Appendix 9B Incorporating Convexity into the Duration Model[1]

In the main body of the chapter, we established these three characteristics of convexity:

1. *Convexity is desirable.* The greater the convexity of a security or a portfolio of securities, the more insurance or interest rate protection an FI manager has against rate increases and the greater the potential gains after interest rate falls.
2. *Convexity and duration.* The larger the interest rate changes and the more convex a fixed-income security or portfolio, the greater the error the FI manager faces in using just duration (and duration matching) to immunize exposure to interest rate shocks.
3. *All fixed-income securities are convex.*[2] To see this, we can take the six-year, 8 percent coupon, 8 percent yield bond and look at two extreme price–yield scenarios. What is the price on the bond if yields falls to zero, and what is its price if yields rise to some very large number?

When $R = 0$:

$$P = \frac{80}{(1+0)} + \ldots + \frac{1{,}080}{(1+0)^6} = 1{,}480$$

The price is just the simple undiscounted sum of the coupon values and the face value. Since yields can never go below zero, \$1,480 is the maximum possible price for the bond.

When R is very large:

$$P \simeq 0$$

As the yield goes to infinity, the bond price falls asymptotically toward zero, but by definition a bond's price can never be negative. Thus, zero must be the minimum bond price (see Figure 9B–1).

Since convexity is a desirable feature for assets, the FI manager might ask: Can we measure convexity? And can we incorporate this measurement into the duration model to adjust for or offset the error in prediction due to its presence? The answer to both questions is yes.

Theoretically speaking, duration is the slope of the price–yield curve, and convexity, or curvature, is the change in the slope of the price–yield curve. Consider the total effect of a change in interest rates on a bond's price as being broken into a number of separate effects. The precise mathematical derivation of these separate effects is based on a Taylor series expansion that you might remember from your math classes. Essentially, the first-order effect (dP/dR) of an interest rate change on the bond's price is the price–yield curve slope effect, which is measured by duration. The second-order effect (d^2P/dR^2) measures the change in the slope of the price–yield curve; this is the curvature, or convexity, effect. There are also third-, fourth-, and higher-order effects from the Taylor series expansion, but for all practical purposes these effects can be ignored.

We have noted that overlooking the curvature of the price–yield curve may cause errors in predicting the interest sensitivity of a portfolio of assets and liabilities, especially when yields change by large amounts. We can adjust for this by explicitly recognizing the second-order effect of yield changes by measuring the change in the slope of the price–yield curve around a given point. Just as D (duration) measures the slope effect (dP/dR), we introduce a new parameter (CX) to measure the curvature effect (d^2P/dR^2) of the price–yield curve.

The resulting equation, predicting the change in a security's price ($\Delta P/P$), is

$$\frac{\Delta P}{P} = -D\frac{\Delta R}{(1+R)} + \frac{1}{2}CX(\Delta R)^2 \qquad (9B.1)$$

or

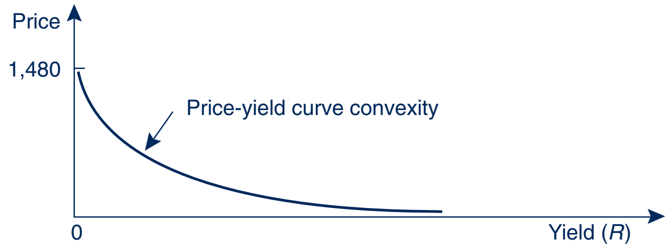
$$\frac{\Delta P}{P} = -MD\Delta R + \frac{1}{2}CX(\Delta R)^2 \qquad (9B.2)$$

The first term in equation 9B.1 is the simple-duration model that over- or underpredicts price changes for large changes in interest rates, and the second term is the second-order effect of interest rate changes, that is, the convexity or curvature adjustment. In equation 9B.1, the first term D can be divided by $1 + R$ to produce what we called earlier *modified duration* (MD). You can see

[1] This section contains more technical details, which may be included or dropped from the chapter reading depending on the rigour of the course.

[2] This applies to fixed-income securities without special option features such as calls and puts.

