis.
- Remind students to use the indicated examples as a reference as they complete questions.
- **Question 5** gives students an opportunity to reflect on each given statement, to reason out possible alternative wording if a statement is false, and to use their connecting and communicating skills to develop and reword any false statements to make them true.
- In **question 8**, all terms should be moved to one side of the equal sign so that the equation is equal to 0. It is beneficial to have a positive leading coefficient.
- **Question 10** requires students to apply their reasoning and proving skills to think through the problem given and to solve it. They will have to select tools and make connections with problem solving strategies they have learned previously to determine the dimensions of the tank.

Student Text Pages
104 to 112

Suggested Timing
40–55 min

Tools
- graphing calculator

Related Resources
- BLM 2–5 Section 2.3 Practice

- A hint for **question 17** part a), is to let $m = x - 1$. For part b), to let $m = x^2 - 4x$.
- A hint for **question 18,** part a), is to use the definition of a root. To solve part b), students should first substitute the value found in part a).
- **Question 19** uses thinking skills of reflecting, reasoning and proving, and problem solving to determine possible dimensions of the open-top boxes. The students will have to show how they can select tools and connect mathematical skills that they have previously learned to solve this problem.
- **Question 20** introduces students to complex numbers. Have students research more information regarding complex numbers and possibly extend the lesson to include complex roots.
- In **question 21,** students are to consider two boxes: a smaller one with dimensions that are consecutive integers and then a larger one with the modified dimensions.
- A hint for **question 22** is to first solve the given polynomial equation.
- Use **BLM 2–5 Section 2.3 Practice** for remediation or extra practice.

Investigate Answers (page 104)

1. a) Window variables: $x \in [-5, 5]$, $y \in [-8, 40]$, Yscl $= 2$

Y1=X^4-13X^2+36   X=1   Y=24
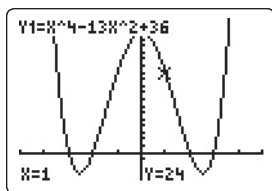
b) The x-intercepts are -3, -2, 2, and 3.
c) $f(x) = (x - 3)(x - 2)(x + 2)(x + 3)$
   The zeros are 3, 2, -2, and -3.
d) The x-intercepts of the graph are the same as the zeros of the function.
2. a) The roots are 3, 2, -2, and -3.
b) The x-intercepts of the graph and the roots of the equation are equal.
c) They are equal.

Communicate Your Understanding Responses (page 109)

C1. A root of an equation is a value for x that produces a result of zero. A root may be real or non-real. A zero is a value of x that makes a function equal to 0. A zero may be real or non-real. An x-intercept is where the graph intersects the x-axis, and is only a real number.
C2. Method 1: Substitute 2, -1, 3, and -2 for x in the left side of the polynomial equation and evaluate to check if the result is 0.

Method 2: Use technology to verify that 2, -1, 3, and -2 are the x-intercepts of the corresponding graph.
C3. 2
C4. Possible methods are: using the factor theorem and long division or synthetic division; using technology to find the x-intercepts (if they are integer values); combining methods (such as factor by grouping, common factor, difference of squares, factor trinomials); and substituting a different letter, such as letting $m = x^2$.
C5. The maximum number of real roots of the corresponding equation is n. The number of x-intercepts is equal to the number of distinct (different) real roots. If a real root is repeated, then the number of x-intercepts is less than the number of real roots.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	10, 19, 21, 22
Reasoning and Proving	5, 10–13, 15–22
Reflecting	5, 12, 13, 19, 22
Selecting Tools and Computational Strategies	3, 4, 6–11, 14–17, 19–22
Connecting	2, 4–9, 10–22
Representing	16
Communicating	12, 13, 16, 18, 20

Families of Polynomial Equations

Teaching Suggestions

• Have students work in pairs or small groups to complete the **Investigate**.
• In addition to using graphing calculators, students may find using *The Geometer's Sketchpad®* useful since colours may be assigned to polynomial functions with the same zeros.
• Refer to **Prerequisite Skills** question 8 for this lesson.
• In **Examples 1** and **2**, point out that the equation for the family of functions must include the constant k to represent the leading coefficient. Encourage students to use fractions and not decimals where appropriate. Apply Chapter 1 skills to graph the polynomials, as shown in **Example 2** part d).
• **Example 3** involves zeros with radicals (or irrational numbers). Do not change these to decimals; leave them as exact values. The solution to part a) may require further scaffolding if students have difficulty simplifying using a difference of squares.
• As students consider the **Communicate Your Understanding** questions, draw out the distinction between the equation for a family of functions and the equation for a member of the family. The similarities and differences in their graphs should be emphasized in terms of the effect of the leading coefficient. It is important for students to understand that infinite functions share the same x-intercepts.
• The factors corresponding to the rational zeros in **questions 10** and **11** should be expressed in the form $ax - b$. All equations should be expanded and simplified.
• The purpose of **question 12** is to have students recall (from Chapter 1) the meaning of "order 2."
• **Question 20** requires students to reason out the actual creation of the box and the wording of the question that will help them establish the functions required. They will need to use connecting skills to produce the functions in parts a) and b) and to compare them, and then they will use representing skills to sketch graphs of all three functions.
• Use **BLM 2–6 Section 2.4 Practice** for remediation or extra practice.

