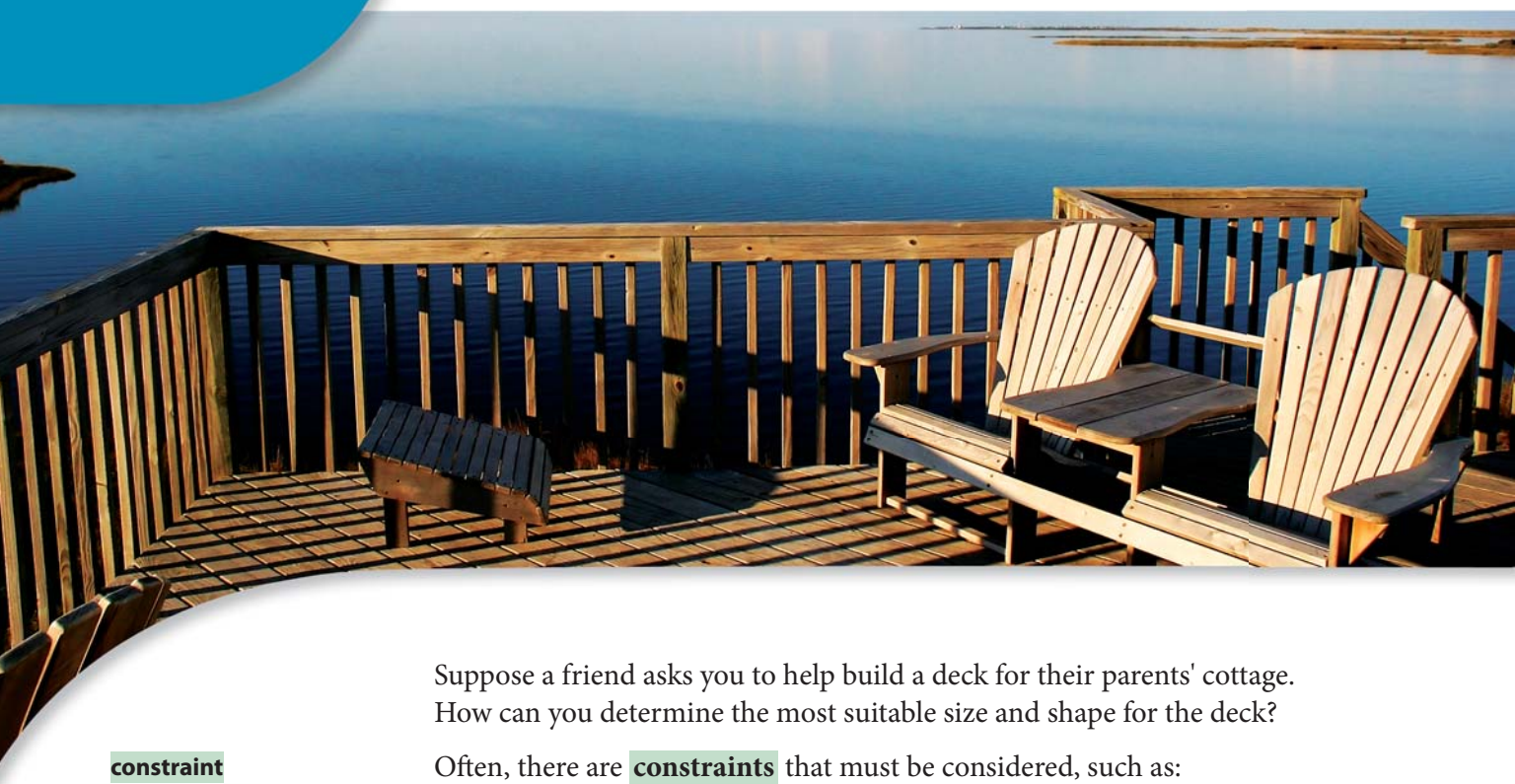
1.4 Optimize Perimeter and Area



Suppose a friend asks you to help build a deck for their parents' cottage. How can you determine the most suitable size and shape for the deck?

constraint
- condition that limits a design option

optimize
- determine the best solution while adhering to given constraints

Often, there are **constraints** that must be considered, such as:
- the budget
- fixed perimeter
- fixed area
- natural or artificial boundaries

An expert in construction can apply measurement and geometry concepts to **optimize** the design of a deck or other structures.