FIGURE 9B–1
The Natural Convexity of Bonds



this in equation 9B.2. This form is more intuitive because we multiply MD by the simple change in R (ΔR) rather than by the discounted change in R [$\Delta R/(1 + R)$]. In the convexity term, the number 1/2 and $(\Delta R)^2$ result from the fact that the convexity effect is the second-order effect of interest rate changes while duration is the first-order effect. The parameter CX reflects the degree of curvature in the price–yield curve at the current yield level, that is, the degree to which the *capital gain effect* exceeds the *capital loss effect* for an equal change in yields up or down. At best, the FI manager can only approximate the curvature effect by using a parametric measure of CX. Even though calculus is based on infinitesimally small changes, in financial markets the smallest change in yields normally observed is one basis point, or a 1/100th of 1 percent change. One possible way to measure CX is introduced next.

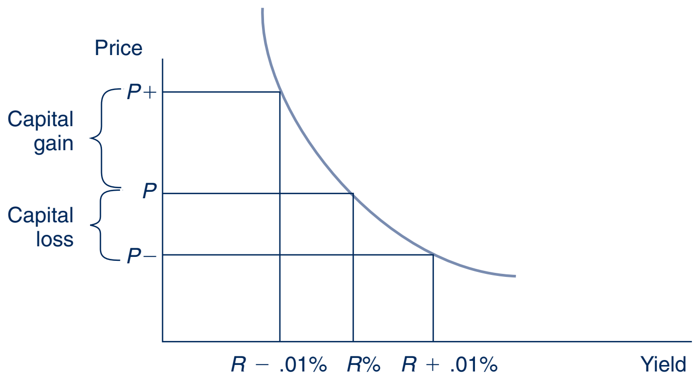
As just discussed, the convexity effect is the degree to which the capital gain effect more than offsets the capital loss effect for an equal increase and decrease in interest rates at the current interest rate level. In Figure 9B–2 we depict yields changing upward by one basis point (R + 0.01%) and downward by one basis point (R − 0.01%). Because convexity measures the curvature of the price–yield curve around the rate level R percent, it intuitively measures the degree to which the capital gain effect of a small yield decrease exceeds the capital loss effect of a small yield increase.[3] Definitionally, the CX parameter equals

$$CX = \begin{matrix}\text{Scaling}\\ \text{factor}\end{matrix}\left[\begin{matrix}\text{Capital}\\ \text{loss from a}\\ \text{one-basis-point}\\ \text{rise in yield}\\ \text{(negative effect)}\end{matrix} \quad + \quad \begin{matrix}\text{Capital}\\ \text{gain from a}\\ \text{one-basis-point}\\ \text{fall in yield}\\ \text{(positive effect)}\end{matrix}\right]$$

The sum of the two terms in the brackets reflects the degree to which the capital gain effect exceeds the capital loss effect for a small one-basis-point interest rate change down and up. The scaling factor normalizes this measure to account for a larger 1 percent change in rates. Remember, when interest rates change by a large amount, the convexity effect is important to measure. A commonly used scaling factor is 10^8 so that:[4]

$$CX = 10^8\left[\frac{\Delta P-}{P} + \frac{\Delta P+}{P}\right]$$

FIGURE 9B–2
Convexity and the Price–Yield Curve



[3] We are trying to approximate as best we can the change in the slope of the price–yield curve at R percent. In theory, the changes are infinitesimally small (dR), but in reality, the smallest yield change normally observed is one basis point (ΔR).

[4] This is consistent with the effect of a 1 percent (100 basis points) change in rates.