Investigate Answers (page 114)

1. b) Similarities: All graphs are parabolas with x-intercepts 1 and -2. In set A, the graphs open upward. In set B, the graphs open downward.

Differences: The parabolas have different y-intercepts, different vertices, and different stretch factors. Set A i) and set B iv) are reflections of each other in the x-axis. This is also true for set A ii) and set B v) and set A iii) and set B vi).

2. All have the same x-intercepts.

3. a) Same factors, but different leading coefficient.

b) They are cubic functions and will have the same x-intercepts. Their shape and direction will be different due to the sign and value of the leading coefficient.

4. a) Window variables: $x \in [-5, 5]$, $y \in [-30, 30]$, Yscl $= 3$

b) Answers may vary.

5. Families have the same x-intercepts and share similar characteristics.

Communicate Your Understanding Responses (page 118)

C1. Infinite number.

C2. The zeros.

C3. A point on the graph, other than an x-intercept.

C4. Similar: Same x-intercepts and same general shape.

Different: y-intercepts and shape is different by a stretch factor.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	18–22
Reasoning and Proving	1–23
Reflecting	18, 21–23
Selecting Tools and Computational Strategies	1–23
Connecting	5–23
Representing	4, 7–10, 18, 19, 20–22
Communicating	2, 3, 4, 17, 18

Achievement Check, question 21, student text page 122

This performance task is designed to assess the specific expectations covered in Section 2.4.

The following Math Process Expectations can be assessed.

• Reasoning and Proving
• Reflecting
• Representing
• Communicating
• Problem Solving
• Selecting Tools
• Connecting

Sample Solution

a) An equation for the family of functions used to create the design is $y = k(x + 3)(x + 1)(x - 5)$ or $y = k(x^3 - x^2 - 17x - 15)$.

b) Using the y-intercepts $(0, 7.5)$, $(0, 15)$, $(0, 22.5)$, $(0, 30)$, $(0, 37.5)$, $(0, 45)$, $(0, -7.5)$, $(0, -15)$, $(0, -22.5)$, $(0, -30)$, $(0, -37.5)$, and $(0, -45)$, the respective k-values are -0.5, -1, -1.5, -2, -2.5, -3, 0.5, 1, 1.5, 2, 2.5, and 3.

c) Answers may vary.

Level 3 Notes

Look for the following:

Part a)

• Most x-intercepts of the graph are correct
• Most factors in the equation for the family of functions correspond with the x-intercepts
• Equation of the family of functions, in factored or expanded form, is mostly accurate

Part b)

• y-intercepts, or other points, of the graphs are mostly accurate
• Corresponding k-values are mostly accurate
• Equations for the graphs in the design are mostly accurate

Part c)

- Design with some degree of complexity
- *x*-intercepts and associated factors for each graph in the design are mostly accurate
- Equations of the graphs in the design are mostly accurate

What Distinguishes Level 2

Part a)

- Some x-intercepts of the graph are correct
- Some factors in the equation for the family of functions correspond with the *x*-intercepts
- Equation for the family of functions, in factored or expanded form, has some errors

Part b)

- *y*-intercepts, or other points, of the graphs are somewhat accurate
- Corresponding *k*-values are somewhat accurate
- Equations for the graphs in the design have some errors

Part c)

- Design is simple
- *x*-intercepts and associated factors for each graph in the design are somewhat accurate
- Equations of the graphs in the design have some errors

What Distinguishes Level 4

Part a)

- *x*-intercepts of the graph are correct
- Factors in the equation for the family of functions correspond with the *x*-intercepts
- Equation of the family of functions is provided, in factored and expanded form, with very minor errors

Part b)

- *y*-intercepts, or other points, of the graphs have only minor errors
- Corresponding *k*-values have only minor errors
- Equations for the graphs in the design have only minor errors

Part c)

- Design has a high degree of complexity
- *x*-intercepts and associated factors for each graph in the design have only minor errors
- Equations of the graphs in the design have only minor errors

