CHAPTER 6

Risk Aversion and Capital Allocation to Risky Assets

The process of constructing an investor portfolio can be viewed as a sequence of two steps: (1) selecting the composition of one's portfolio of risky assets such as stocks and long-term bonds and (2) deciding how much to invest in that risky portfolio versus in a safe asset such as short-term Treasury bills. Obviously, an investor cannot decide how to allocate investment funds between the risk-free asset and that risky portfolio without knowing its expected return and degree of risk, so a fundamental part of the asset allocation problem is to characterize the risk-return tradeoff for this portfolio.

While the task of constructing an optimal risky portfolio is technically complex, it can be delegated to a professional since it largely entails well-defined optimization techniques. In contrast, the decision of how much to invest in that portfolio depends on an investor's *personal* preferences about risk versus expected return, and therefore it cannot easily be delegated. As we will see in the chapter on behavioural finance, many investors stumble over this cardinal step. We therefore begin our journey into portfolio theory by establishing a framework to explore this fundamental decision, namely, capital allocation between the risk-free and the risky portfolio.

We begin by introducing two themes in portfolio theory that are centred on risk. The first is the tenet that investors will avoid risk unless they can anticipate a reward for engaging in risky investments. The second theme allows us to quantify investors' personal tradeoffs between portfolio risk and expected return. To do this we introduce a personal *utility function*, which allows each investor to assign welfare or "utility" scores to alternative portfolios based on expected return and risk and choose the portfolio with the highest score. We elaborate on the historical and empirical basis for the utility model in the appendix to this chapter.

Armed with the utility model, we can resolve the investment decision that is most consequential to investors, that is, how much of their wealth to put at risk for the greater expected return that can thus be achieved. We assume that the construction of the risky portfolio from the universe of available risky assets has already taken place and defer the discussion of how to construct that risky portfolio to the next chapter. At this point the investor can assess the expected return and risk of the overall portfolio. Using the expected return and risk parameters in the utility model yields the optimal allocation of capital between the risky portfolio and the risk-free asset.

PART TWO

6.1 RISK AND RISK AVERSION

In Chapter 5 we introduced the concepts of the holding-period return (HPR) and the excess return over the risk-free rate. We also discussed estimation of the **risk premium** (the *expected* excess return) and the standard deviation of the rate of return, which we use as the measure of portfolio risk. We demonstrated these concepts with a scenario analysis of a specific risky portfolio (Spreadsheet 5.1). To emphasize that bearing risk typically must be accompanied by a reward in the form of a risk premium, we first distinguish between speculation and gambling.

Risk, Speculation, and Gambling

One definition of *speculation* is "the assumption of considerable business risk in obtaining commensurate gain." Although this definition is fine linguistically, it is useless without first specifying what is meant by "commensurate gain" and "considerable risk."

By *commensurate gain* we mean a positive expected profit beyond the risk-free alternative. This is the risk premium, the incremental expected gain from taking on the risk. By *considerable risk* we mean that the risk is sufficient to affect the decision. An individual might reject a prospect that has a positive risk premium because the added gain is insufficient to make up for the risk involved.

To gamble is "to bet or wager on an uncertain outcome." If you compare this definition to that of speculation, you will see that the central difference is the lack of "good profit." Economically speaking, a gamble is the assumption of risk for no purpose but enjoyment of the risk itself, whereas speculation is undertaken because one perceives a favourable risk-return tradeoff. To turn a gamble into a speculative prospect requires an adequate risk premium for compensation to risk-averse investors for the risks that they bear. Hence *risk aversion and speculation are not inconsistent.*

In some cases a gamble may appear to the participants as speculation. Suppose that two investors disagree sharply about the future exchange rate of the Canadian dollar against the British pound. They may choose to bet on the outcome. Suppose that Paul will pay Mary $100 if the value of one pound exceeds $2 one year from now, whereas Mary will pay Paul if the pound is worth less than $2. There are only two relevant outcomes: (1) the pound will exceed $2, or (2) it will fall below $2. If Paul and Mary agree on the probabilities of the two possible outcomes, and if neither party anticipates a loss, it must be that they assign $p = .5$ to each outcome. In that case the expected profit to both is zero and each has entered one side of a gambling prospect.

What is more likely, however, is that the bet results from differences in the probabilities that Paul and Mary assign to the outcome. Mary assigns it $p > .5$, whereas Paul's assessment is $p < .5$. They perceive, subjectively, two different prospects. Economists call this case of differing belief's *heterogeneous expectations.* In such cases investors on each side of a financial position see themselves as speculating rather than gambling.

Both Paul and Mary should be asking, "Why is the other willing to invest in the side of a risky prospect that I believe offers a negative expected profit?" The ideal way to resolve heterogeneous beliefs is for Paul and Mary to "merge their information," that is, for each party to verify that he or she possesses all relevant information and processes the information properly. Of course, the acquisition of information and the extensive communication that is required to eliminate all heterogeneity in expectations is costly, and thus, up to a point, heterogeneous expectations cannot be taken as irrational. If, however, Paul and Mary enter such contracts frequently, they would recognize the information problem in one of two

CC 1

Concept Check

Assume that dollar-denominated T-bills in Canada and pound-denominated bills in the United Kingdom offer equal yields to maturity. Both are short-term assets, and both are free of default risk. Neither offers investors a risk premium. However, a Canadian investor who holds U.K. bills is subject to exchange rate risk since the pounds earned on the U.K. bills eventually will be exchanged for dollars at the future exchange rate. Is the Canadian investor engaging in speculation or gambling?

ways: either they will realize that they are creating gambles when each wins half of the bets, or the consistent loser will admit that he or she has been betting on inferior forecasts.

Risk Aversion and Utility Values

We have discussed risk with simple prospects and how risk premiums bear on speculation. A prospect that has a zero-risk premium is called a **fair game**. Investors who are *risk-averse* reject investment portfolios that are fair games or worse. Risk-averse investors are willing to consider only risk-free or speculative prospects. Loosely speaking, a risk-averse investor "penalizes" the expected rate of return of a risky portfolio by a certain percentage (or penalizes the expected profit by a dollar amount) to account for the risk involved. The greater the risk the investor perceives, the larger the penalization. (One might wonder why we assume risk aversion as fundamental. We believe that most investors accept this view from simple introspection, but we discuss the question more fully in the appendix at the end of this chapter.)

We can formalize this notion of a risk-penalty system. To do so, we will assume that each investor can assign a welfare, or **utility**, score to competing investment portfolios based on the expected return and risk of those portfolios. The utility score may be viewed as a means of ranking portfolios. Higher utility values are assigned to portfolios with more attractive risk-return profiles. Portfolios receive higher utility scores for higher expected returns and lower scores for higher volatility. Many particular "scoring" systems are legitimate. One reasonable function that is commonly employed by financial theorists and the CFA Institute assigns a portfolio with expected return $E(r)$ and variance of returns σ^2 the following utility score:

$$U = E(r) - \frac{1}{2} A\sigma^2 \tag{6.1}$$

where U is the utility value and A is an index of the investor's aversion to taking on risk. (The factor of ½ is a scaling convention that will simplify calculations in later chapters. It has no economic significance, and we could eliminate it simply by defining a "new" A with half the value of the A used here.)

Equation 6.1 is consistent with the notion that utility is enhanced by high expected returns and diminished by high risk. The extent to which variance lowers utility depends on A, the investor's degree of risk aversion. More risk-averse investors (who have the larger A's) penalize risky investments more severely. Investors choosing among competing investment portfolios will select the one providing the highest utility level. Notice in equation 6.1 that the utility provided by a risk-free portfolio is simply the rate of return on the portfolio, since there is no penalization for risk.

Risk aversion obviously will have a major impact on the investor's appropriate risk-return tradeoff. The next boxed article discusses some techniques that financial advisors use to gauge the risk aversion of their clients.

Time for Investing's Four-Letter Word

What four-letter word should pop into mind when the stock market takes a harrowing nose dive?

No, not those. R-I-S-K.

Risk is the potential for realizing low returns or even losing money, possibly preventing you from meeting important objectives, like sending your kids to the college of their choice or having the retirement lifestyle you crave.

But many financial advisers and other experts say that these days investors aren't taking the idea of risk as seriously as they should, and they are overexposing themselves to stocks.

"The market has been so good for years that investors no longer believe there's risk in investing," says Gary Schatsky, a financial adviser in New York.

So before the market goes down and stays down, be sure that you understand your tolerance for risk and that your portfolio is designed to match it.

Assessing your risk tolerance, however, can be tricky. You must consider not only how much risk you can afford to take but also how much risk you can stand to take.

Determining how much risk you can stand—your temperamental tolerance for risk—is more difficult. It isn't quantifiable.

To that end, many financial advisers, brokerage firms and mutual-fund companies have created risk quizzes to help people determine whether they are conservative, moderate or aggressive investors. Some firms that offer such quizzes include Merrill Lynch, T. Rowe Price Associates Inc., Baltimore, Zurich Group Inc.'s Scudder Kemper Investments Inc., New York, and Vanguard Group in Malvern, Pa.

Typically, risk questionnaires include seven to 10 questions about a person's investing experience, financial security and tendency to make risky or conservative choices.

The benefit of the questionnaires is that they are an objective resource people can use to get at least a rough idea of their risk tolerance. "It's impossible for someone to assess their risk tolerance alone," says Mr. Bernstein. "I may say I don't like risk, yet will take more risk than the average person."

Many experts warn, however, that the questionnaires should be used simply as a first step to assessing risk tolerance. "They are not precise," says Ron Meier, a certified public accountant.

The second step, many experts agree, is to ask yourself some difficult questions, such as: How much you can stand to lose over the long term?

"Most people can stand to lose a heck of a lot temporarily," says Mr. Schatsky. The real acid test, he says, is how much of your portfolio's value you can stand to lose over months or years.

As it turns out, most people rank as middle-of-the-road risk-takers, say several advisers. "Only about 10% to 15% of my clients are aggressive," says Mr. Roge.

What's Your Risk Tolerance?

Circle the letter that corresponds to your answer.

1. Just 60 days after you put money into an investment its price falls 20 percent. Assuming none of the fundamentals have changed, what would you do?
   a. Sell to avoid further worry and try something else
   b. Do nothing and wait for the investment to come back
   c. Buy more. It was a good investment before; now it's a cheap investment too
2. Now look at the previous question another way. Your investment fell 20 percent, but it's part of a portfolio being used to meet investment goals with three different time horizons.

2A. What would you do if the goal were five years away?
   a. Sell
   b. Do nothing
   c. Buy more

continued

EXAMPLE 6.1 Evaluating Investments by Using Utility Scores

Consider three investors with different degrees of risk aversion: $A_1 = 2$, $A_2 = 3.5$, and $A_3 = 5$, all of whom are evaluating the three portfolios in Table 6.1. Since the risk-free rate is assumed to be 5 percent, equation 6.1 implies that all three investors would assign a utility score of .05 to the risk-free alternative. Table 6.2 presents the utility scores that would be assigned by each investor to each portfolio. The portfolio with the highest utility score for each investor appears in bold. Notice that the high-risk portfolio, H, would be chosen only by the investor with the lowest degree of risk aversion, $A_1 = 2$, while the low-risk portfolio, L, would be passed over even by the most risk-averse of our three investors. All three portfolios beat the risk-free alternative for the investors with levels of risk aversion given in the table.

CC 2 Concept Check

A portfolio has an expected rate of return of .20 and standard deviation of .20. Bills offer a sure rate of return of .07. Which investment alternative will be chosen by an investor whose $A = 4$? What if $A = 8$?

2B. What would you do if the goal were 15 years away?
  a. Sell
  b. Do nothing
  c. Buy more

2C. What would you do if the goal were 30 years away?
  a. Sell
  b. Do nothing
  c. Buy more

3. The price of your retirement investment jumps 25% a month after you buy it. Again, the fundamentals haven't changed. After you finish gloating, what do you do?
  a. Sell it and lock in your gains
  b. Stay put and hope for more gain
  c. Buy more; it could go higher

4. You're investing for retirement, which is 15 years away. Which would you rather do?
  a. Invest in a money-market fund or guaranteed investment contract, giving up the possibility of major gains, but virtually assuring the safety of your principal
  b. Invest in a 50-50 mix of bond funds and stock funds, in hopes of getting some growth, but also giving yourself some protection in the form of steady income
  c. Invest in aggressive growth mutual funds whose value will probably fluctuate significantly during the year, but have the potential for impressive gains over five or 10 years

5. You just won a big prize! But which one? It's up to you.
  a. $2,000 in cash
  b. A 50% chance to win $5,000
  c. A 20% chance to win $15,000

6. A good investment opportunity just came along. But you have to borrow money to get in. Would you take out a loan?
  a. Definitely not
  b. Perhaps
  c. Yes

7. Your company is selling stock to its employees. In three years, management plans to take the company public. Until then, you won't be able to sell your shares and you will get no dividends. But your investment could multiply as much as 10 times when the company goes public. How much money would you invest?
  a. None
  b. Two months' salary
  c. Four months' salary

Scoring Your Risk Tolerance

To score the quiz, add up the number of answers you gave in each category *a*–*c*, then multiply as shown to find your score.