Investigate

Tools

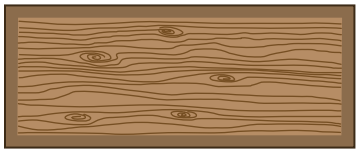
- toothpicks, geoboards, colour tiles, or linking cubes
- grid paper

Optional
- computer
- graphing calculator or spreadsheet software
- *The Geometer's Sketchpad®*
- string or yarn with markings

What is the maximum rectangular area for a given perimeter?

Part A: Enclose Four Sides

Soonji has forty 1-m segments of wood railing with which to surround a patio. Her patio is to be in the shape of a rectangle, with the fencing placed around the perimeter.



1. Assume that Soonji uses all 40 sections of railing and that the sections cannot be cut into smaller segments. Find the dimensions of several different rectangular patio designs. Determine the area of each design. Copy and complete the table.

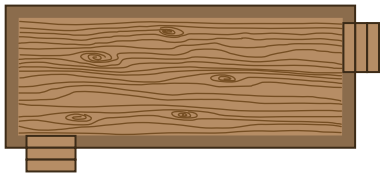
Sketch	Length (m)	Width (m)	Perimeter (m)	Area (m^2)
			40	
			40	
			40	

2. a) Determine the dimensions of the rectangle with the maximum possible patio area.

b) What is the maximum area?

c) Is there anything unique about this rectangle? Explain.

3. Suppose that Soonji decides to leave two 2-m gaps in the outside rail for two staircases.



a) Will the addition of stairs increase or decrease the maximum possible area of the patio? Explain your thinking.

b) Do you think the maximum area can be obtained by placing the two staircases:
- on adjacent sides of the patio, as shown?
- on parallel sides of the patio?
- on the same side of the patio?

Explain your prediction.

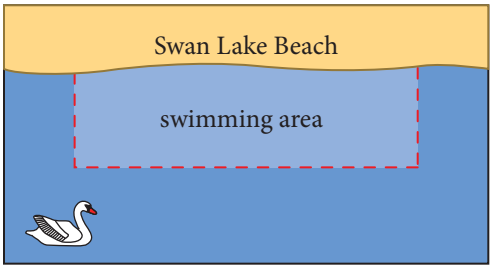
4. a) Determine the maximum area of a patio with two staircases. How does your answer compare to your prediction from step 3, part b)?

b) Sketch a diagram of the patio design with two staircases that provides the maximum area. Indicate the dimensions and calculate the maximum area.

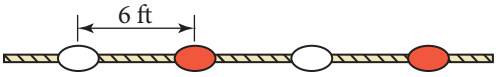
5. Reflect What type of rectangle will typically provide the maximum area for a given perimeter? Do you think this is always true? Explain.

Part B: Enclose Three Sides

Ryan has a summer job working at a campground. He is roping off a rectangular swimming area on the beach at Swan Lake, as shown.



In the equipment shed, Ryan found a rope with 50 buoys, equally spaced at 6-ft intervals, and two support stakes to anchor the buoy rope.



Ryan does not need to run the buoys along the edge of the beach, so he only needs to make three sides of a rectangle in the water.

1. a) Find the dimensions of several different rectangular swimming areas. Determine the area of each design. Copy and complete the table.

Sketch	Length (ft)	Width (ft)	Perimeter (ft)	Area (ft^2)

b) What is the maximum possible swimming area?

2. Sketch a diagram that shows where the stakes should be anchored to provide the maximum swimming area. Label the dimensions of the swimming area.

3. What is the shape of the rectangular swimming area? Explain.

4. **Reflect** Is the type of rectangle that gives a maximum area for a given perimeter the same when enclosing three sides as when enclosing four sides? Explain.

Example

Minimum Perimeter for a Given Area

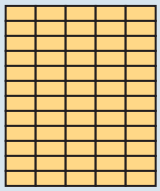
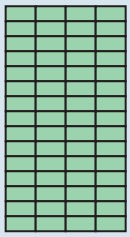
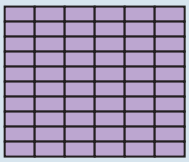
Latoya wants her new vegetable garden to have an area of 60 m^2. The garden will be rectangular and surrounded by a fence. What dimensions will require the least amount of fencing, to the nearest hundredth of a metre?