Solving Inequalities Using Technology

Teaching Suggestions

- In this section, non-factorable polynomial inequalities are solved using technology. The next section focuses on factorable polynomial inequalities, which are solved algebraically.
- Have students work in pairs or small groups to complete the **Investigate**. The purpose of the **Investigate** is to help students make the connection between an inequality and y-values of the graph. In particular, the part above the x-axis corresponds to $>$ and the part below the x-axis corresponds to $<$. The x-axis divides the graph into two parts; one part is above the x-axis, and the other is below. Thus, the x-intercepts define the intervals where a graph may change from positive to negative or vice versa.
- Refer to **Prerequisite Skills** question 9 for this lesson.
- Discuss with students the concepts presented on page 125 before presenting the examples. It is important to define the intervals from left to right along the x-axis. The x-intercepts of the graph are required to determine the intervals. These are found by solving the equation that corresponds to the inequality.
- **Example 1** demonstrates how to use a graphing calculator to determine the x-intercepts needed to solve a non-factorable inequality. The graph can then be used to "see" which part lies above or below the x-axis.
- **Example 2** demonstrates how to use a CAS and test values to solve an inequality, without graphing first. If needed, use **T–4 The Computer Algebra System (CAS) on the TI-89 Calculator** to support this activity.
- **Example 3** provides two methods for determining the x-intercept. These methods were indicated in Section 2.3, Example 4.
- **Example 3** requires students to select tools when reflecting on the wording of the question to show how best to solve the problem of determining when the yearly sales will be $100 million or more. They will have to reason through which method they prefer to use and use their connecting skills to use the graphing calculator in the most appropriate way to solve the problem.
- As students consider the **Communicate Your Understanding** questions, draw out that in order to solve an inequality it is necessary first to determine the x-intercepts of the corresponding graph. The intervals that are possible solutions to the inequality are based on the values of the x-intercepts. The solutions can easily be found by looking at the parts of the graph that are below or above the x-axis.
- The graphs in **question 6** have integer valued x-intercepts.
- Any method can be used to solve **question 9**.
- The purpose of **questions 11** and **12** is for students to set up an inequality, similar to **Example 3**.
- **Question 13** gives students the opportunity to reflect, reason, and problem solve when trying to create the three cubic polynomial inequalities required. They will then have to select appropriate tools and use connecting skills to solve the inequalities.
- Use **BLM 2–7 Section 2.5 Practice** for remediation or extra practice.

Use **Think-Pair-Share** to help students consolidate their knowledge of this section.

COMMON ERRORS

- Students point the inequality symbols in the wrong direction.

R_x Emphasize that the symbols always point to the smallest value and in the same direction, so $3 < x < 7$ is correct but $3 < x > 7$ is incorrect. Refer to a number line to emphasize these points.

- Students have difficulty identifying the intervals for the solution.

R_x Point out that the x-axis divides the graph in half and they need only look for the intervals along the x-axis that correspond to the parts of the curve that are either above or below the x-axis and possibly include the x-intercepts (depending on the inequality being solved).

Investigate Answers (pages 123–124)

1. a) $x \leq 3$
   b) $x > -1$
   c) $-5 < x \leq 7$

2. a) above x-axis for $x < -5$ or $x > 3$; positive
   b) same
   c) below x-axis for $x < -4$ or $x > 5$; negative
   d) same

3. The x-intercepts divide the x-axis into intervals that correspond to the solution to the inequality.

4. a) $-3 \leq x \leq 1$ or $x \geq 4$
   b) $x < -3$ or $1 < x < 4$
   c) **i)** The solution to this inequality is the set of x-values that correspond to the part of the graph that lies above the x-axis and includes the x-intercepts.
      ii) The solution to this inequality is the set of x-values that correspond to the part of the graph that lies below the x-axis, not including the x-intercepts.

Communicate Your Understanding Responses (page 129)

C1. Examples may vary. A polynomial equation is solved by finding the roots. The number of roots, real and non-real, corresponds to the degree of the function. A polynomial inequality is solved by finding the intervals where the curve (or y-values), lies above or below the x-axis (and may or may not include the x-intercepts).

C2. The solution to an inequality depends on the intervals determined by the x-intercepts.

C3. The real roots of the polynomial equation are the x-intercepts of the graph.