(*a*) answers ______ × 1 = ______ points
(*b*) answers ______ × 2 = ______ points
(*c*) answers ______ × 3 = ______ points

YOUR SCORE ______ points

If you scored . . .	**You may be a(n):**
9–14 points	Conservative investor
15–21 points	Moderate investor
22–27 points	Aggressive investor

Table 6.1 Available Risky Portfolios (risk-free rate = 5%)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Table 6.2 Utility Scores of Alternative Portfolios for Investors with Varying Degrees of Risk Aversion

Investor Risk Aversion (*A*)	Utility Score of Portfolio *L* [$E(r) = .07$; $\sigma = .05$]	Utility Score of Portfolio *M* [$E(r) = .09$; $\sigma = .10$]	Utility Score of Portfolio *H* [$E(r) = .13$; $\sigma = .20$]
2.0	$.07 - \frac{1}{2} \times 2 \times .05^2 = .0675$	$.09 - \frac{1}{2} \times 2 \times .1^2 = .0800$	$\mathbf{.13 - \frac{1}{2} \times 2 \times .2^2 = .09}$
3.5	$.07 - \frac{1}{2} \times 3.5 \times .05^2 = .0656$	$\mathbf{.09 - \frac{1}{2} \times 3.5 \times .1^2 = .0725}$	$.13 - \frac{1}{2} \times 3.5 \times .2^2 = .06$
5.0	$.07 - \frac{1}{2} \times 5 \times .05^2 = .0638$	$\mathbf{.09 - \frac{1}{2} \times 5 \times .1^2 = .0650}$	$.13 - \frac{1}{2} \times 5 \times .2^2 = .03$

Because we can compare utility values to the rate offered on risk-free investments when choosing between a risky portfolio and a safe one, we may interpret a portfolio's utility value as its "certainty equivalent" rate of return to an investor. That is, the **certainty equivalent rate** of a portfolio is the rate that risk-free investments would need to offer with certainty to be considered equally attractive to the risky portfolio.

Now we can say that a portfolio is desirable only if its certainty equivalent return exceeds that of the risk-free alternative. A sufficiently risk-averse investor may assign any risky portfolio, even one with a positive risk premium, a certainty equivalent rate of return that is below the risk-free rate, which will cause the investor to reject the portfolio. At the same time, a less risk-averse (more risk-tolerant) investor will assign the same portfolio a certainty equivalent rate that exceeds the risk-free rate and thus will prefer the portfolio to the risk-free alternative. If the risk premium is zero or negative to begin with, any downward adjustment to utility only makes the portfolio look worse. Its certainty equivalent rate will be below that of the risk-free alternative for all risk-averse investors.

In contrast to risk-averse investors, **risk-neutral** investors judge risky prospects solely by their expected rates of return. The level of risk is irrelevant to the risk-neutral investor, meaning that there is no penalization for risk. For this investor, a portfolio's certainty equivalent rate is simply its expected rate of return.

A **risk lover** is willing to engage in fair games and gambles; this investor adjusts the expected return upward to take into account the "fun" of confronting the prospect's risk. Risk lovers always will take a fair game because their upward adjustment of utility for risk gives the fair game a certainty equivalent that exceeds the alternative of the risk-free investment.

We can depict the individual's tradeoff between risk and return by plotting the characteristics of potential investment portfolios that the individual would view as equally attractive on a graph with axes measuring the expected value and standard deviation of portfolio returns. Figure 6.1 plots the characteristics of one portfolio.

Portfolio P, which has expected return $E(r_p)$ and standard deviation σ_p, is preferred by risk-averse investors to any portfolio in quadrant IV because it has an expected return equal to or greater than any portfolio in that quadrant and a standard deviation equal to or smaller than any portfolio in that quadrant. Conversely, any portfolio in quadrant I is preferable to portfolio P because its expected return is equal to or greater than P's and its standard deviation is equal to or smaller than P's.

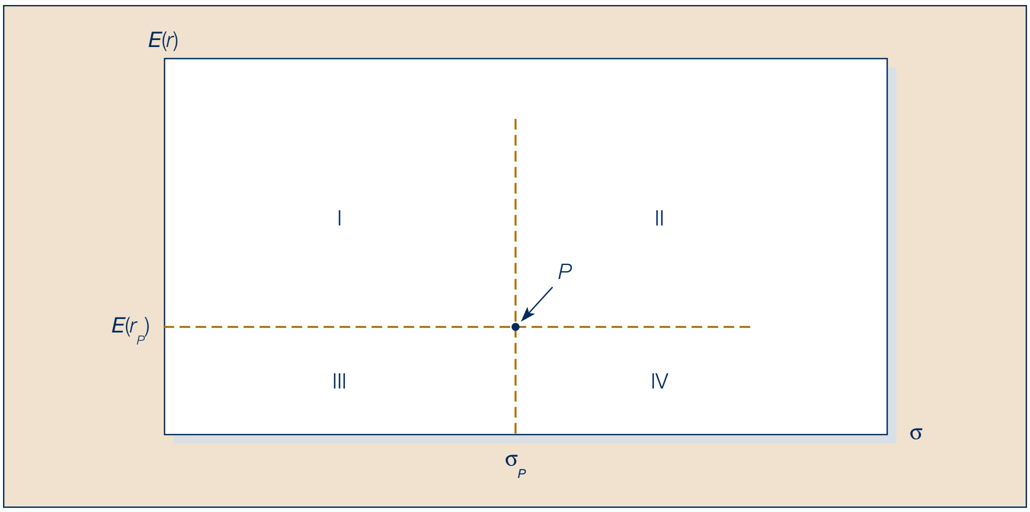

Figure 6.1

The tradeoff between risk and return of a potential investment portfolio.

This is the mean-standard deviation, or equivalently, **mean-variance (M-V) criterion**. It can be stated as A dominates *B* if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

and at least one inequality is strict.

In the expected return–standard deviation graph, the preferred direction is northwest, because in this direction we simultaneously increase the expected return *and* decrease the variance of the rate of return. This means that any portfolio that lies northwest of *P* is superior to *P*.

What can be said about the portfolios in quadrants II and III? Their desirability, compared with *P*, depends on the exact nature of the investor's risk aversion. Suppose an investor identifies all portfolios that are equally attractive as portfolio *P*. Starting at *P*, an increase in standard deviation lowers utility; it must be compensated for by an increase in expected return. Thus, point *Q* is equally desirable to this investor as *P*. Investors will be equally attracted to portfolios with high risk and high expected returns compared with other portfolios with lower risk but lower expected returns.

These equally preferred portfolios will lie on a curve in the mean-standard deviation graph that connects all portfolio points with the same utility value (Figure 6.2). This is called the **indifference curve**.

To determine some of the points that appear on the indifference curve, examine the utility values of several possible portfolios for an investor with $A = 4$, presented in Table 6.3. Note that each

Figure 6.2
The indifference curve.

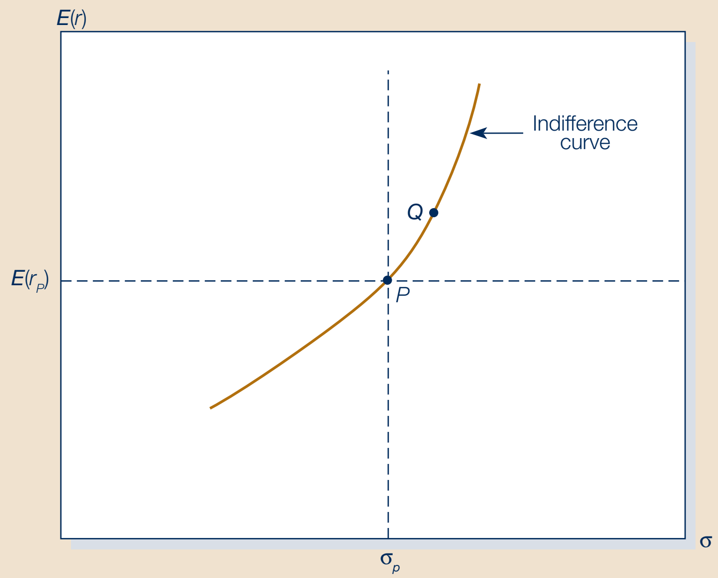

Table 6.3
Utility Values of Possible Portfolios for Investors with $A = 4$

Expected Return, $E(r)$	Standard Deviation, σ	Utility = $E(r) - \frac{1}{2}A\sigma^2$
.10	.200	$.10 - .5 \times 4 \times .04 = .02$
.15	.255	$.15 - .5 \times 4 \times .065 = .02$
.20	.300	$.20 - .5 \times 4 \times .09 = .02$
.25	.339	$.25 - .5 \times 4 \times .115 = .02$

portfolio offers identical utility, since the high-return portfolios also have high risk. Although in practice the exact indifference curves of various investors cannot be known, this analysis can take us a long way in determining appropriate principles for portfolio selection strategy.

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Concept Check

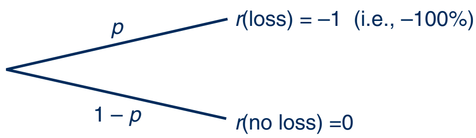
a. How will the indifference curve of a less risk-averse investor compare to the indifference curve drawn in Figure 6.2?

b. Draw both indifference curves passing through point *P*.

Estimating Risk Aversion

How might we go about estimating the levels of risk aversion we might expect to observe in practice? One way is to observe individuals' decisions when confronted with risk. For example, we can observe how much people are willing to pay to avoid risk, such as when they buy insurance against large losses. Consider an investor with risk aversion, A, whose entire wealth is in a piece of real estate. Suppose that in any given year there is a probability, p, of a disaster such as a mudslide that will destroy the real estate and wipe out the investor's entire wealth. Such an event would amount to a rate of return of -100 percent. Otherwise, with probability $1 - p$, the real estate remains intact, and we will assume that its rate of return is zero.

We can describe the probability distribution of the rate of return on this so-called simple prospect with the following diagram (with returns expressed in decimals):

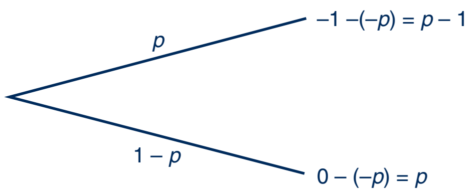


The expected rate of return of this prospect is

$$E(r) = p \times (-1) + (1 - p) \times 0 = -p \tag{6.2}$$

In other words, the expected loss is a fraction p of the value of the real estate.

What about variance and standard deviation of the investor's position? The deviations from expectation, $r - E(r)$, for each outcome are



The variance of the rate of return equals the expectation of the squared deviation:

$$\sigma^2(r) = p \times (p - 1)^2 + (1 - p) \times p^2 = p(1 - p) \tag{6.3}$$

To calculate the utility score of this simple prospect we use the risk-aversion coefficient, A, the expected return, $E(r)$ (from equation 6.2), and the variance, $\sigma^2(r)$ (from equation 6.3) in equation 6.1 and obtain

$$U = E(r) - ½A\sigma^2(r) \qquad (6.4)$$
$$= -p - ½Ap(1 - p)$$

Now we will relate the risk-aversion parameter to the amount that an individual would be willing to pay for insurance against the potential loss. Suppose an insurance company offers to cover any loss over the year for a fee of v dollars per dollar of insured property. The individual who pays $\$v$ per dollar of real estate value to the insurance company will face no risk—the insurance company will reimburse any losses, so the real estate will be worth its original value at year-end. Taking out such a policy amounts to a sure negative rate of return of $-v$, with a utility score: $U = -v$.

How much will our investor pay for the policy, that is, what is the maximum value of v he or she will be willing to pay? To find this value, we equate the utility score of the uninsured land (given in equation 6.4) to that of the insured land (which is $-v$):

$$U = -p - ½Ap(1 - p) = -v \qquad (6.5)$$

We can solve equation 6.5 for the policy cost at which the investor would be indifferent between purchasing insurance or going uninsured. This is the maximum amount that he or she will be willing pay for the insurance policy:

$$v = p[1 + ½A(1 - p)] \qquad (6.6)$$

Remember that the expected loss on the land is p. Therefore, the term in the square brackets in equation 6.6 tells us the *multiple* of the expected loss, p, the investor is willing to pay for the policy. Obviously, a risk-neutral investor, with $A = 0$, will be willing to pay no more than the expected loss, $v = p$. With $A = 1$, the term in square brackets is almost 1.5 (because p is small), so v will be close to $1.5p$. In other words, the investor is willing to pay almost 50 percent more than the expected loss for the policy. For each additional increment to the degree of risk aversion ($A = 2$, 3, and so on), the investor is willing to add (almost) another 50 percent of the expected loss to the insurance premium.