Calculation of CX

To calculate the convexity of the 8 percent coupon, 8 percent yield, six-year maturity Eurobond that had a price of $1,000:[5]

$$CX = 10^8\left[\underbrace{\frac{999.53785 - 1{,}000}{1{,}000}}_{\substack{\text{Capital loss from}\\ \text{a one-basis-point}\\ \text{increase in rates}}} + \underbrace{\frac{1{,}000.46243 - 1{,}000}{1{,}000}}_{\substack{\text{Capital gain from}\\ \text{a one-basis-point}\\ \text{decrease in rates}}}\right]$$

$$CX = 10^8[0.00000028]$$

$$CX = 28$$

This value for CX can be inserted into the bond price prediction equation 9B.2 with the convexity adjustment:

$$\frac{\Delta P}{P} = -MD\Delta R + \frac{1}{2}(28)\Delta R^2$$

Assuming a 2 percent increase in R (from 8 to 10 percent),

$$\begin{aligned}\frac{\Delta P}{P} &= -\left[\frac{4.993}{1.08}\right]0.02 + \frac{1}{2}(28)(0.02)^2\\ &= -0.0925 + 0.0056\\ &= -0.0869 \text{ or } -8.69\%\end{aligned}$$

We have adjusted headings here. Please confirm if this fine.

The simple duration model (the first term) predicts that a 2 percent rise in interest rates will cause the bond's price to fall 9.25 percent. However, for large changes in yields, the duration model overpredicts the price fall. The duration model with the second-order convexity adjustment predicts a price fall of 8.69 percent; it adds back 0.56 percent because of the convexity effect. This is much closer to the true fall in the six-year, 8 percent coupon bond's price if we calculated this using 10 percent to discount the coupon and face value cash flows on the bond. The true value of the bond price fall is 8.71 percent. That is, using the convexity adjustment reduces the error between predicted value and true value to just a few basis points.[6]

In Table 9B–1 we calculate various properties of convexity, where

N = Time to maturity
R = Yield to maturity
C = Annual coupon
D = Duration
CX = Convexity

Part 1 of Table 9B–1 shows that as the bond's maturity (N) increases, so does its convexity (CX). As a result, long-term bonds have more convexity—which is a desirable property—than do

TABLE 9B–1 Properties of Convexity

1. Convexity Increases with Bond Maturity			2. Convexity Varies with Coupon		3. For Same Duration, Zero-Coupon Bonds Are Less Convex Than Coupon Bonds	
Example			Example		Example	
A	B	C	A	B	A	B
$N = 6$	$N = 18$	$N = \infty$	$N = 6$	$N = 6$	$N = 6$	$N = 5$
$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$
$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 0\%$	$C = 8\%$	$C = 0\%$
$D = 5$	$D = 10.12$	$D = 13.5$	$D = 5$	$D = 6$	$D = 5$	$D = 5$
$CX = 28$	$CX = 130$	$CX = 312$	$CX = 28$	$CX = 36$	$CX = 28$	$CX = 25.72$

[5] You can easily check that $999.53785 is the price of the six-year bond when rates are 8.01 percent and $1,000.46243 is the price of the bond when rates fall to 7.99 percent. Since we are dealing in small numbers and convexity is sensitive to the number of decimal places assumed, we use at least five decimal places in calculating the capital gain or loss. In fact, the more decimal places used, the greater the accuracy of the *CX* measure.

[6] It is possible to use the third moment of the Taylor series expansion to reduce this small error (8.71 percent versus 8.69 percent) even further. In practice, few people do this.

short-term bonds. This property is similar to that possessed by duration.[7]

Part 2 of Table 9B–1 shows that coupon bonds of the same maturity (N) have less convexity than do zero-coupon bonds. However, for coupon bonds and discount or zero-coupon bonds of the same duration, part 3 of the table shows that the coupon bond has more convexity. We depict the convexity of both in Figure 9B–3.

Finally, before leaving convexity, we might look at one important use of the concept by managers of insurance companies, pension funds, and mutual funds. Remembering that convexity is a desirable form of interest rate risk insurance, FI managers could structure an asset portfolio to maximize its desirable effects. Consider a pension fund manager with a 15-year payout horizon. To immunize the risk of interest rate changes, the manager purchases bonds with a 15-year duration. Consider two alternative strategies to achieve this:

Strategy 1: Invest 100 percent of resources in a 15-year deep-discount bond with an 8 percent yield.

Strategy 2: Invest 50 percent in the very short-term money market and 50 percent in 30-year deep-discount bonds with an 8 percent yield.