C4. First, graph the function. Then, determine the x-intercepts. Write the intervals that correspond to the parts of the graph that lie below the x-axis, including the x-intercepts.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	13–15, 17
Reasoning and Proving	3, 4, 5, 10–17
Reflecting	13–15
Selecting Tools and Computational Strategies	1–12, 16, 17
Connecting	5–17
Representing	3, 6
Communicating	4

Solving Factorable Polynomial Inequalities Algebraically

Teaching Suggestions

- Whereas in Section 2.5 technology was used to solve non-factorable polynomial inequalities, this section develops algebraic methods to solve factorable polynomial inequalities.
- In the solution for **Example 1**, part b), be sure to explain why the inequality must be reversed. Remind students to use an open (or hollow) circle when showing a number line solution for an inequality with $>$ or $<$ and a solid circle for solutions to inequalities with $\leq$ or $\geq$.
- In **Example 2**, part a), Method 1, remind students to look for values of x that make the product positive, since > 0 means positive. There are many values that will make this happen and so they must consider all possibilities. In Method 2, the number line intervals are based on the roots of the corresponding equation.
- In **Example 2**, part b), the polynomial must be factored first to obtain a product before it can be solved. Method 2 achieves the same result, but by using intervals and test values. The chart is an efficient way to summarize the results. Remind students that the results will be the same for any test value in the interval.
- In the solution of **Example 3**, the quadratic $w^2 + 19w + 176$ cannot be factored since the discriminant is not a perfect square; in fact, $b^2 - 4ac = -343$ is negative, so there are no other real roots.
- **Example 3** requires students to reflect on the wording of the question and reason out how to put the words into an inequality to solve the problem, finding the minimum dimensions. Once the inequality is produced, they will use selecting tools and connecting of concepts previously learned to work out the identity, using two cases in order to solve the problem.
- As students consider the **Communicate Your Understanding** questions, draw out that solving a polynomial inequality involves factoring it to break it down to a product of linear inequalities, and then determining the solution by considering solutions to the linear equalities. When discussing question C3, point out that the higher the degree of the polynomial the more cases that must be considered. The interval method is more efficient to organize the information.
- Students may choose their preferred method for **questions 4** and **7**.
- **Question 8** uses reflecting, reasoning and proving, and problem solving skills to determine the minimum dimensions of the box. The wording has to be considered, tools have to be selected, and connections have to be made with previously learned mathematical concepts to solve the problem.
- The point of **questions 11** and **12** is for students to apply the most efficient method based on the degree of the polynomial inequality.
- Use **BLM 2–8 Section 2.6 Practice** for remediation or extra practice.

COMMON ERRORS

- Students have difficulty determining the cases.

R_x Point out that the cases depend on the number of factors. Remind students to think of the factors in the inequality in terms of the product of two, three, or four numbers, such as ab or abc or $abcd$ as they consider cases.

- Students have difficulty determining the intervals.

R_x Students should first find the zeros of the function, indicate these in numerical order on the number line, and then use them to determine the intervals from left to right.

ONGOING ASSESSMENT

Achievement Check, question 10, on student text page 139.

Communicate Your Understanding Responses (page 138)

C1. Examples may vary. The inequality must be reversed to make the statement true. For instance, suppose we choose a value, $x = 3$, and substitute it in $-2x < 16$. The result is $-6 < 16$, which is a true statement. If we use $x > -8$, which is the simplified form of $-2x < 16$, with the inequality reversed, the result is $3 > -8$, which is also true. However, if the inequality is not reversed, the simplified form would be $x < -8$ and then $3 < -8$, which is not a true statement.

C2. A polynomial inequality is solved by using the factors, each of which is linear. The results of these solutions must then be combined to give the final solution.

C3. Both methods are efficient for polynomials of degree one or two, however for polynomials of degree three or more, there are more cases to be considered and so the interval method is more efficient.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	8, 11, 12, 14
Reasoning and Proving	8, 9, 11–14
Reflecting	8, 9, 11, 14
Selecting Tools and Computational Strategies	2, 4, 7–9, 11–14
Connecting	2–14
Representing	1–3, 5
Communicating	10–12

Achievement Check, question 10, student text page 139

This performance task is designed to assess the specific expectations covered in Section 2.6.

The following Math Process Expectations can be assessed.

- Reasoning and Proving
- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting

Sample Solution

a) Factor $x^3 - 5x^2 + 2x + 8$ using the factor theorem.

$x^3 - 5x^2 + 2x + 8 = (x - 4)(x - 2)(x + 1)$

Solve $(x + 1)(x - 2)(x - 4) < 0$.

i)

Factor \ Interval	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$2 < x < 4$	$x = 4$	$x > 4$
$(x + 1)$	−	0	+	+	+	+	+
$(x - 2)$	−	−	−	0	+	+	+
$(x - 4)$	−	−	−	−	−	0	+
$(x + 1)(x - 2)(x - 4)$	−	0	+	0	−	0	+

The solution is $x < -1$ or $2 < x < 4$.

ii) Case 1

$(x + 1) < 0$ $(x - 2) < 0$ $(x - 4) < 0$

    $x < -1$ $x < 2$ $x < 4$

So, $x < -1$ is a solution.