Table 6.4 shows how many multiples of the expected loss the investor is willing to pay for insurance for two values of the probability of disaster, p, as a function of the degree of risk aversion. Based on individuals' actual willingness to pay for insurance against catastrophic loss as in this example, economists estimate that investors seem to exhibit degrees of risk aversion in the range of 2 to 4, that is, would be likely to be willing to pay as much as two to three times the expected loss but not much more.

By the way, this analysis also tells you something about the merits of competitive insurance markets. Insurance companies that are able to share their risk with many co-insurers will be willing to offer coverage for premiums that are only slightly higher than the expected loss, even though

Table 6.4

Investor's Willingness to Pay for Catastrophe Insurance

	Expected Rate of Loss, $p = .0001$	Expected Rate of Loss, $p = .01$
Investor Risk Aversion, A	**Maximum Premium, v, as a Multiple of Expected Loss, p**	**Maximum Premium, v, as a Multiple of Expected Loss, p**
0	1.0000	1.0000
1	1.5000	1.4950
2	1.9999	1.9900
3	2.4999	2.4850
4	2.9998	2.9800
5	3.4998	3.4750

each investor may *value* the coverage at several multiples of the expected loss. The large savings that investors thus derive from competitive insurance markets are analogous to the consumer surplus derived from competition in other markets.

More support for the hypothesis that A is somewhere in the range of 2 to 4 can be obtained from estimates of the expected rate of return and risk on a broad stock-index portfolio. We will present this argument shortly after we describe how investors might determine their optimal allocation of wealth to risky assets.

6.2 CAPITAL ALLOCATION ACROSS RISKY AND RISK-FREE PORTFOLIOS

History shows us that long-term bonds have been riskier investments than investments in Treasury bills and that stock investments have been riskier still. On the other hand, the riskier investments have offered higher average returns. Investors, of course, do not make all-or-nothing choices from these investment classes. They can and do construct their portfolios using securities from all asset classes. Some of the portfolio may be in risk-free Treasury bills, and some in high-risk stocks.

The most straightforward way to control the risk of the portfolio is through the fraction of the portfolio invested in Treasury bills and other safe money market securities versus risky assets. This *capital allocation decision* is an example of an *asset allocation* choice—a choice among broad investment classes, rather than among the specific securities within each asset class. Most investment professionals consider asset allocation to be the most important part of portfolio construction (see the box here). Therefore, we start our discussion of the risk-return tradeoff available to investors by examining the most basic asset allocation choice: the choice of how much of the portfolio to place in risk-free money market securities versus in other risky asset classes.

We will denote the investor's portfolio of risky assets as P, and the risk-free asset as F. We will assume for the sake of illustration that the risky component of the investor's overall portfolio comprises two mutual funds: one invested in stocks and the other invested in long-term bonds. For now, we take the composition of the risky portfolio as given and focus only on the allocation between it and risk-free securities. In later sections, we turn to asset allocation and security selection across risky assets.

When we shift wealth from the risky portfolio to the risk-free asset, we do not change the relative proportions of the various risky assets within the risky portfolio. Rather, we reduce the relative weight of the risky portfolio as a whole in favour of risk-free assets.

For example, assume that the total market value of an initial portfolio is \$300,000, of which \$90,000 is invested in the Ready Asset money market fund, a risk-free asset for practical purposes. The remaining \$210,000 is invested in risky equity securities—\$113,400 in equities (E) and \$96,600 in long-term bonds (B). The E and B holding is "the" risky portfolio, 54 percent in E and 46 percent in B:

$$\text{E:} \qquad w_1 = \frac{113{,}400}{210{,}000} = .54$$

$$\text{B:} \qquad w_2 = \frac{96{,}600}{210{,}000} = .46$$

Why No Index-Based Asset Allocation Programs?

If asset allocation explains nearly everything about variance, it logically follows that security selection explains nearly nothing. For about a decade, various investment firms have been cranking out marketing material quoting academic work done in the 1980s and 1990s by researchers led by Gary Brinson about asset allocation and variance for large U.S. pension funds. Most get it wrong.

It is perhaps the most misquoted and misunderstood research in the history of capital markets research. If the findings could be summed up in a sentence, it would likely be that asset allocation explains more than 90% of portfolio variability in returns on average.

Two observations pop up. One, the study's primary finding is about variability, sometimes called standard deviation and referred to as risk. Two, the phrase "on average" acknowledges there are times when the actual experience may be considerably better or considerably worse.

Mr. Brinson suggests active management (security selection) has no measurable impact on variance. Product manufacturers drone on about what asset allocation does explain without ever referencing what it doesn't.

Here's what Mr. Brinson has said: "Our study does not mean that if you got a return of 10%, then 9% is due to asset allocation. What it means is that if you looked at the ups and downs and zigs and zags of your portfolio across time, I could explain on average 90% of those zigs and zags if I know your asset allocation. But I can't tell you anything about the return you'll achieve."

This research is most often used in support of wrap account products that aim to optimize risk-adjusted returns by assigning clients an off-the-shelf portfolio that offers an asset mix that is purportedly customized to the client's unique circumstances. Product manufacturers usually have four-to-eight model portfolios available and clients end up in one of them based on their answers to a relatively generic questionnaire.

These programs almost exclusively use higher-cost actively managed funds as the portfolio building blocks. Actively managed funds are not only more expensive, they also tend to have higher portfolio turnover and tend to be at least somewhat impure, so the prescribed asset allocation is often not the actual asset allocation. As such, active funds could materially compromise the asset mix.

Separate research by Bill Sharpe and others has repeatedly shown that most active managers lag their benchmarks. If the product manufacturers genuinely understood and believed both pieces of research, they would at least consider using cheap, pure and tax-effective index products in the construction of their portfolios. Instead, consumers are fed selective information that maximizes corporate profit (Brinson on variability) without being told about other material aspects that might harm profit (Sharpe on return).

In a court of law, people are required to tell the truth, the whole truth and nothing but the truth. We can now go to regulators and politicians and ask them to insist there be more complete disclosure in prospectuses, too. Cherry-picking material facts should not be tolerated in a profession where the practitioners are expected to put the client's interests first.

Source: John De Goey, "Why No Index-Based Asset Allocation Programs?," *National Post* [National Edition], April 25, 2005, p. FP.9.

The weight of the risky portfolio, P, in the **complete portfolio**, including risk-free investments, is denoted by y:

$$y = \frac{210{,}000}{300{,}000} = .7 \text{ (Risky assets)}$$

$$1 - y = \frac{90{,}000}{300{,}000} = .3 \text{ (Risk-free assets)}$$

The weights of each stock in the complete portfolio are as follows:

$$\text{E:} \quad \frac{\$113{,}400}{\$300{,}000} = .378$$

$$\text{B:} \quad \frac{\$96{,}600}{\$300{,}000} = .322$$

$$\text{Risky portfolio} \quad = .700$$

The risky portfolio is 70 percent of the complete portfolio.

EXAMPLE 6.2 The Risky Portfolio

Suppose that the owner of this portfolio wishes to decrease risk by reducing the allocation to the risky portfolio from $y = .7$ to $y = .56$. The risky portfolio would total only $168,000 (.56 × $300,000 = $168,000), requiring the sale of $42,000 of the original $210,000 risky holdings, with the proceeds used to purchase more shares in Ready Asset (the money market fund). Total holdings in the risk-free asset will increase to 300,000 (1 − .56) = $132,000, or the original holdings plus the new contribution to the money market fund:

$$\$90{,}000 + \$42{,}000 = \$132{,}000$$

The key point, however, is that we leave the proportions of each stock in the risky portfolio unchanged. Because the weights of E and B in the risky portfolio are .54 and .46, respectively, we sell .54 × $42,000 = $22,680 of E and .46 × $42,000 = $19,320 of B. After the sale, the proportions of each share in the risky portfolio are in fact unchanged:

$$\text{E:} \qquad w_1 = \frac{113{,}400 - 22{,}680}{210{,}000 - 42{,}000} = .54$$

$$\text{B:} \qquad w_2 = \frac{96{,}600 - 19{,}320}{210{,}000 - 42{,}000} = .46$$

Rather than thinking of our risky holdings as E and B stock separately, we may view our holdings as if they were in a single fund that holds E and B in fixed proportions. In this sense we treat the risky fund as a single risky asset, that asset being a particular bundle of securities. As we shift in and out of safe assets, we simply alter our holdings of that bundle of securities commensurately.

Given this assumption, we now can turn to the desirability of reducing risk by changing the risky/risk-free asset mix, that is, reducing risk by decreasing the proportion y. As long as we do not alter the weights of each stock within the risky portfolio, the probability distribution of the rate of return on the risky portfolio remains unchanged by the asset reallocation. What will change is the probability distribution of the rate of return on the complete portfolio that consists of the risky asset and the risk-free asset.

CC 4

Concept Check

What will be the dollar value of your position in E and its proportion in your overall portfolio if you decide to hold 50 percent of your investment budget in Ready Asset?

6.3 THE RISK-FREE ASSET

By virtue of its power to tax and control the money supply, only the government can issue default-free bonds. Actually, the default-free guarantee by itself is not sufficient to make the bonds risk-free in real terms. The only risk-free asset in real terms would be a perfectly price-indexed bond. Moreover, a default-free perfectly indexed bond offers a guaranteed real rate to an investor only if the maturity of the bond is identical to the investor's desired holding period. Even indexed bonds are subject to interest rate risk, because real interest rates change unpredictably through time. When future real rates are uncertain, so is the future price of perfectly indexed bonds.

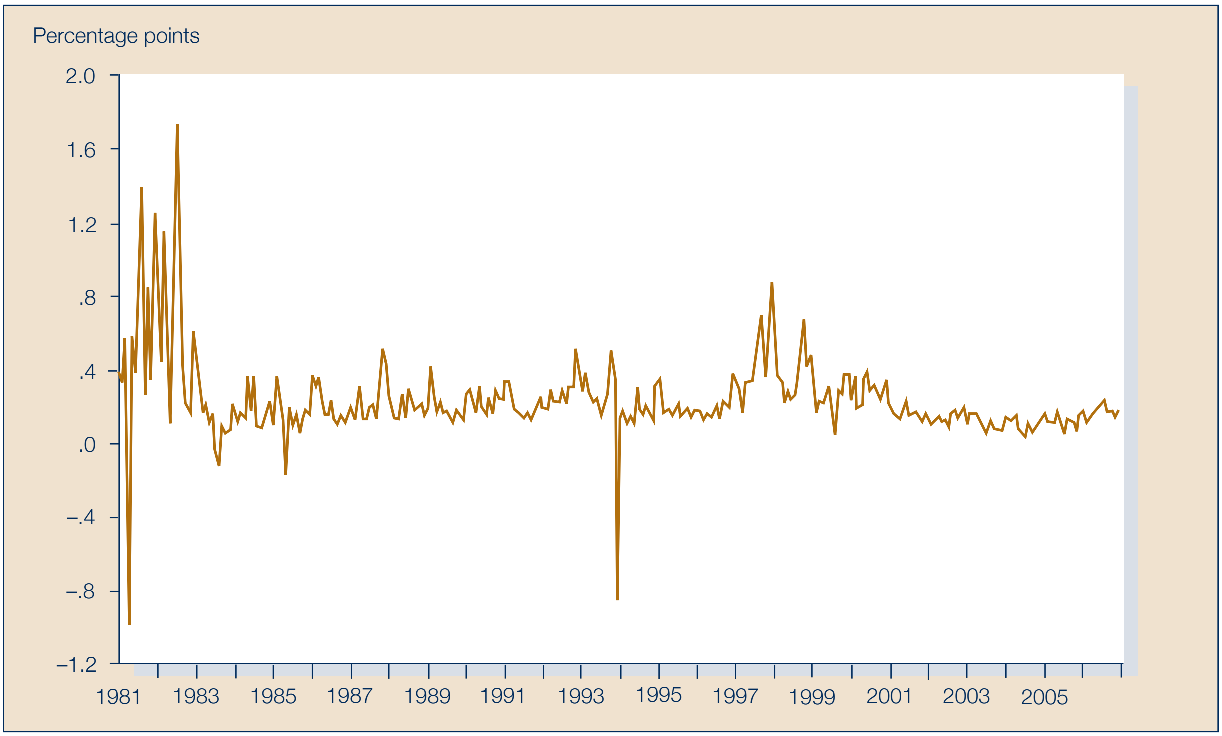
Nevertheless, it is common practice to view Treasury bills as "the" risk-free asset. Their short-term nature makes their values insensitive to interest rate fluctuations. Indeed, an investor can lock in a short-term nominal return by buying a bill and holding it to maturity. The inflation uncertainty over the course of a few weeks, or even months, is negligible compared with the uncertainty of stock market returns.

In practice, most investors use a broader range of money market instruments as a risk-free asset. All the money market instruments are virtually free of interest rate risk because of their short maturities and are fairly safe in terms of default or credit risk.