The duration (D) and convexities (CX) of these two asset portfolios are

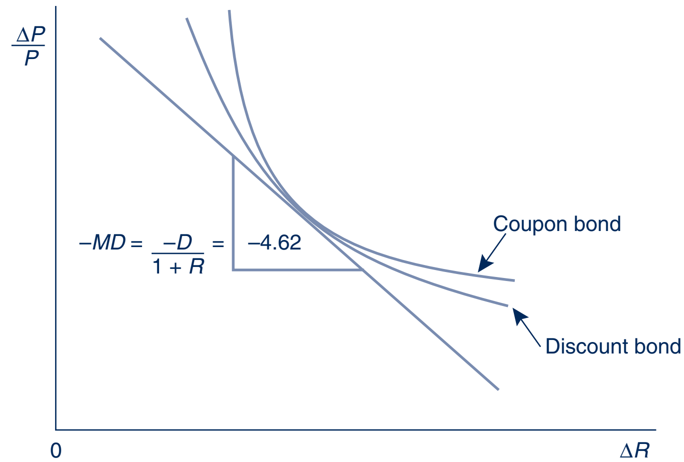
Strategy 1: $D = 15$, $CX = 206$

Strategy 2:[8] $D = ½(0) + ½(30) = 15$,
$CX = ½(0) + ½(797) = 398.5$

Strategies 1 and 2 have the same durations, but strategy 2 has a greater convexity. Strategy 2 is often called a *barbell portfolio*, as shown in Figure 9B–4 by the shaded bars.[9] Strategy 1 is the unshaded bar. To the extent that the market does not price (or fully price) convexity, the barbell strategy dominates the direct duration-matching strategy (strategy 1).[10]

More commonly, an FI manager may seek to attain greater convexity in the asset portfolio than in the liability portfolio, as shown in Figure 9B–5. As a result, both positive and negative shocks to

FIGURE 9B–3
Convexity of a Coupon versus a Discount Bond with the Same Duration



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[7] Note that the CX measure differs according to the level of interest rates. For example, we are measuring CX in Table 9B–1 when yields are 8 percent. If yields were 12 percent, the CX number would change. This is intuitively reasonable, as the curvature of the price–yield curve differs at each point on the price–yield curve. Note that duration also changes with the level of interest rates.

[8] The duration and convexity of one-day funds are approximately zero.

[9] This is called a barbell because the weights are equally loaded at the extreme ends of the duration range, or bar, as in weightlifting.

[10] In a world in which convexity is priced, the long-term 30-year bond's price would rise to reflect the competition among buyers to include this more convex bond in their barbell asset portfolios. Thus, buying bond insurance—in the form of the barbell portfolio—would involve an additional cost to the FI manager. In addition, for the FI to be hedged in both a duration sense and a convexity sense, the manager should not choose the convexity of the asset portfolio without seeking to match it to the convexity of the liability portfolio. For further discussion of the convexity trap that results when an FI mismatches its asset and liability convexities, see J. H. Gilkeson and S. D. Smith, "The Convexity Trap: Pitfalls in Financing Mortgage Portfolios and Related Securities," Federal Reserve Bank of Atlanta, *Economic Review,* November–December 1992, pp. 17–27.