Solution

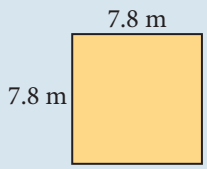
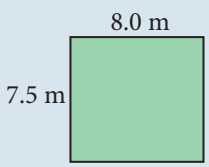
To have the least amount of fencing, determine the dimensions that minimize the perimeter for the given area.

Method 1: Use systematic trial.

Determine the different dimensions for rectangles with an area of 60 m^2. Organize the results in a table.

Model	Length (m)	Width (m)	Perimeter (m) $P = 2(l + w)$	Analysis
	12	5	$2(12 + 5) = 34$	Try to obtain a smaller perimeter with a different rectangle.
	15	4	38	Wrong direction. Try a more square-like rectangle.
	10	6	32	This is better. Try a more square-like rectangle.

No other rectangles with whole number side lengths are possible. $\sqrt{60} \doteq 7.746$, so try a side length of 7.746 m. Then check a slightly different rectangle to verify.

Model	Length (m)	Width (m)	Perimeter (m) $P = 2(l + w)$	Analysis
	7.746	7.746	30.98	This is the smallest perimeter so far. Try a slightly different rectangle to verify.
	8.0	7.5	31.0	This gives a slightly larger perimeter.

The method of systematic trial can be cumbersome, depending on how accurate the answer must be. An approach involving algebraic reasoning can lead more directly to a precise answer.

Method 2: Apply algebraic and graphical reasoning.

Rearrange the formula for the area of a rectangle to isolate w.

$$A = lw$$

$$60 = lw \qquad \text{Substitute the known area.}$$

$$\frac{60}{l} = \frac{lw}{l} \qquad \text{Divide both sides by } l.$$

$$w = \frac{60}{l}$$

Substitute this expression into the formula for the perimeter of a rectangle.

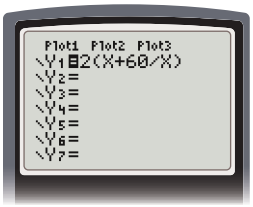
$$P = 2(l + w)$$

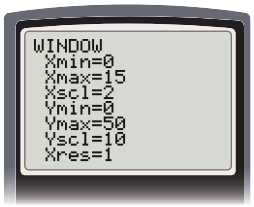
$$P = 2\left(l + \frac{60}{l}\right)$$

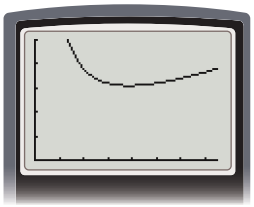
Use technology to graph this relation. Locate the value of l that gives a minimum value of P.

Use a TI-84 Plus graphing calculator.

Use reasoning and systematic trial to choose the window settings.

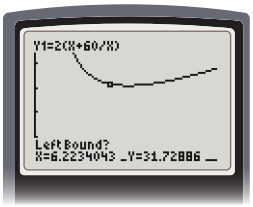


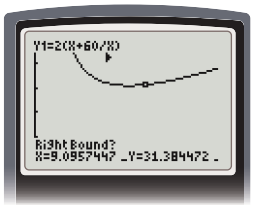


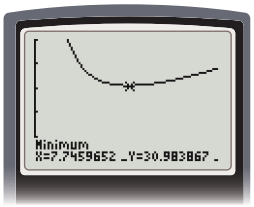


Determine the coordinates of the minimum point on this graph:

- Press 2nd [CALC].
- Select **3:minimum**.
- Follow the prompts, pressing ENTER after each step.







From the graph, the minimum perimeter of 30.98 m occurs when the length is 7.75 m.

The dimensions of Latoya's garden that require the least fencing are 7.75 m by 7.75 m.

Technology Tip

By default, the graphing calculator uses x for the independent variable and Y_1 as the dependent variable, instead of l and P.

$$P = 2\left(l + \frac{60}{l}\right)$$

$$\Updownarrow$$

$$Y_1 = 2\left(x + \frac{60}{x}\right)$$

Technology Tip

When prompted for the left bound, use the arrow keys to move the cursor to the left of the lowest point on the graph. For the right bound, move the cursor to the right of the minimum.