Most money market funds hold, for the most part, three types of securities: Treasury bills, bearer deposit notes (BDNs), and commercial paper (CP), differing slightly in their default risk. The yields to maturity on BDNs and CP for identical maturity, for example, are always slightly higher than those of T-bills. The pattern of this yield spread for short-term high-quality commercial paper is shown in Figure 6.3.

Money market funds have changed their relative holdings of these securities over time, but by and large, T-bills make up only about 15 percent of their portfolios. Nevertheless, the risk of such blue-chip short-term investments as BDNs and CP is minuscule compared with that of most other assets, such as long-term corporate bonds, common stocks, or real estate. Hence, we treat money market funds as the most easily accessible risk-free asset for most investors.

Figure 6.3 Yield spread between 3-month corporate paper and T-bills.



Source: Data from Scotia Capital, *Debt Market Indices*, various years. Available www.scotiacapital.com.

6.4 PORTFOLIOS OF ONE RISKY ASSET AND ONE RISK-FREE ASSET

In this section, we examine the risk-return combinations available to investors. This is the "technological" part of asset allocation; it deals with only the opportunities available to investors given the features of the broad asset markets in which they can invest. In the next section, we will address the "personal" part of the problem—the specific individual's choice of the best risk-return combination from the set of feasible combinations.

Suppose that the investor already has decided on the composition of the optimal risky portfolio. The investment proportions in all the available risky assets are known. Now the final concern is with the proportion of the investment budget, y, to be allocated to the risky portfolio, P. The remaining proportion, $1 - y$, is to be invested in the risk-free asset, F.

Denote the risky rate of return by r_P and denote the expected rate of return on P by $E(r_P)$ and its standard deviation by σ_P. The rate of return on the risk-free asset is denoted as r_f. In the numerical example we assume that $E(r_P) = 15$ percent, $\sigma_P = 22$ percent, and the risk-free rate is $r_f = 7$ percent. Thus, the risk premium on the risky asset is $E(r_P) - r_f = 8$ percent.

With a proportion y in the risky portfolio and $1 - y$ in the risk-free asset, the rate of return on the *complete* portfolio, denoted C, is r_C where

$$r_C = yr_P + (1 - y)r_f \tag{6.7}$$

Taking the expectation of this portfolio's rate of return,

$$\begin{aligned} E(r_C) &= yE(r_P) + (1 - y)r_f \\ &= r_f + y[E(r_P) - r_f] \\ &= .07 + y(.15 - .07) \end{aligned} \tag{6.8}$$

This result is easily interpreted. The base rate of return for any portfolio is the risk-free rate. In addition, the portfolio is *expected* to earn a risk premium that depends on the risk premium of the risky portfolio, $E(r_P) - r_f$, and the investor's exposure to the risky asset, denoted by y. Investors are assumed to be risk-averse and thus unwilling to take on a risky position without a positive risk premium.

When we combine a risky asset and a risk-free asset in a portfolio, the standard deviation of that portfolio is the standard deviation of the risky asset multiplied by the weight of the risky asset in that portfolio. In our case, the complete portfolio consists of the risky asset and the risk-free asset. Since the standard deviation of the risky portfolio is $\sigma_P = .22$,

$$\begin{aligned} \sigma_C &= y\sigma_P \\ &= .22y \end{aligned} \tag{6.9}$$

which makes sense because the standard deviation of the portfolio is proportional to both the standard deviation of the risky asset and the proportion invested in it. In sum, the rate of return of the complete portfolio will have expected return $E(r_C) = r_f + y[E(r_P) - r_f] = .07 + .08y$ and standard deviation $\sigma_C = .22y$.

The next step is to plot the portfolio characteristics (as a function of y) in the expected return–standard deviation plane. This is done in Figure 6.5. The expected return–standard deviation combination for the risk-free asset, F, appears on the vertical axis because the standard deviation is zero. The risky asset, P, is plotted with a standard deviation, $\sigma_P = .22$, and expected return of .15. If an investor chooses to invest solely in the risky asset, then $y = 1.0$, and the resulting portfolio is P. If the chosen position is $y = 0$, then $1 - y = 1.0$, and the resulting portfolio is the risk-free portfolio F.

What about the more interesting midrange portfolios where y lies between zero and 1? These portfolios will graph on the straight line connecting points F and P. The slope of that line is simply $[E(r_P) - r_f]/\sigma_P$ (or rise/run), in this case .08/.22.

The conclusion is straightforward. Increasing the fraction of the overall portfolio invested in the risky asset increases the expected return by the risk premium of equation 6.1, which is .08. It also increases portfolio standard deviation according to equation 6.9 at the rate of .22. The extra return per extra risk is thus .08/.22 = .36.

To derive the exact equation for the straight line between F and P, we rearrange equation 6.9 to find that $y = \sigma_C/\sigma_P$, and substitute for y in equation 6.8 to describe the expected return–standard deviation tradeoff:

$$\begin{aligned} E[r_C(y)] &= r_f + y[E(r_P) - r_f] \\ &= r_f + \frac{\sigma_C}{.22}[E(r_P) - r_f] \\ &= .07 + \frac{.08}{.22}\sigma_C \end{aligned} \tag{6.10}$$

Thus, the expected return of the portfolio as a function of its standard deviation is a straight line, with intercept r_f and slope as follows:

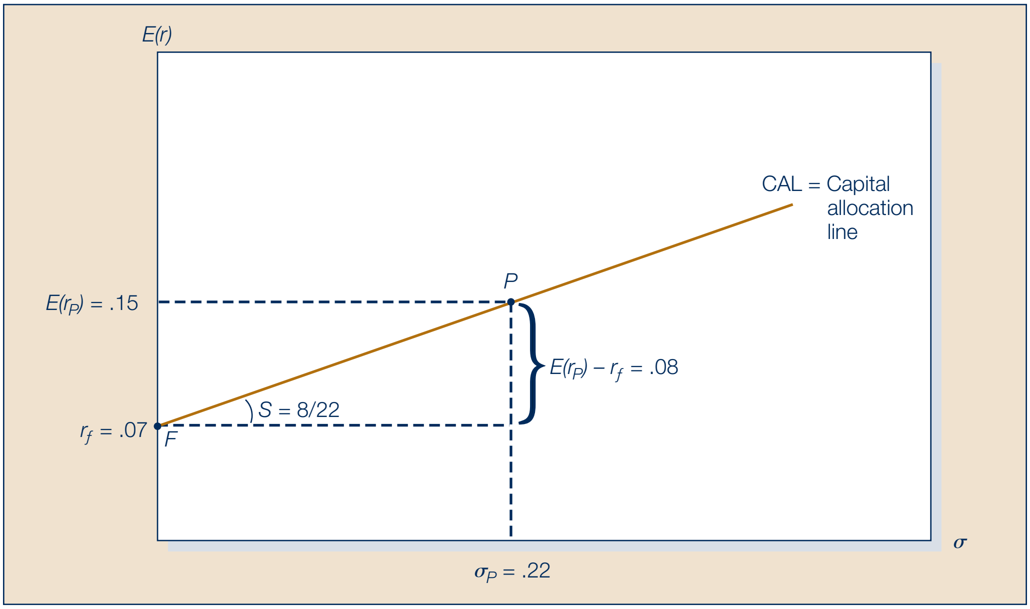
$$\begin{aligned} S &= \frac{E(r_P) - r_f}{\sigma_P} \\ &= \frac{.08}{.22} \end{aligned} \tag{6.11}$$

Figure 6.4 graphs the *investment opportunity set,* which is the set of feasible expected return and standard deviation pairs of all portfolios resulting from different values of y. The graph is a straight line originating at r_f and going through the point labelled P.

This straight line is called the **capital allocation line (CAL)**. It depicts all the risk-return combinations available to investors. The slope of the CAL, S, equals the increase in the expected

Figure 6.4

The investment opportunity set with a risky asset and a risk-free asset.



return of the chosen portfolio per unit of additional standard deviation—in other words, the measure of extra return per extra risk. For this reason, the slope also is called the **reward-to-variability ratio.**

A portfolio equally divided between the risky asset and the risk-free asset, that is, where $y = .5$, will have an expected rate of return of $E(r_C) = .07 + .5 \times .08 = .11$, implying a risk premium of 4 percent, and a standard deviation of $\sigma_C = .5 \times .22 = .11$, or 11 percent. It will plot on the line FP midway between F and P. The reward-to-variability ratio will be $S = .04/.11 = .36$, same as that of portfolio P.

CC 5

Concept Check

Can the reward-to-variability ratio, $S = [E(r_C) - r_f]/\sigma_C$, of any combination of the risky asset and the risk-free asset be different from the ratio for the risky asset taken alone, $[E(r_P) - r_f]/\sigma_P$, which in this case is .36?

What about points on the line to the right of portfolio P in the investment opportunity set? If investors can borrow at the (risk-free) rate of $r_f = 7$ percent, they can construct portfolios that may be plotted on the CAL to the right of P.

EXAMPLE 6.3 Leverage

Suppose the investment budget is $300,000, and our investor borrows an additional $120,000, investing the total available funds in the risky asset. This is a *leveraged* position in the risky asset; it is financed in part by borrowing. In that case

$$y = \frac{420{,}000}{300{,}000}$$
$$= 1.4$$

and $1 - y = 1 - 1.4 = -.4$, reflecting a short position in the risk-free asset, which is a borrowing position. Rather than lending at a 7 percent interest rate, the investor borrows at 7 percent. The distribution of the portfolio rate of return still exhibits the same reward-to-variability ratio:

$$E(r_C) = .07 + (1.4 \times .08) = .182$$
$$\sigma_C = 1.4 \times .22 = .308$$
$$S = \frac{E(r_C) - r_f}{\sigma_C}$$
$$= \frac{.182 - .07}{.308} = .36$$

As one might expect, the leveraged portfolio has a higher standard deviation than does an unleveraged position in the risky asset.

Of course, nongovernment investors cannot borrow at the risk-free rate. The risk of a borrower's default causes lenders to demand higher interest rates on loans. Therefore, the nongovernment investor's borrowing cost will exceed the lending rate of $r_f = 7$ percent. Suppose that

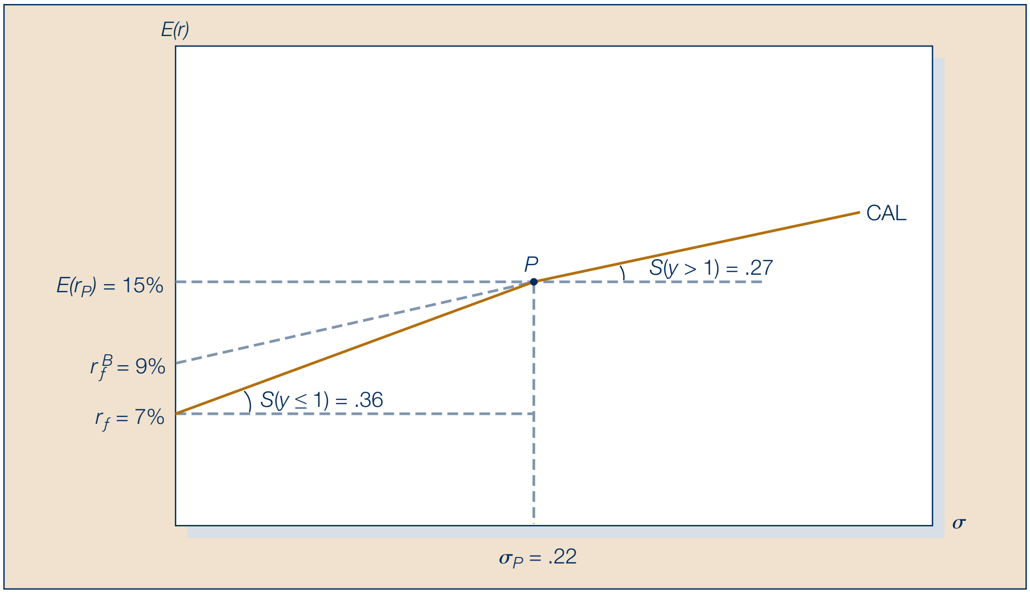


Figure 6.5 The opportunity set with differential borrowing and lending rates.

the borrowing rate is $r^B{}_f = 9$ percent. Then, in the borrowing range the reward-to-variability ratio, the slope of the CAL, will be $[E(r_P) - r^B{}_f]/\sigma_P = .06/.22 = .27$. The CAL therefore will be "kinked" at point P as shown in Figure 6.5. To the left of P the investor is lending at 7 percent, and the slope of the CAL is .36. To the right of P, where $y > 1$, the investor is borrowing to finance extra investments in the risky asset, and the slope is .27.

In practice, borrowing to invest in the risky portfolio is easy and straightforward if you have a margin account with a broker. All you have to do is tell your broker that you want to buy "on margin." Margin purchases may not exceed 70 percent of the purchase value. Therefore, if your net worth in the account is $300,000, the broker is allowed to lend you up to $300,000 to purchase additional stock.[1] You would then have $600,000 on the asset side of your account and $300,000 on the liability side, resulting in $y = 2.0$.