Case 2

$(x + 1) > 0$ $(x - 2) > 0$ $(x - 4) < 0$

    $x > -1$ $x > 2$ $x < 4$

So, $2 < x < 4$ is a solution.

Case 3

$(x + 1) > 0$ $(x - 2) < 0$ $(x - 4) > 0$

    $x > -1$ $x < 2$ $x > 4$

Case 3 has no solution.

Case 4

$(x + 1) < 0$ $(x - 2) > 0$ $(x - 4) > 0$

    $x < -1$ $x > 2$ $x > 4$

Case 4 has no solution.

Combining the results, the solution is $x < -1$ or $2 < x < 4$.

b) The methods are the same in that the polynomial must be factored and the zeros of the linear factors are used to determine the solution. The methods are different in that a single number line and test values are used in the interval method, whereas the case method requires listing all the possible solutions within each and then considering if their common values meet the criteria of the inequality.

Level 3 Notes

Look for the following:

- Factored form of equation, chart intervals, and factors are mostly accuracy
- Understanding of chart method is evident and final solution is mostly accurate
- Most of the cases are identified and solved fairly accurately
- Results from cases are interpreted fairly accurately
- Justification for how the two methods are the same and how they are different is mostly valid

What Distinguishes Level 2

- Factored form of equation, chart intervals, and factors are somewhat accurate
- Understanding of chart method is evident and final solution is somewhat accurate
- Most of the cases are identified and solved with some accuracy
- Results from cases are interpreted with some errors
- Justification for how the two methods are the same and how they are different is somewhat valid

What Distinguishes Level 4

- Factored form of equation, chart intervals, and factors have only minor errors
- Understanding of chart method is evident and final solution has only minor errors
- All of the cases are identified and solved with a high degree of accuracy
- Results from cases are interpreted with only minor errors
- Justification for how the two methods are the same and how they are different is accurate

Review

Study Guide

Use the following study guide to direct students who have difficulty with specific questions to appropriate examples to review.

| Question	Section(s)	Refer to
1	2.1	Example 1 (page 85), Example 2 (page 86), Example 4 (page 89)
2	2.1	Example 5 (page 90)
3	2.1	Example 5 (page 90)
4	2.2	Example 2 (pages 97–98)
5	2.2	Example 3 (page 99)
6	2.2	Example 4 (pages 100–101)
7	2.2	Example 1 (pages 95–96)
8	2.3	Example 4 (pages 108–109)
9	2.3	Example 3 (pages 107–108)
10	2.3	Example 2 (pages 105–106), Example 4 (pages 108–109)
11	2.3	Example 3 (pages 107–108)
12	2.4	Investigate (page 114)
13	2.4	Example 3 (page 117)
14	2.4	Example 4 (pages 117–118)
15	2.5	Example 1 (page 126), Example 2 (pages 126–127)
16	2.5	Example 3 (page 128)
17	2.6	Example 1 (page 132)
18	2.6	Example 2 (pages 133–137)

Problem Wrap-Up

Using the Chapter Problem

• Students may work on the Chapter Problem individually or in pairs.
• Remind students to keep track of their solutions to the Chapter Problem
  questions. Discuss how the Chapter Problem questions relate to the
  Chapter Problem Wrap-Up.
• The use of technology is recommended when creating the bottle designs
  for the Chapter Problem Wrap-Up. Students who have computers at home
  may work on the design using free graphing software, such as Winplot. If
  an in-class computer is available, you may wish to have them display their
  design electronically (the use of animation and colour can result in some
  very attractive design).
• The following tips may benefit all students or just those having difficulty
  getting started with their designs.
  – Choose a degree of your polynomial function.
  – Select the x-intercepts.
  – Graph the function.
  – Reflect it in the x-axis.
  – Change the leading coefficient to create a family of functions.
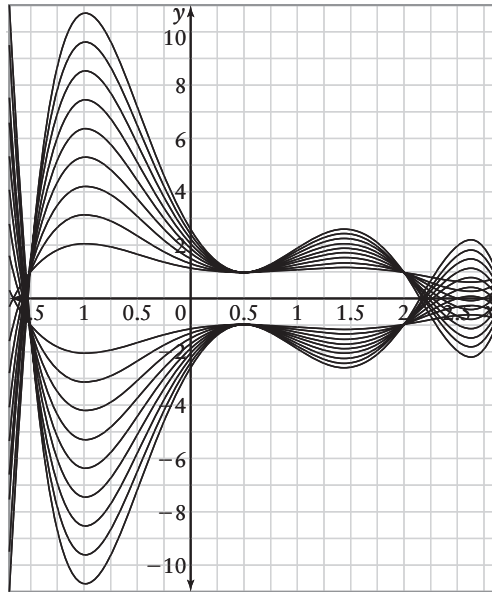 – Apply your knowledge of transformations to create interesting designs.