Use a TI-Nspire™ CAS graphing calculator.

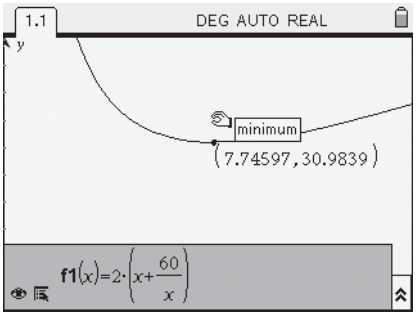
Turn on the calculator, and open a new **Graphs & Geometry** page.

Enter the formula: $2(x + 60/x)$ for $f1(x)$.

Press (menu), and select **4:Window**. Select **1:Window Settings**.
Set **XMin** to 0, **XMax** to 15, **YMin** to 0, and **YMax** to 50.
Tab down to **OK** and press (enter). The curve will be displayed.

Press (menu), and select **6:Points & Lines**. Select **2:Point On**.

If necessary, move the cursor up to the curve. Press (enter) to plot the point.



Press (esc). A hand will appear over the point.
Press (ctrl), and (·). The hand will close to grab the point.

Use the cursor keys to drag the point towards the minimum of the curve. When you reach the minimum, the label "minimum" will appear in a box, as shown. Note the coordinates of the minimum.

The minimum perimeter of 30.98 m occurs when the length is 7.75 m.

The dimensions of Latoya's garden that require the least amount of fencing are 7.75 m by 7.75 m.

Key Concepts

- When enclosing four sides, the maximum area for a given rectangular area is obtained by forming a square.
- When enclosing three sides, the maximum area for a given rectangular area is obtained by forming a rectangle whose length is twice its width.
- The minimum perimeter for a given rectangular area is obtained by forming a square.

Discuss the Concepts

D1. **a)** Do all rectangles with a perimeter of 10 m have the same area? Explain.

b) Does your conclusion from part a) apply to rectangles with a different perimeter? If yes, explain why. If no, provide a **counter-example**.

counter-example
- an example that contradicts a proposed truth

D2. Suppose you receive this email from a friend:

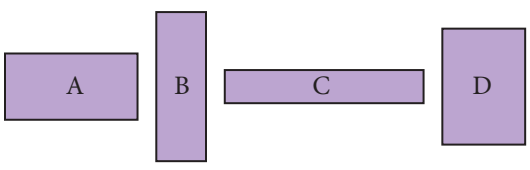
> Subject: New Garden
> To: Recipient <recipient@domain.com>
>
> Hi;
> I'm helping my mom make a new garden. She wants it as big as possible. I've got 10 m of chicken wire for fencing. Can you help me? Do you think I should back it against the garage? Any suggestions?
> Thanks!

Discuss an appropriate response to your friend.

Practise A

Use this information to answer questions 1 and 2.

These rectangles all have the same perimeter.

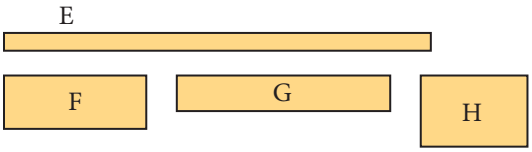


1. Order the rectangles from greatest to least area.

2. Is it possible to draw a rectangle with the same perimeter but with a greater area than these rectangles? Explain using words and a diagram.

Use this information to answer questions 3 and 4.

These rectangles all have the same area.



3. Order the rectangles from least to greatest perimeter.

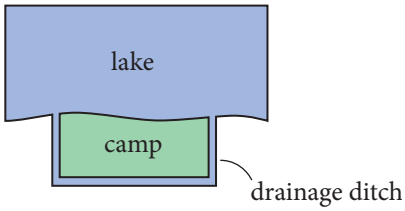
4. Is it possible to draw a rectangle with the same area that has a smaller perimeter than any of the given rectangles? Explain using words and a diagram.