CC 6

Concept Check

Suppose that there is a shift upward in the expected rate of return on the risky asset, from 15 percent to 17 percent. If all other parameters remain unchanged, what will be the slope of the CAL for $y \leq 1$ and $y > 1$?

6.5 RISK TOLERANCE AND ASSET ALLOCATION

We have shown how to develop the CAL, the graph of all feasible risk-return combinations available from different asset-allocation choices. The investor confronting the CAL now must choose one optimal combination from the set of feasible choices. This choice entails a tradeoff between risk and return. Individual investor differences in risk aversion imply that, given an identical

[1]Margin purchases require the investor to maintain the securities in a margin account with the broker. If the value of the securities declines below a maintenance margin, a margin call is sent out, requiring a deposit to bring the net worth of the account up to the appropriate level. If the margin call is not met, regulations mandate that some or all of the securities be sold by the broker and the proceeds used to reestablish the required margin. See Chapter 3, Section 3.5, for a further discussion. As we will see in Chapter 20, futures contracts also offer leverage if the risky portfolio is an index fund on which a contract trades.

opportunity set (as described by a risk-free rate and a reward-to-variability ratio), different investors will choose different positions in the risky asset. In particular, the more risk-averse investors will choose to hold less of the risky asset and more of the risk-free asset.

An investor who faces a risk-free rate, r_f, and a risky portfolio with expected return $E(r_P)$ and standard deviation σ_P will find that, for any choice of y, the expected return of the complete portfolio is given by equation 6.8:

$$E(r_C) = r_f + y[E(r_P) - r_f]$$

From equation 6.9, the variance of the overall portfolio is

$$\sigma^2{}_C = y^2\sigma^2{}_P$$

The investor attempts to maximize his or her utility level, U, by choosing the best allocation to the risky asset, y. The utility function is given by equation 6.1 as $U = E(r) - ½ A\sigma^2$. As the allocation to the risky asset increases (higher y), expected return increases, but so does volatility, so utility can increase or decrease. To illustrate, Table 6.5 shows utility levels corresponding to different values of y. Initially, utility increases as y increases, but eventually it declines.

Figure 6.6 is a plot of the utility function from Table 6.5. The graph shows that utility is highest at $y = .41$. When y is less than .41, investors are willing to assume more risk to increase expected return. But at higher levels of y, risk is higher, and additional allocations to the risky asset are undesirable—beyond this point, further increases in risk dominate the increase in expected return and reduce utility.

To solve the utility maximization problem more generally, we write the problem as follows:

$$\underset{y}{\text{Max}}\ U = E(r_C) - ½A\sigma_C^2 = r_f + y[E(r_P) - r_f] - ½Ay^2\sigma_P^2$$

Students of calculus will remember that the maximization problem is solved by setting the derivative of this expression to zero. Doing so and solving for y yields the optimal position for risk-averse investors in the risky asset, y^*, as follows:[2]

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2} \tag{6.12}$$

Table 6.5

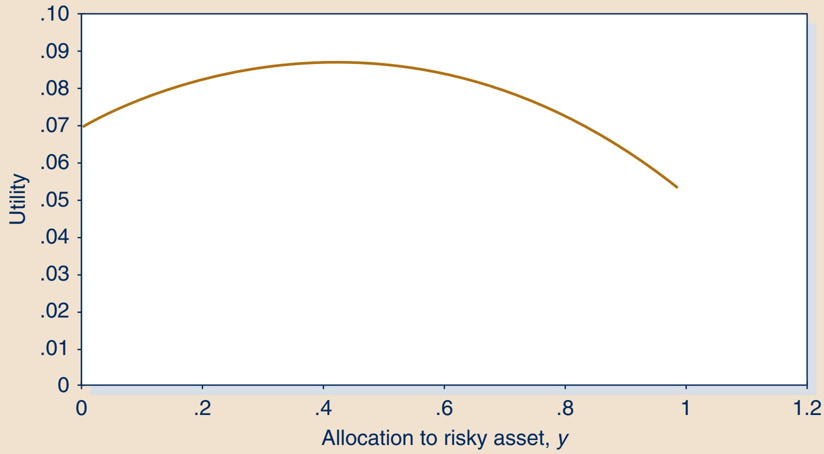
Utility Levels for Various Positions in Risky Assets (y) for Investor with Risk Aversion $A = 4$

(1) y	(2) $E(r_C)$	(3) σ_C	(4) $U = E(r) - ½ A\sigma^2$
0	.070	0	.0700
0.1	.078	.022	.0770
0.2	.086	.044	.0821
0.3	.094	.066	.0853
0.4	.102	.088	.0865
0.5	.110	.110	.0858
0.6	.118	.132	.0832
0.7	.126	.154	.0786
0.8	.134	.176	.0720
0.9	.142	.198	.0636
1.0	.150	.220	.0532

[2]The derivative with respect to y equals $E(r_P) - r_f - yA\sigma^2{}_P$. Setting this expression equal to zero and solving for y yields equation 6.12.

Figure 6.6

Utility as a function of allocation to the risky asset, *y*.



This solution shows that the optimal position in the risky asset is, as one would expect, *inversely* proportional to the level of risk aversion and the level of risk (as measured by the variance) and directly proportional to the risk premium offered by the risky asset.

EXAMPLE 6.4 Capital Allocation

Using our numerical example (r_f = 7%, $E(r_P)$ = 15%, and σ_P = 22%), and expressing all returns as decimals, the optimal solution for an investor with a coefficient of risk aversion A = 4 is

$$y^* = \frac{.15 - .07}{4 \times .22^2} = .41$$

In other words, this particular investor will invest 41 percent of the investment budget in the risky asset and 59 percent in the risk-free asset. As we saw in Figure 6.6, this is the value of y at which utility is maximized.

With 41 percent invested in the risky portfolio, the expected return and standard deviation of the complete portfolio are

$$E(r_C) = 7 + [.41 \times (15 - 7)] = 10.28\%$$
$$\sigma_C = .41 \times 22 = 9.02\%$$

The risk premium of the complete portfolio is $E(r_C) - r_f$ = 3.28%, which is obtained by taking on a portfolio with a standard deviation of 9.02 percent. Notice that 3.28/9.02 = .36, which is the reward-to-variability (Sharpe) ratio assumed for this example.

A graphical way of presenting this decision problem is to use indifference curve analysis. To illustrate how to build an indifference curve, consider an investor with risk aversion A = 4 who currently holds all her wealth in a risk-free portfolio yielding r_f = 5%. Because the variance of such a portfolio is zero, equation 6.1 tells us that its utility value is U = .05. Now we find the expected return the investor would require to maintain the *same* level of utility when holding a risky portfolio, say with σ = 1%. We use equation 6.1 to find how much $E(r)$ must increase to compensate for the higher value of σ:

$$U = E(r) - \frac{1}{2} A\sigma^2$$

$$.05 = E(r) - \frac{1}{2} \times 4 \times .01^2$$

This implies that the necessary expected return increases to

$$\text{Required } E(r) = .05 + \frac{1}{2} \times A\sigma^2 = .05 + \frac{1}{2} \times 4 \times .01^2 = .0502 \qquad (6.13)$$

We can repeat this calculation for many other levels of σ, each time finding the value of $E(r)$ necessary to maintain $U = .05$. This process will yield all combinations of expected return and volatility with utility level of .05; plotting these combinations gives us the indifference curve.

We can readily generate an investor's indifference curves using a spreadsheet. Table 6.6 contains risk-return combinations with utility values of .05 and .09 for two investors, one with $A = 2$ and the other with $A = 4$. For example, column 2 uses equation 6.13 to calculate the expected return that must be paired with the standard deviation in column 1 for an investor with $A = 2$ to derive a utility value of $U = .05$. Column 3 repeats the calculations for a higher utility value, $U = .09$. The plot of these expected return–standard deviation combinations appears in Figure 6.7 as the two curves labelled $A = 2$. Notice that the intercepts of the indifference curves are at .05 and .09, exactly the level of utility corresponding to the two curves.

Given the choice, any investor would prefer a portfolio on the higher indifference curve, the one with a higher certainty equivalent (utility). Portfolios on higher indifference curves offer a higher expected return for any given level of risk. For example, both indifference curves for $A = 2$ have the same shape, but for any level of volatility, a portfolio on the curve with utility of .09 offers an expected return 4 percent greater than the corresponding portfolio on the lower curve, for which $U = .05$.

Columns 4 and 5 of Table 6.6 repeat this analysis for a more risk-averse investor, with $A = 4$. The resulting pair of indifference curves in Figure 6.7 demonstrates that more risk-averse investors have steeper indifference curves than less risk-averse investors. Steeper curves mean that investors require a greater increase in expected return to compensate for an increase in portfolio risk.

Higher indifference curves correspond to higher levels of utility. The investor thus attempts to find the complete portfolio on the highest possible indifference curve. When we superimpose plots of indifference curves on the investment opportunity set represented by the capital allocation line as in Figure 6.8, we can identify the *highest possible* indifference curve that

Table 6.6

Spreadsheet Calculations of Indifference Curves (entries in columns 2–4 are expected returns necessary to provide specified utility value)

	A = 2		A = 4	
σ	U = .05	U = .09	U = .05	U = .09
0	.0500	.0900	.050	.090
.05	.0525	.0925	.055	.095
.10	.0600	.1000	.070	.110
.15	.0725	.1125	.095	.135
.20	.0900	.1300	.130	.170
.25	.1125	.1525	.175	.215
.30	.1400	.1800	.230	.270
.35	.1725	.2125	.295	.335
.40	.2100	.2500	.370	.410
.45	.2525	.2925	.455	.495
.50	.3000	.3400	.550	.590

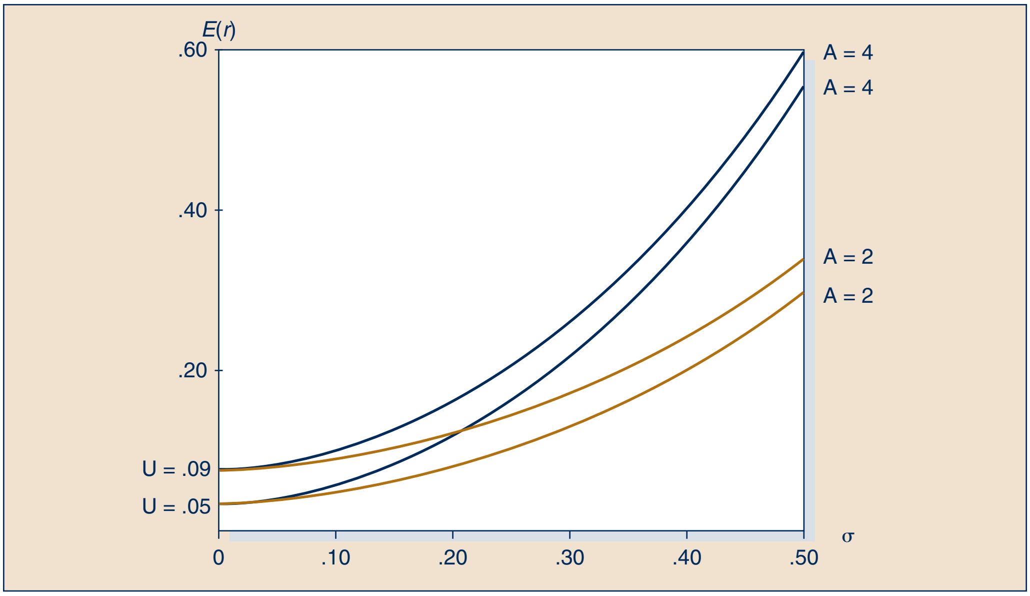


Figure 6.7
Indifference curves for $U = .05$ and $U = .09$ with $A = 2$ and $A = 4$.

still touches the CAL. That indifference curve is tangent to the CAL, and the tangency point corresponds to the standard deviation and expected return of the optimal complete portfolio.