Level 3 Sample Response

Bottle Design for Ladies' Perfume

Design without a grid.
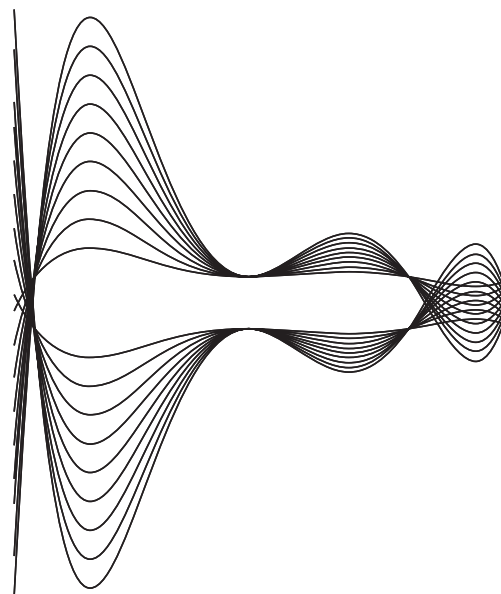
Design on a grid.

The family of functions that was used to create this design is
$y = a(x - 3)(x - 2)(2x + 3)(2x - 1)^2$.

To obtain the design, I used corresponding positive and negative values of the leading coefficient a. I also applied vertical translations to get the top and bottom part of the design.

Equations used for the top part of the design:
$y = a(x - 3)(x - 2)(2x + 3)(2x - 1)^2 + 1$, with $a = 0.01, 0.02, 0.03, 0.04,$ $0.05, 0.06, 0.07, 0.08, 0.09$.

Equations used for the bottom part of the design:
$y = a(x - 3)(x - 2)(2x + 3) (2x - 1)^2 - 1$, with $a = -0.01, -0.02, -0.03,$ $-0.04, -0.05, -0.06, -0.07, -0.08, -0.09$.

To get the shape of the bottle, I used the domain restrictions $[-1.7, 2.9]$.

The x-intercepts of the family of functions $y = a(x - 3)(x - 2)(2x + 3)(2x - 1)^2$ are $x = -1.5$, $x = 0.5$ (order 2), $x = 2$, and $x = 3$. The x-intercepts of the graph correspond to the real roots of this family of functions. When the graphs are translated 1 unit up or down, they no longer intersect the x-axis at these values.

i) For $y = 0.09(x - 3)(x - 2)(2x + 3)(2x - 1)^2 + 1$, solve
    $0 = 0.09(x - 3)(x - 2)(2x + 3)(2x - 1)^2 + 1$.
    Using a graphing calculator, the x-intercepts are $x = -1.52$, $x = 2.16$, and
    $x = 3.15$. The x-intercept 3.15 is not in the restricted domain $[-1.7, 2.9]$.
ii) For $y = 0.07(x - 3)(x - 2)(2x + 3)(2x - 1)^2 + 1$, solve
    $0 = 0.07(x - 3)(x - 2)(2x + 3)(2x - 1)^2 + 1$.
    Using a graphing calculator, the x-intercepts are $x = -1.52$, $x = 2.108$,
    and $x = 2.926$.
    The x-intercept 2.926 is not in the restricted domain $[-1.7, 2.9]$.
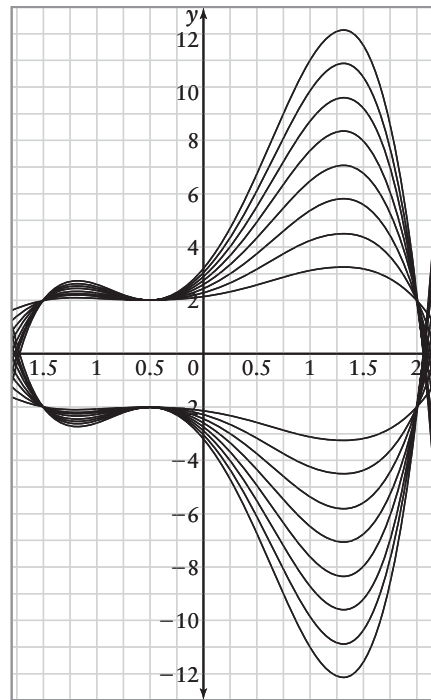
Use the above equations to write two inequalities.
i) $0.09(x - 3)(x - 2)(2x + 3)(2x - 1)^2 + 1 \geq 0$
    Use the x-intercepts: $-1.52, 2.16, 3.15$.
    From the graph we can see that $y \geq 0$ for $x \in [-1.52, 2.16]$ and $x \in (3.15, \infty)$.
ii) $-0.07(x - 3)(x - 2)(2x + 3)(2x - 1)^2 - 1 \leq 0$
    Use the x-intercepts: $-1.52, 2.108, 2.926$.
    From the graph we can see that $y \leq 0$ for $x \in [-1.52, 2.108]$ and $x \in (2.926, \infty)$.