Apply B

5. a) Farmer Fran has 30 segments of 2-m fence rails to build a pen for her sheep. What are the dimensions of the pen with the maximum possible grazing area for the sheep, assuming the fence rails cannot be cut?

b) How does your answer change if the fence rails can be cut? How much additional area does this provide?

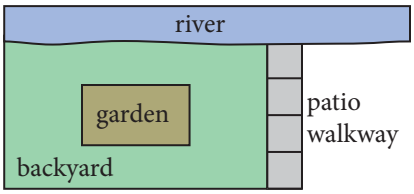
6. A scout camp is being built on the shore of a lake. The scouts are digging a narrow, 60-m long drainage ditch to surround the rectangular camp on three sides, as shown.



Determine the length and width of a ditch that will provide the maximum camp area. Use words and diagrams to show your reasoning.

Use this information to answer questions 7 to 10.

Ashraf is building a garden in his large backyard. He has 144 ft of fencing, with which to enclose his garden. He also has the option of using the river or the patio walkway (or both) instead of fencing.



7. Ashraf decides to build the garden somewhere in the middle of the backyard, as shown above. What are the dimensions of the rectangular garden with maximum area? What is the maximum area in this case?

8. Ashraf decides to build the garden against *either* the river *or* the patio walkway. What are the dimensions of the rectangular garden with maximum area? What is the maximum area in this case?

9. Ashraf decides to build the garden against *both* the river *and* the patio walkway. What are the dimensions of the rectangular garden with maximum area? What is the maximum area in this case?

10. **a)** Order the three gardens from questions 7 to 9 from greatest to least area.

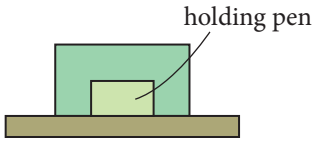
b) Which garden design do you think Ashraf should choose? Why?

11. Julia is an electronics technologist. She is designing a circuit board that has an area of 40 cm^2. It must fit snugly inside a computer console.

a) What dimensions of the circuit board will have a minimum perimeter?

b) How would your answer change if the circuit board has to fit in a slot which is only 5 cm wide? Justify your answer.

12. A marine biologist wants to use netting to build a rectangular holding pen by the beach to study marine wildlife. The pen must have an area of 120 m^2.



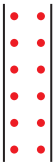
a) Determine the dimensions of the pen that require the least amount of netting.

b) Determine the total length of netting required to build the pen from part a).

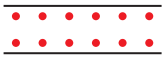
Chapter Problem

13. The lift line area of Horstman Glacier is not very large and can become quite crowded. Riders wait for and ride the T-bar lift in pairs. Rail barriers are placed to organize the waiting riders.

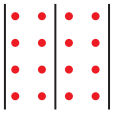
Set-Up 1
One very long line.



Set-up 2
One very wide line.



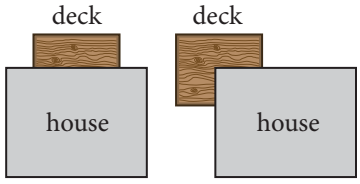
Set-Up 3
Multiple lines that arrange the riders in a square.



a) For each of the three set-ups, describe how the skiers would proceed through the waiting area. Include diagrams to help you explain.

b) Which lift line set-up do you think is best for organizing people who are waiting for the T-bar lift? Explain your reasoning.

14. Angelina wants to build a 140-ft^2 deck against her house. She is considering the options shown.



a) Assume Angelina does not have to install railing along the sides of the house. Which option minimizes the amount of railing that Angelina needs to enclose her deck?

b) What is the minimum length of railing needed?

c) What other considerations might Angelina think about when choosing her design?

Extend C

15. Ryan has found three additional support stakes at the campground. Can Ryan make a swimming area with a greater area than the rectangular one if he uses the stakes to make a different shape with the buoy rope?

a) Determine which shape maximizes the swimming area.

b) What is the maximum swimming area that can be obtained?

c) Compare your results with those of several classmates. What shapes seem to produce the greatest area?

16. Elias bought 30 cm of flexible plastic edging to enclose a flowerbed. He can bend the edging into any shape. He is considering an equilateral triangle, a square, and a circle as possible shapes.

a) Predict which of these shapes will maximize the area of the flowerbed.

b) Use a graphing calculator or dynamic geometry software to test your prediction in part a).