To illustrate, Table 6.7 provides calculations for four indifference curves (with utility levels of .07, .078, .08653, and .094) for an investor with $A = 4$. Columns 2–5 use equation 6.13 to calculate the expected return that must be paired with the standard deviation in column 1 to provide the utility value corresponding to each curve. Column 6 uses equation 6.10 to calculate $E(r_C)$ on the CAL for the standard deviation σ_C in column 1:

$$E(r_C) = r_f + [E(r_P) - r_f]\frac{\sigma_C}{\sigma_P} = .07 + [.15 - .07]\frac{\sigma_C}{.22}$$

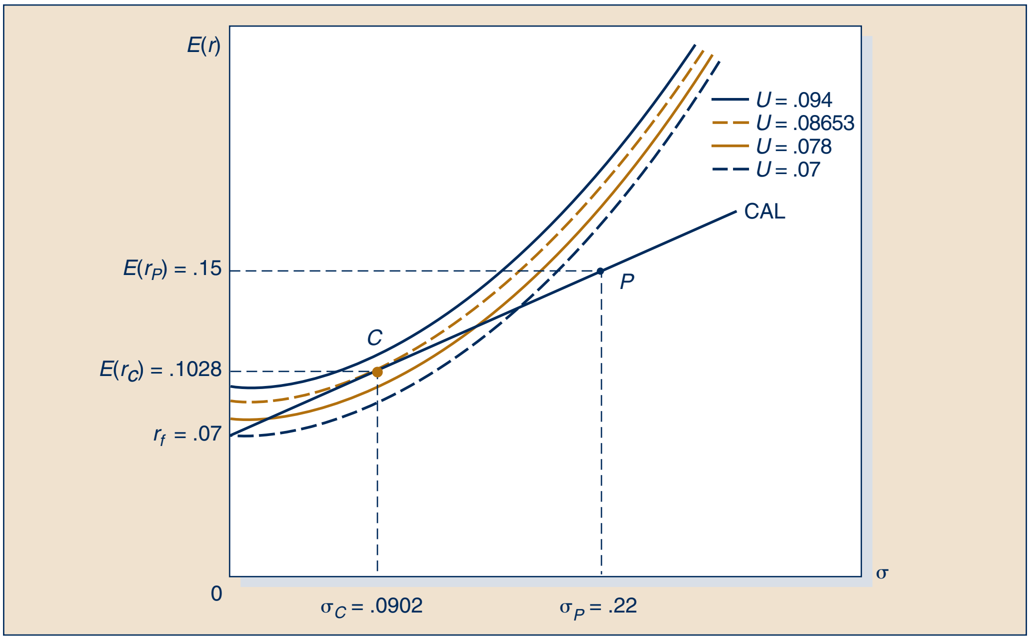


Figure 6.8
Finding the optimal complete portfolio by using indifference curves.

Table 6.7

Expected Returns on Four Indifference Curves and the CAL

σ	*U* = .07	*U* = .078	*U* = .08653	*U* = .094	CAL
0	.0700	.0780	.0865	.0940	.0700
.02	.0708	.0788	.0873	.0948	.0773
.04	.0732	.0812	.0897	.0972	.0845
.06	.0772	.0852	.0937	.1012	.0918
.08	.0828	.0908	.0993	.1068	.0991
.0902	.0863	.0943	.1028	.1103	.1028
.10	.0900	.0980	.1065	.1140	.1064
.12	.0988	.1068	.1153	.1228	.1136
.14	.1092	.1172	.1257	.1332	.1209
.18	.1348	.1428	.1513	.1588	.1355
.22	.1668	.1748	.1833	.1908	.1500
.26	.2052	.2132	.2217	.2292	.1645
.30	.2500	.2580	.2665	.2740	.1791

E-INVESTMENTS
Risk Aversion

There is a difference between an investor's *willingness* to take risk and his or her *ability* to take risk. Take the quizzes offered at the Web sites below and compare the results. If they are significantly different, which one would you use to determine an investment strategy?

http://mutualfunds.about.com/library/personalitytests/blrisktolerance.htm

http://mutualfunds.about.com/library/personalitytests/blriskcapacity.htm

Figure 6.8 graphs the four indifference curves and the CAL. The graph reveals that the indifference curve with $U = .08653$ is tangent to the CAL; the tangency point corrsponds to the complete portfolio that maximizes utility. The tangency point occurs at $\sigma_C = 9.02\%$ and $E(r_C) = 10.28\%$, the risk-return parameters of the optimal complete portfolio with $y^* = .41$. These values match our algebraic solution using equation 6.12.

We conclude that the choice for y^*, the fraction of overall investment funds to place in the risky portfolio versus the safer but lower expected-return risk-free asset, is in large part a matter of risk aversion.

CC 7

Concept Check

a. If an investor's coefficient of risk aversion is $A = 3$, how does the optimal asset mix change? What are the new $E(r_C)$ and σ_C?

b. Suppose that the borrowing rate, $r^B_f = 9$ percent, is greater than the lending rate, $r_f = 7$ percent. Show, graphically, how the optimal portfolio choice of some investors will be affected by the higher borrowing rate. Which investors will not be affected by the borrowing rate?

6.6 PASSIVE STRATEGIES: THE CAPITAL MARKET LINE

The CAL is derived with the risk-free asset and "the" risky portfolio *P*. Determination of the assets to include in risky portfolio *P* may result from a passive or an active strategy. A **passive strategy** describes a portfolio decision that avoids *any* direct or indirect security analysis.[3] At

[3]By "indirect security analysis" we mean the delegation of that responsibility to an intermediary, such as a professional money manager.

first blush, a passive strategy would appear to be naïve. As will become apparent, however, forces of supply and demand in large capital markets may make such a strategy a reasonable choice for many investors.

A natural candidate for a passively held risky asset would be a well-diversified portfolio of common stocks. We already have said that a passive strategy requires that we devote no resources to acquiring information on any individual stock or group of stocks, so we must follow a "neutral" diversification strategy. One way is to select a diversified portfolio of stocks that mirrors the value of the corporate sector of the Canadian economy. This results in a value-weighted portfolio in which, for example, the proportion invested in Nortel's stock will be the ratio of Nortel's total market value to the market value of all listed stocks.

www.tsx.com

The most frequently used value-weighted stock portfolio in Canada is the Toronto Stock Exchange's composite index of the largest capitalization Canadian corporations[4] (S&P/TSX Composite). Table 6.8 shows the historical record of this portfolio. The last columns show the average risk premium over T-bills, its standard deviation, and the reward-to-variability (Sharpe) ratio. The risk premium of 4.38 percent and standard deviation of 16.78 percent over the entire period correspond to the figures of 8 percent and 22 percent we assumed for the risky portfolio example in Section 6.4.

The Sharpe ratio is .26 for the entire 50-year period. It varies between .12 and .34 over the various subperiods. These numbers are clearly lower than the corresponding Sharpe ratios for the U.S. large stocks that make up the S&P 500 index. Over the 80-year 1926–2005 period, these stocks have a Sharpe ratio of .41, while over different subperiods it varies from a low of .15 to a high of .74.

We call the capital allocation line provided by one-month T-bills and a broad index of common stocks the **capital market line (CML)**. A passive strategy generates an investment opportunity set that is represented by the CML.

How reasonable is it for an investor to pursue a passive strategy? Of course, we cannot answer such a question without comparing the strategy to the costs and benefits accruing to an active portfolio strategy. Some thoughts are relevant at this point, however.

First, the alternative active strategy is not free. Whether you choose to invest the time and cost to acquire the information needed to generate an optimal active portfolio of risky assets, or whether you delegate the task to a professional who will charge a fee, construction of an active portfolio is more expensive than construction of a passive one. The passive portfolio requires only small commissions on purchases of T-bills (or zero commissions if you purchase bills directly from the government) and management fees to a mutual fund company that offers a market index fund to the public. First Canadian's Equity Index Fund, for example, mimics

Table 6.8

Annual Rates of Return for Common Stock and Three-Month T-Bills, Standard Deviations, and Sharpe Ratios of Stock Risk Premiums over Time

	Stocks Mean	T-Bills Mean	Risk Premiums Mean	St. Dev.	Sharpe Ratio
1957–1972	9.68	4.37	5.31	16.24	0.33
1973–1988	12.44	10.26	2.18	18.57	0.12
1989–2006	11.25	5.73	5.52	16.38	0.34
1957–2006	11.13	6.74	4.38	16.78	0.26

Source: Modified from Scotia Capital Inc., *Debt Market Indices*, various years.

[4]For a discussion of value-weighted Canadian stock portfolios in asset allocation, see Paul Potvin, "Passive Management, the TSE 300 and the Toronto 35 Stock Indexes," *Canadian Investment Review* 5, no. 1 (Spring 1992).

Criticisms of Indexing Don't Hold Up

Amid the stock market's recent travails, critics are once again taking aim at index funds. But like the firing squad that stands in a circle, they aren't making a whole lot of sense.

Indexing, of course, has never been popular in some quarters. Performance-hungry investors loathe the idea of buying index funds and abandoning all chance of beating the market averages. Meanwhile, most Wall Street firms would love indexing to fall from favor because there isn't much money to be made running index funds.

But the latest barrage of nonsense also reflects today's peculiar stock market. Here is a look at four recent complaints about index funds:

They're undiversified. Critics charge that the most popular index funds, those that track the Standard & Poor's 500-stock index, are too focused on a small number of stocks and a single sector, technology.

S&P 500 funds currently have 25.3% of their money in their 10-largest stockholdings and 31.1% of assets in technology companies. This narrow focus made S&P 500 funds especially vulnerable during this year's market swoon.

But the same complaint could be leveled at actively managed funds. According to Chicago researchers Morningstar Inc., diversified U.S. stock funds have an average 36.2% invested in their 10-largest stocks, with 29.1% in technology.

They're top-heavy. Critics also charge that S&P 500 funds represent a big bet on big-company stocks. True enough. I have often argued that most folks would be better off indexing the Wilshire 5000, which includes most regularly traded U.S. stocks, including both large and small companies.

But let's not get carried away. The S&P 500 isn't that narrowly focused. After all, it represents some 77.2% of U.S. stock-market value.

Whether you index the S&P 500 or the Wilshire 5000, what you are getting is a fund that pretty much mirrors the U.S. market. If you think index funds are undiversified and top-heavy, there can only be one reason: The market is undiversified and top heavy.

They're chasing performance. In the 1990s, the stock market's return was driven by a relatively small number of sizzling performers. As these hot stocks climbed in value, index funds became more heavily invested in these companies, while lightening up on lackluster performers.

That, complain critics, is the equivalent of buying high and selling low. A devastating criticism? Hardly. This is what all investors do. When Home Depot's stock climbs 5%, investors collectively end up with 5% more money riding on Home Depot's shares.

You can do better. Sure, there is always a chance you will get lucky and beat the market. But don't count on it.

As a group, investors in U.S. stocks can't outperform the market because, collectively, they are the market. In fact, once you figure in investment costs, active investors are destined to lag behind Wilshire 5000-index funds, because these active investors incur far higher investment costs.

But this isn't just a matter of logic. The proof is also in the numbers. Over the past decade, only 28% of U.S. stock funds managed to beat the Wilshire 5000, according to Vanguard.

The problem is, the long-term argument for indexing gets forgotten in the rush to embrace the latest, hottest funds. An indexing strategy will beat most funds in most years. But in any given year, there will always be some funds that do better than the index. These winners garner heaps of publicity, which whets investors' appetites and encourages them to try their luck at beating the market.

Source: Jonathan Clements, "Criticisms of Indexing Don't Hold Up," *The Wall Street Journal*, April 25, 2000. Reprinted by permission of The Wall Street Journal,

the S&P/TSX Composite index. It purchases shares of the firms constituting the Composite in proportion to the market values of the outstanding equity of each firm, and therefore essentially replicates it. The fund thus duplicates its performance. It has low operating expenses (as a percentage of assets) when compared to other mutual stock funds precisely because it requires minimal managerial effort.

A second reason supporting a passive strategy is the free-rider benefit. If we assume there are many active, knowledgeable investors who quickly bid up prices of undervalued assets and bid down overvalued assets (by selling), we have to conclude that at any time most assets will be fairly priced. Therefore, a well-diversified portfolio of common stock will be a reasonably fair buy, and the passive strategy may not be inferior to that of the average active investor. (We will explain this assumption and provide a more comprehensive analysis of the relative success of passive strategies in later chapters.) See also the box here.

To summarize, however, a passive strategy involves investment in two passive portfolios: virtually risk-free, short-term T-bills (or, alternatively, a money market fund), and a fund of common stocks that mimics a broad market index. The capital allocation line representing such a strategy is called the *capital market line*. Historically, based on data from 1957 to 2006, the passive risky portfolio offered an average risk premium of 4.38 percent and a standard deviation of 16.78 percent, resulting in a reward-to-variability ratio of .26. Passive investors allocate their investment budgets among instruments according to their degree of risk aversion.[5]

CC 8

Concept Check

Suppose that expectations about the S&P/TSX Composite index and the T-bill rate are the same as they were in 2006, but you find that today a greater proportion is invested in T-bills than in 2006. What can you conclude about the change in risk tolerance over the years since 2006?