Bottle Design for Men's Cologne

Design without a grid.
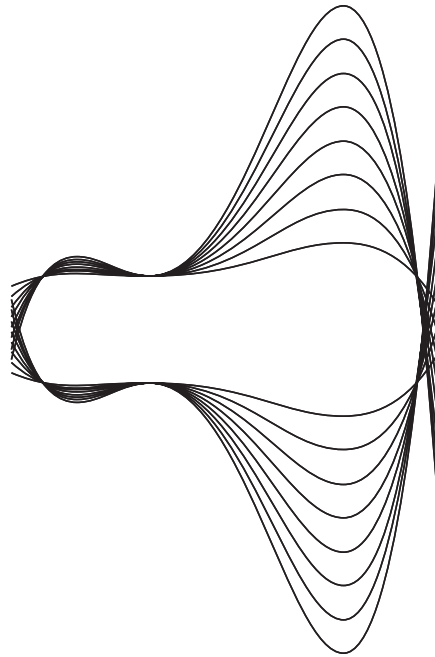
Design with a grid.

The family of functions that was used to create this design is
$y = a(2x + 3)(x - 2)(2x + 1)^2$.

To obtain the design, I used corresponding positive and negative values of the leading coefficient a. I also applied vertical translations to get the top and bottom part of the design.

Equations used for the top part of the design: $y = a(2x + 3)(x - 2)(2x + 1)^2 + 2$, with $a = -0.200, -0.175, -0.150, -0.125, -0.100, -0.075, -0.050, -0.0250$.

Equations used for the bottom part of the design:
$y = a(2x + 3)(x - 2)(2x + 1)^2 - 2$, with a = 0.200, 0.175, 0.150, 0.125, 0.100, 0.075, 0.050, 0.0250.

To get the shape of the bottle, with a base and lid, I used the domain restrictions $[-1.8, 2.2]$.

The x-intercepts of the family of functions $y = a(2x + 3)(x - 2)(2x + 1)^2$ are $x = -1.5$, $x = -0.5$ (order 2), and $x = 2$. The x-intercepts of the graph correspond to the real roots of this family of functions. When the graphs are translated up by 2, or down by 2, they no longer intersect the x-axis at these values.

The x-intercepts and real roots of the two outer functions that define the shape of the bottle are:

i) For $y = -0.2(2x + 3)(x - 2)(2x + 1)^2 + 2$, solve
    $0 = -0.2(2x + 3)(x - 2)(2x + 1)^2 + 2$.
    Using a graphing calculator, the x-intercepts are: $x = -1.72$ and $x = 2.05$.
ii) For $y = 0.1(2x + 3)(x - 2)(2x + 1)^2 - 2$, solve
    $0 = 0.1(2x + 3)(x - 2)(2x + 1)^2 - 2$.
    Using a graphing calculator, the x-intercepts are $x = -1.85$ and $x = 2.1$.

Use the above equations to write two inequalities.

i) $-0.2(2x + 3)(x - 2)(2x + 1)^2 + 2 \geq 0$
    Use the x-intercepts: -1.72, 2.05.
    From the graph we can see that $y \geq 0$ for $x \in [-1.22, 2.05]$.
ii) $0.1(2x + 3)(x - 2)(2x + 1)^2 - 2 \leq 0$
    Use the x-intercepts: -1.85, 2.1.
    From the graph we can see that $y \leq 0$ for $x \in [-1.85, 2.1]$.

Level 3 Notes

- Equations include a family of polynomial functions of some complexity
- Interesting and somewhat complex design created with numerous curves
- Solutions to most parts of the question are provided
- Solutions may contain very minor errors
- Fairly good understanding of how to determine the real roots of the equations and their connection to the x-intercepts of the corresponding graphs
- Good understanding of techniques to solve equations and inequalities
- Fairly organized and logical solution
- Clear justification or reasoning for choices made

What Distinguishes Level 2

- Equations include a family of polynomial functions of some complexity
- Simple and not very complex design created with a few curves
- Solutions to some parts of the question are provided
- Solutions may contain some errors
- Some understanding of how to determine the real roots of the equations and their connection to the x-intercepts of the corresponding graphs
- Some understanding of techniques to solve equations and inequalities
- Somewhat organized and logical solution
- Somewhat appropriate solutions with some significant errors
- Little justification or reasoning for choices made

What Distinguishes Level 4

- Equations include a family of polynomial functions of fair complexity
- Very interesting, creative, and complex design created with a variety of curves
- Solutions to all parts of the question are provided
- Solutions are all accurate
- High degree of understanding of how to determine the real roots of the equations and their connection to the x-intercepts of the corresponding graphs
- Excellent understanding of techniques to solve equations and inequalities
- Thoroughly organized and logical solution
- Clear justification or reasoning for choices made

Study Guide

Use the following study guide to direct students who have difficulty with specific questions to appropriate examples to review.