SUMMARY

1. Speculation is the undertaking of a risky investment for its risk premium. The risk premium has to be large enough to compensate a risk-averse investor for the risk of the investment.
2. A fair game is a risky prospect that has a zero risk premium. It will not be undertaken by a risk-averse investor.
3. Investors' preferences toward the expected return and volatility of a portfolio may be expressed by a utility function that is higher for higher expected returns and lower for higher portfolio variances. More risk-averse investors will apply greater penalties for risk. We can describe these preferences graphically using indifference curves.
4. The desirability of a risky portfolio to a risk-averse investor may be summarized by the certainty equivalent value of the portfolio. The certainty equivalent rate of return is a value that, if it is received with certainty, would yield the same utility as the risky portfolio.
5. Shifting funds from the risky portfolio to the risk-free asset is the simplest way to reduce risk. Other methods involve diversification of the risky portfolio and hedging. We take up these methods in later chapters.
6. T-bills provide a perfectly risk-free asset in nominal terms only. Nevertheless, the standard deviation of real rates on short-term T-bills is small compared to that of other assets such as long-term bonds and common stocks, so for the purpose of our analysis we consider T-bills as the risk-free asset. Money market funds hold, in addition to T-bills, short-term relatively safe obligations such as CP and CDs. These entail some default risk, but again, the additional risk is small relative to most other risky assets. For convenience, we often refer to money market funds as risk-free assets.
7. An investor's risky portfolio (the risky asset) can be characterized by its reward-to-variability ratio, $S = [E(r_P) - r_f]/\sigma_P$. This ratio is also the slope of the CAL, the line that, when graphed, goes from the risk-free asset through the risky asset. All combinations of the risky asset and the risk-free asset lie on this line. Other things equal, an investor would prefer a steeper-sloping CAL, because that means higher expected return for any level of risk. If the borrowing rate is greater than the lending rate, the CAL will be "kinked" at the point of the risky asset.
8. The investor's degree of risk aversion is characterized by the slope of his or her indifference curve. Indifference curves show, at any level of expected return and risk, the required risk premium for taking on one additional percentage point of standard deviation. More risk-averse investors have steeper indifference curves; that is, they require a greater risk premium for taking on more risk.
9. The optimal position, y^*, in the risky asset, is proportional to the risk premium and inversely proportional to the variance and degree of risk aversion:
$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$
Graphically, this portfolio represents the point at which the indifference curve is tangent to the CAL.
10. A passive investment strategy disregards security analysis, targeting instead the risk-free asset and a broad portfolio of risky assets such as the S&P/TSX Composite stock portfolio.

[5]Several studies that take into account the full range of available assets place the degree of risk aversion for the representative investor in the range of 2.0 to 4.0. See, for example, I. Friend and M. Blume, "The Demand for Risky Assets," *American Economic Review* 65 (1975), pp. 900–922.

KEY TERMS

risk premium 172
fair game 173
utility 173
certainty equivalent rate 176
risk-neutral 176
risk lover 176
mean-variance (M-V) criterion 177
indifference curve 177
complete portfolio 181
capital allocation line (CAL) 185
reward-to-variability ratio 186
passive strategy 192
capital market line (CML) 193

SELECTED READINGS

The classic article describing the asset allocation choice, whereby investors choose the optimal fraction of their wealth to place in risk-free assets, is:

Tobin, James. "Liquidity Preference as Behavior Towards Risk." *Review of Economic Studies* 25 (February 1958).

Practitioner-oriented approaches to asset allocation may be found in:

Maginn, John L., and Donald L. Tuttle. *Managing Investment Portfolios: A Dynamic Process*, 2nd ed. New York: Warren, Gorham, & Lamont, Inc., 1990.

Similar practitioner-oriented Canadian contributions include:

Auger, Robert, and Denis Parisien. "Understanding Asset Allocation." *Canadian Investment Review* 4, no. 1 (Spring 1991).

Potvin, Paul. "Passive Management, the TSE 300, and the Toronto 35 Stock Indexes." *Canadian Investment Review* 5, no. 1 (Spring 1992).

Two frequently cited papers on the impact of diversification on portfolio risk are:

Evans, John L., and Stephen H. Archer. "Diversification and the Reduction of Dispersion: An Empirical Analysis." *Journal of Finance*, December 1968.

Wagner, W. H., and S. C. Lau. "The Effect of Diversification on Risk." *Financial Analysts Journal*, November/December 1971.

The seminal works on portfolio selection are:

Markowitz, Harry M. "Portfolio Selection." *Journal of Finance*, March 1952.

Markowitz, Harry M. *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley & Sons, Inc., 1959.

Also see:

Samuelson, Paul A. "Risk & Uncertainty: A Fallacy of Large Numbers." *Scientia* 98 (1963).

PROBLEMS

1. Consider a risky portfolio. The end-of-year cash flow derived from the portfolio will be either \$70,000 or \$200,000 with equal probabilities of .5. The alternative risk-free investment in T-bills pays 6 percent per year.

a. If you require a risk premium of 8 percent, how much will you be willing to pay for the portfolio?

b. Suppose that the portfolio can be purchased for the amount you found in (*a*). What will be the expected rate of return on the portfolio?

c. Now suppose that you require a risk premium of 12 percent. What is the price that you will be willing to pay?

d. Comparing your answers to (*a*) and (*c*), what do you conclude about the relationship between the required risk premium on a portfolio and the price at which the portfolio will sell?

2. Consider a portfolio that offers an expected rate of return of 12 percent and a standard deviation of 18 percent. T-bills offer a risk-free 7 percent rate of return. What is the maximum level of risk aversion for which the risky portfolio is still preferred to bills?

3. Draw the indifference curve in the expected return–standard deviation plane corresponding to a utility level of .05 for an investor with a risk aversion coefficient of 3. *Hint:* Choose several possible standard deviations, ranging from .05 to .25, and find the expected rates of return providing a utility level of .05. Then plot the expected return–standard deviation points so derived.

4. Now draw the indifference curve corresponding to a utility level of .04 for an investor with risk aversion coefficient $A = 4$. Comparing your answers to problems 2 and 3, what do you conclude?

5. Draw an indifference curve for a risk-neutral investor providing a utility level of .05.

6. What must be true about the sign of the risk aversion coefficient, A, for a risk lover? Draw the indifference curve for a utility level of .05 for a risk lover.

Use the following data in answering problems 7, 8, and 9.

Utility Formula Data		
Investment	Expected Return $E(r)$	Standard Deviation (σ)
1	12%	30%
2	15	50
3	21	16
4	24	21

$$U = E(r) - \frac{1}{2}A\sigma^2 \text{ where } A = 4.0$$

7. On the basis of the utility formula above, which investment would you select if you were risk-averse?

8. On the basis of the utility formula above, which investment would you select if you were risk-neutral?

9. The variable A in the utility formula represents the
   a. Investor's return requirement
   b. Investor's aversion to risk
   c. Certainty-equivalent rate of the portfolio
   d. Preference for one unit of return per four units of risk

Consider the historical data of Table 5.2, showing that the average annual rate of return on the S&P/TSX Composite portfolio over the past 50 years has averaged about 4.38 percent more than the Treasury bill return and that the Composite standard deviation has been about 16.12 percent per year. Assume that these values are representative of investors' expectations for future performance and that the current T-bill rate is 5 percent. Use these values to answer problems 10 to 12.

10. Calculate the expected return and standard deviation of portfolios invested in T-bills and the Composite index with weights as follows:

W_{bills}	W_{market}
0	1.0
0.2	0.8
0.4	0.6
0.6	0.4
0.8	0.2
1.0	0

11. Calculate the utility levels of each portfolio of problem 10 for an investor with $A = 3$. What do you conclude?

12. Repeat problem 11 for an investor with $A = 5$. What do you conclude?

You manage a risky portfolio with an expected rate of return of 18 percent and a standard deviation of 28 percent. The T-bill rate is 8 percent. Use these data for problems 13–22.

13. Your client chooses to invest 70 percent of a portfolio in your fund and 30 percent in a T-bill money market fund. What is the expected value and standard deviation of the rate of return on your client's portfolio?

14. Suppose that your risky portfolio includes the following investments in the given proportions:

    Stock *A:* 27 percent
    Stock *B:* 33 percent
    Stock *C:* 40 percent

    What are the investment proportions of your client's overall portfolio, including the position in T-bills?

15. What is the reward-to-variability ratio (S) of your risky portfolio? Your client's?

16. Draw the CAL of your portfolio on an expected return–standard deviation diagram. What is the slope of the CAL? Show the position of your client on your fund's CAL.

17. Suppose that your client decides to invest in your portfolio a proportion y of the total investment budget so that the overall portfolio will have an expected rate of return of 16 percent.
    a. What is the proportion y?
    b. What are your client's investment proportions in your three stocks and the T-bill fund?
    c. What is the standard deviation of the rate of return on your client's portfolio?

18. Suppose that your client prefers to invest in your fund a proportion y that maximizes the expected return on the overall portfolio subject to the constraint that the overall portfolio's standard deviation will not exceed 18 percent.
    a. What is the investment proportion (y)?
    b. What is the expected rate of return on the overall portfolio?

19. Your client's degree of risk aversion is $A = 3.5$.

a. What proportion (y) of the total investment should be invested in your fund?

b. What is the expected value and standard deviation of the rate of return on your client's optimized portfolio?

You estimate that a passive portfolio (i.e., one invested in a risky portfolio that mimics the S&P/TSX Composite index) yields an expected rate of return of 13 percent with a standard deviation of 25 percent. Continue to assume that $r_f = 8$ percent.

20. Draw the CML and your fund's CAL on an expected return–standard deviation diagram.

a. What is the slope of the CML?

b. Characterize in one short paragraph the advantage(s) of your fund over the passive fund.

21. Your client ponders whether to switch the 70 percent that is invested in your fund to the passive portfolio.

a. Explain to your client the disadvantage(s) of the switch.

b. Show your client the maximum fee you could charge (as a percentage of the investment in your fund deducted at the end of the year) that would still leave him or her at least as well off investing in your fund as in the passive one. (*Hint*: The fee will lower the slope of your client's CAL by reducing the expected return net of the fee.)

22. Consider the client in problem 19 with $A = 3.5$.

a. If the client chose to invest in the passive portfolio, what proportion (y) would be selected?

b. Is the fee (percentage of the investment in your fund, deducted at the end of the year) that you can charge to make the client indifferent between your fund and the passive strategy affected by her capital allocation decision?

Problems 23–26 are based on the following assumptions. Suppose that the lending rate is $r_f = 5$ percent, while the borrowing rate that your client faces is 9 percent. Continue to assume that the S&P/TSX Composite index has an expected return of 13 percent and a standard deviation of 25 percent. Your fund here has $r_p = 11$ percent and $\sigma_p = 15$ percent.

23. Draw a diagram of the CML your client faces with the borrowing constraints. Superimpose on it two sets of indifference curves, one for a client who will choose to borrow, and one for a client who will invest in both the index fund and a money market fund.

24. What is the range of risk aversion for which the client will neither borrow nor lend, that is, for which $y = 1$?

25. Solve problems 23 and 24 for a client who uses your fund rather than an index fund.

26. Amend your solution to problem 22(*b*) for clients in the risk-aversion range that you found in problem 24.

27. Look at the data in Table 6.8 regarding the average risk premium of the S&P/TSX Composite over T-bills and the standard deviation of that risk premium. Suppose that the S&P/TSX Composite is your risky portfolio.

a. If your risk-aversion coefficient is 2 and you believe that the entire 1957–2006 period is representative of future expected performance, what fraction of your portfolio should be allocated to T-bills and what fraction to equity?

b. What if you believe that the most recent subperiod period is representative?

c. What do you conclude upon comparing your answers to (*a*) and (*b*)?

28. What do you think would happen to the expected return on stocks if investors perceived higher volatility in the equity market? Relate your answer to equation 6.12.

29. You manage an equity fund with an expected risk premium of 10 percent and an expected standard deviation of 14 percent. The rate on Treasury bills is 6 percent. Your client chooses to invest \$60,000 of her portfolio in your equity fund and \$40,000 in a T-bill money market fund. What is the expected return and standard deviation of return on your client's portfolio?

30. What is the reward-to-variability ratio for the *equity fund* in problem 29?

31. Given \$100,000 to invest, what is the expected risk premium in dollars of investing in equities versus risk-free T-bills based on the following table?

Action	Probability	Expected Return
Invest in equities	.6	50,000
	.4	−\$30,000
Invest in risk-free T-bills	1.0	\$ 5,000

32. The change from a straight to a kinked capital allocation line is a result of the

a. Reward-to-variability ratio increasing

b. Borrowing rate exceeding the lending rate

c. Investor's risk tolerance decreasing

d. Increase in the portfolio proportion of the risk-free asset

STANDARD &POOR'S

1. Go to **www.mcgrawhill.ca/edumarketinsight** (have you remembered to bookmark this page?) and link to *Company*, then *Population*. Select a company of interest to you and link to the *Company Research* page. Observe the menu of company information reports on the left. Link to the *Recent News* and review the most recent *Business Wire* articles. What recent event or information release had an apparent impact upon your company's stock price? (Review the *Key Items Chart* under the *Excel Analytics, Daily Adjusted Prices.*)
2. Go to **www.mcgrawhill.ca/edumarketinsight** and link to *Industry*. From the pull-down menu, link to an industry that is of interest to you. From the menu on the left side, select the *S&P 500* report under *Industry GICS Sub-Industry Financial Highlights*. How many companies from this industry are in the S&P 500? What percentage of the Main Industry Group does this Industry Group represent in the S&P 500? Look at the ratios provided for the industry and their comparisons to the *GICS Sub-Industry Benchmarks*. How did the industry perform relative to S&P 500 companies during the last year?

E-INVESTMENTS
Market Crashes

See the **http://mutualfunds.about.com/cs/history/a/marketcrash.htm** site to get a historical perspective on the market crashes that have happened in the United States since 1900. When did the crashes occur? How long did each of the crashes last? How much value was lost in each crash?

What has the general trend of the market been since 1900? How do you feel about investing in the stock market knowing that you might experience one or more market crashes during your investing lifetime?

APPENDIX 6A: RISK AVERSION AND EXPECTED UTILITY

We digress here to examine the rationale behind our contention that investors are risk-averse. Recognition of risk aversion as central in investment decisions goes back at least to 1738. Daniel Bernoulli, one of a famous Swiss family of distinguished mathematicians, spent the years 1725 through 1733 in St. Petersburg, where he analyzed the following coin-toss game. To enter the game one pays an entry fee. Thereafter, a coin is tossed until the *first* head appears. The number of tails, denoted by n, that appear until the first head is tossed is used to compute the payoff, $\$R$, to the participant, as

$$R(n) = 2^n$$

The probability of no tails before the first head ($n = 0$) is ½ and the corresponding payoff is $2^0 = \$1$. The probability of one tail and then heads ($n = 1$) is ½ × ½ with payoff $2^1 = \$2$, the probability of two tails and then heads ($n = 2$) is ½ × ½ × ½, and so forth.

The following table illustrates the probabilities and payoffs for various outcomes:

Tails	Probability	Payoff = $\$R(n)$	Probability × Payoff
0	$\frac{1}{2}$	\$1	\$1/2
1	$\frac{1}{4}$	\$2	\$1/2
2	$\frac{1}{8}$	\$4	\$1/2
3	$\frac{1}{16}$	\$8	\$1/2
—	—	—	—
—	—	—	—
n	$(\frac{1}{2})^{n+1}$	$\$2^n$	\$1/2

The expected payoff is therefore

$$\begin{aligned} E(R) &= \sum_{n=0}^{\infty} Pr(n)R(n) \\ &= \tfrac{1}{2} + \tfrac{1}{2} + \ldots \\ &= \infty \end{aligned}$$

This game is called the "St. Petersburg Paradox." Although the expected payoff is infinite, participants obviously will be willing to purchase tickets to play the game only at a finite, and possibly quite modest, entry fee.

Bernoulli resolved the paradox by noting that investors do not assign the same value per dollar to all payoffs. Specifically, the greater their wealth, the less their "appreciation" for each extra dollar. We can make this insight mathematically precise by assigning a welfare or utility value to any level of investor wealth. Our utility function should increase as wealth is higher, but each extra dollar of wealth should increase utility by progressively smaller amounts.[6] (Modern economists would say that investors exhibit "decreasing marginal utility" from an additional payoff dollar.) One particular function that assigns a subjective value to the investor from a payoff of $\$R$, which has a smaller value per dollar the greater the payoff, is the function $\log(R)$. If this function measures utility values of wealth, the subjective utility value of the game is indeed finite.[7] The certain wealth level necessary to yield this utility value is \$2, because log (2.00) = .693. Hence the certainty equivalent value of the risky payoff is \$2, which is the maximum amount that this investor will pay to play the game.

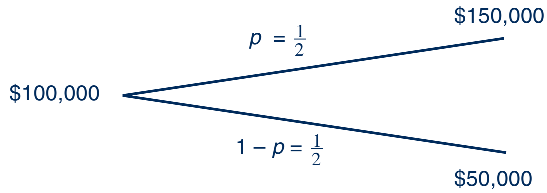
Von Neumann and Morgenstern adapted this approach to investment theory in a complete axiomatic system in 1946. Avoiding unnecessary technical detail, we restrict ourselves here to an intuitive exposition of the rationale for risk aversion.

Imagine two individuals who are identical twins, except that one of them is less fortunate than the other. Peter has only \$1,000 to his name while Paul has a net worth of \$200,000. How many hours of work would each twin be willing to offer to earn one extra dollar? It is likely that Peter (the poor twin) has more essential uses for the extra money than does Paul. Therefore, Peter will offer more hours. In other words, Peter derives a greater personal welfare or assigns a greater "utility" value to the 1,001st dollar than Paul does to the 200,001st.

Figure 6A.1 depicts graphically the relationship between wealth and the utility value of wealth that is consistent with this notion of decreasing marginal utility.

Individuals have different rates of decrease in their marginal utility of wealth. What is constant is the *principle* that per-dollar utility decreases with wealth. Functions that exhibit the property of decreasing per-unit value as the number of units grows are called *concave*. A simple example is the log function, familiar from high school mathematics. Of course, a log function will not fit all investors, but it is consistent with the risk aversion that we assume for all investors.

Now consider the following simple prospect:



This is a fair game in that the expected profit is zero. Suppose, however, that the curve in Figure 6A.1 represents the investor's utility value of wealth, assuming a log utility function. Figure 6A.2 shows this curve with the numerical values marked.

Figure 6A.2 shows that the loss in utility from losing \$50,000 exceeds the gain from winning \$50,000. Consider the gain first. With probability $p = .5$, wealth goes from \$100,000 to \$150,000. Using the log utility function, utility goes from log(100,000) = 11.51 to log(150,000) = 11.92, the distance G on the graph. This gain is $G = 11.92 - 11.51 = .41$. In expected utility terms, then, the gain is $pG = .5 \times .41 = .21$.

Now consider the possibility of coming up on the short end of the prospect. In that case, wealth goes from \$100,000 to \$50,000. The loss in utility, the distance L on the graph, is $L = \log(100{,}000) - \log(50.000) = 11.51 - 10.82 = .69$. Thus the loss in expected utility terms is $(1 - p)L = .5 \times .69 = .35$, which exceeds the gain in expected utility from the possibility of winning the game.

We compute the expected utility from the risky prospect as follows:

$$\begin{aligned} E[U(W)] &= pU(W_1) + (1 - p)U(W_2) \\ &= \tfrac{1}{2}\log(50{,}000) + \tfrac{1}{2}\log(150{,}000) \\ &= 11.37 \end{aligned}$$

If the prospect is rejected, the utility value of the (sure) \$100,000 is log(100,000) = 11.51, which is greater than that of the fair game (11.37). Hence the risk-averse investor will reject the fair game.

Using a specific investor utility function (such as the log utility) allows us to compute the certainty equivalent

[6] This utility function is similar in spirit to the one that assigns a satisfaction level to portfolios with given risk-and-return attributes. However, the utility function here refers not to investors' satisfaction with alternative portfolio choices but only to the subjective welfare they derive from different levels of wealth.

[7] If we substitute the "utility" value, $\log(R)$, for the dollar payoff, R, to obtain an expected utility value of the game (rather than expected dollar value), we have, calling $V(R)$ the expected utility,

$$V(R) = \sum_{n=0}^{\infty} Pr(n)\log[R(n)] = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} \log(2^n) = .693$$

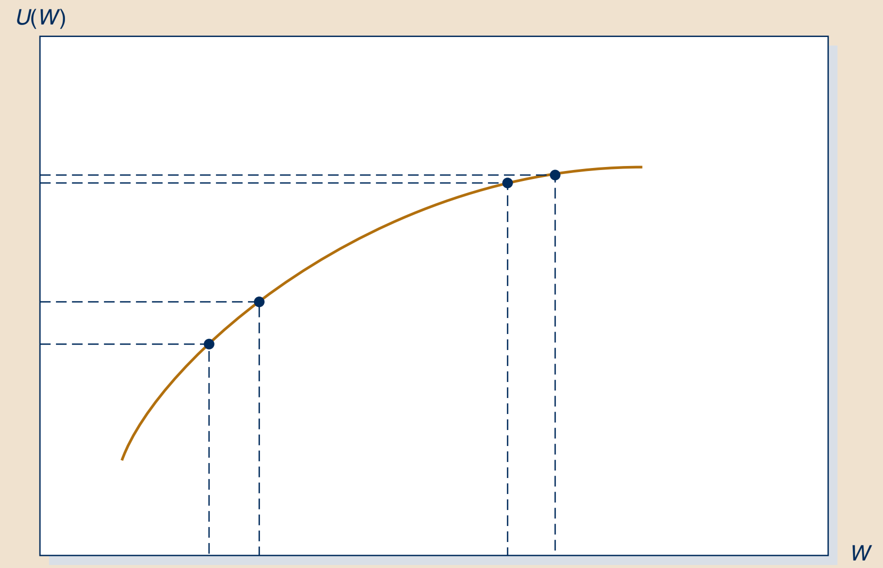


Figure 6A.1

Utility of wealth with a log utility function.

value of the risky prospect to a given investor. This is the amount that, if received with certainty, the investor would consider equally attractive as the risky prospect.

If log utility describes the investor's preferences toward wealth outcomes, then Figure 6A.2 also can tell us what is, for her, the dollar value of the prospect. We ask: What sure level of wealth has a utility value of 11.37 (which equals the expected utility from the prospect)? A horizontal line drawn at the level 11.37 intersects the utility curve at the level of wealth W_{CE}. This means that

$$\log(W_{CE}) = 11.37$$

which implies that

$$\begin{aligned} W_{CE} &= e^{11.37} \\ &= \$86{,}681.86 \end{aligned}$$

W_{CE} is therefore the certainty equivalent of the prospect. The distance Y in Figure 6A.2 is the penalty, or the downward adjustment, to the expected profit that is attributable to the risk of the prospect:

$$\begin{aligned} Y &= E(W) - W_{CE} \\ &= \$100{,}000 - \$86{,}681.86 \\ &= \$13{,}318.14 \end{aligned}$$

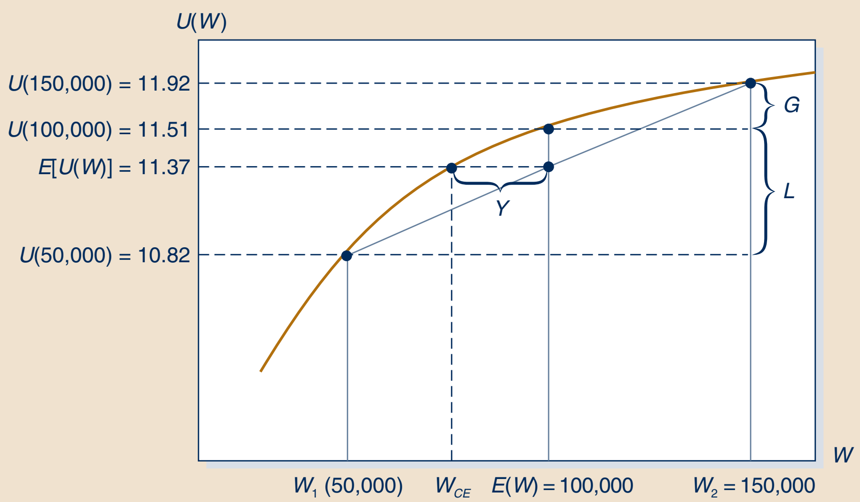


Figure 6A.2

Fair games and expected utility.

The investor views $86,681.86 for certain as being equal in utility value as $100,000 at risk. Therefore, she would be indifferent between the two.

Does revealed behaviour of investors demonstrate risk aversion? Looking at prices and past rates of return in financial markets, we can answer with a resounding yes. With remarkable consistency, riskier bonds are sold at lower prices than safer ones with otherwise similar characteristics. Riskier stocks also have provided higher average rates of return over long periods of time than less risky assets such as T-bills. For example, over the 1957–2006 period, the average rate of return on the S&P/TSX Composite portfolio exceeded the T-bill return by about 4.38 percent per year.

It is abundantly clear from financial data that the average, or representative, investor exhibits substantial risk aversion. For readers who recognize that financial assets are priced to compensate for risk by providing a risk premium and at the same time feel the urge for some gambling, we have a constructive recommendation: Direct your gambling desire to investment in financial markets. As Von Neumann once said, "The stock market is a casino with the odds in your favour." A small risk-seeking investor may provide all the excitement you want with a positive expected return to boot!

CC A1

Concept Check

Suppose the utility function is $U(W) = \sqrt{w}$.

a. What is the utility level at wealth levels $50,000 and $150,000?

b. What is expected utility if p still equals .5?

c. What is the certainty equivalent of the risky prospect?

d. Does this utility function also display risk aversion?

e. Does this utility function display more or less risk aversion than the log utility function?

PROBLEMS

1. Suppose that your wealth is $250,000. You buy a $200,000 house and invest the remainder in a risk-free asset paying an annual interest rate of 6 percent. There is a probability of .001 that your house will burn to the ground and its value will be reduced to zero. With a log utility of end-of-year wealth, how much would you be willing to pay for insurance (at the beginning of the year)? (Assume that if the house does not burn down, its end-of-year value still will be $200,000.)

2. If the cost of insuring your house is $1 per $1,000 of value, what will be the certainty equivalent of your end-of-year wealth if you insure your house at

   a. ½ its value?

   b. Its full value?

   c. 1½ times its value?