Question	Section(s)	Refer to
1	2.1	Example 1 (page 85), Example 4 (page 89)
2	2.2	Example 1 (pages 95–96)
3	2.2	Example 2 (pages 97–98)
4	2.1	Example 2 (page 86)
5	2.1	Example 5 (page 90)
6	2.2	Example 2 (pages 97–98), Example 3 (page 99)
7	2.3	Investigate (page 104)
8	2.3	Example 1 (page 105)
9	2.3	Example 2 (pages 105–106)
10a)	2.5	Key Concepts (page 129)
10b)	2.3	Investigate (page 104)
11	2.4	Example 3 (page 117)
12	2.4	Example 4 (pages 117–118)
13	2.3	Example 3 (pages 107–108)
14	2.5	Example 1 (page 126)
15	2.5	Example 3 (page 128)
16	2.6	Example 2 (pages 133–137)
17	2.6	Example 3 (pages 137–138)

Can students do each of the following?
• Apply the remainder theorem to determine the remainder when a polynomial is divided by a binomial
• Apply the factor theorem to factor polynomials in one variable of degree greater than two
• Determine the equation of a family of polynomial functions that satisfy given conditions
• Solve polynomial equations using a variety of strategies
• Describe the connection between the real roots of a polynomial equation and the x-intercepts of the graph of the corresponding function
• Solve linear and factorable polynomial inequalities and represent the solutions on a number line
• Explain the difference between the solution to a polynomial equation and a polynomial inequality
• Solve polynomial inequalities algebraically and using technology

Can You Tell Just by Looking?

Teaching Suggestions

Have the students read the question in pairs but then go back to their desks with their graphing technology and work on their own. This will help ESL and ELL students, as well as give students another pair of eyes to interpret the questions so there is no mistaking what is being asked. The task could be assigned as an in-class assignment or as an independent assignment to be completed outside of class.

As student responses for the level of mathematical understanding they represent. As you assess each response, consider the following questions:

• Has the student used a labelled diagram to determine the roots of the equations?
• Did the student comprehend the given information?
• Has the student provided written calculations to determine the roots of the equations?
• Has the student provided clearly communicated responses to parts a), b), c), d), and e)?

Level 3 Sample Response

a) If you are just looking at the page and determining the roots then they may not be very accurate. Using the factor theorem,
  i) The roots are $x = 2$ and $x = 1$. So, the factors are $(x - 2)$ and $(x - 1)$.
  ii) The roots are $x = 3$, $x = 2$, and $x = 1$. So, the factors are $(x - 3)$, $(x - 2)$, and $(x - 1)$.
  iii) The roots are $x = 4$, $x = 3$, $x = 2$, and $x = 1$. So, the factors are $(x - 4)$, $(x - 3)$, $(x - 2)$, and $(x - 1)$.
  iv) The roots are $x = 5$, $x = 4$, $x = 3$, $x = 2$, and $x = 1$. So, the factors are $(x - 5)$, $(x - 4)$, $(x - 3)$, $(x - 2)$, and $(x - 1)$.
b) The roots can be found using graphing technology and the Zero operation on the calculator.
  i) $x = 1$ and $x = 2$
  ii) $x = 1$, $x = 2$, and $x = 3$
  iii) $x = 1$, $x = 2$, $x = 3$, and $x = 4$
  iv) $x = 1$, $x = 2$, $x = 3$, $x = 4$, and $x = 5$
c) The number of roots equals the degree of the equation and the roots are the factorial of the degree (2!, 3!, 4!, and 5!).
d) The constant term in each equation is the product of the roots, with an even-degree function having a positive sign and an odd-degree function having a negative sign. For example, in i) the constant is $2 = 1 \times 2$, in ii) the constant is $-6 = -(1 \times 2 \times 3)$, etc.
e) I conjecture that i) and iii) will have roots but ii) will not.
  i) $x = -3$, $x = 1$, and $x = 2$
  ii) $x = -3$, $x = 1$, and $x = 2$
  iii) $x = 1$, $x = 2$, $x = 3$, $x = 4$, $x = 5$, and $x = 6$