FIGURE 9B–4
Barbell Strategy

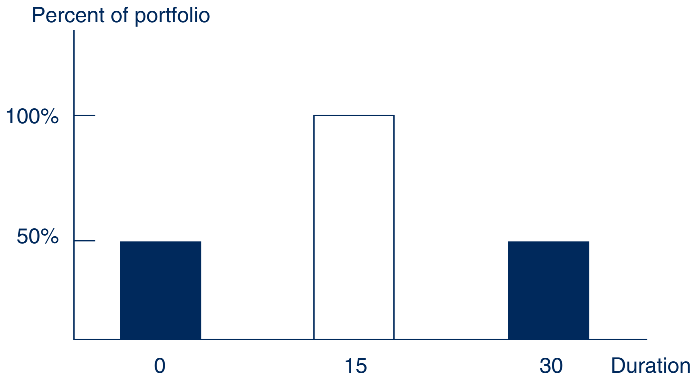
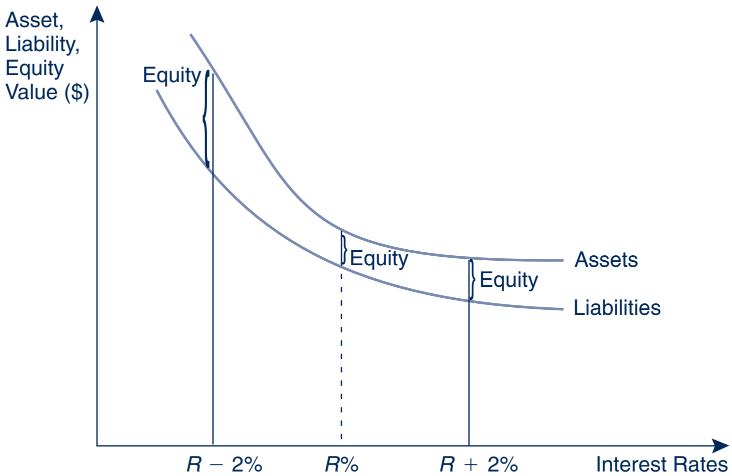


FIGURE 9B–5
Assets Are More Convex Than Liabilities



interest rates would have beneficial effects on the FI's net worth.[11]

THE PROBLEM OF THE FLAT TERM STRUCTURE

We have been calculating simple, or Macaulay, duration, which was named after an economist who was among the first to develop the *duration* concept. A key assumption of the simple duration model is that the yield curve or the term structure of interest rates is flat and that when rates change, the yield curve shifts in a parallel fashion. We show this in Figure 9B–6.

In the real world, the yield curve can take many shapes and at best may only approximate a flat yield curve. If the yield curve is not flat, using simple duration could be a potential source of error in predicting asset and liability interest rate sensitivities. Many models can deal with this problem. These models differ according to the shapes and shocks to the yield curve that are assumed.

Suppose the yield curve is not flat but shifts in such a manner that the yields on different maturity discount bonds change in a proportional fashion.[12] Consider calculating the duration of a six-year Eurobond when the yield curve is not flat at 8 percent. Instead, the yield curve looks like the one in Figure 9B–7.

[11] Another strategy would be for the FI to issue callable bonds as liabilities. Callable bonds have limited upside capital gains because if rates fall to a low level, then the issuer calls the bond in early (and reissues new lower-coupon bonds). The effect of limited upside potential for callable bond prices is that the price–yield curve for such bonds exhibits negative convexity. Thus, if asset investments have positive convexity and liabilities negative convexity, then yield shocks (whether positive or negative) are likely to produce net worth gains for the FI.

[12] We are interested in the yield curve on discount bonds because these yields reflect the time value of money for single payments at different maturity dates. Thus, we can use these yields as discount rates for cash flows on a security to calculate appropriate present values of its cash flows and its duration.

FIGURE 9B–6
Yield Curve Underlying Macaulay Duration

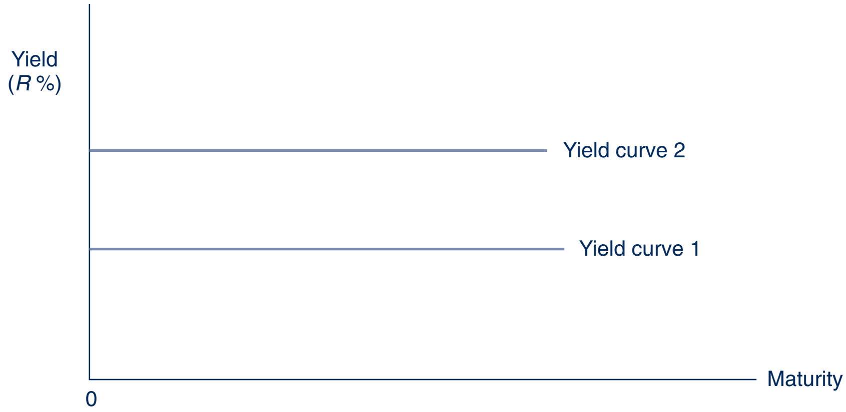
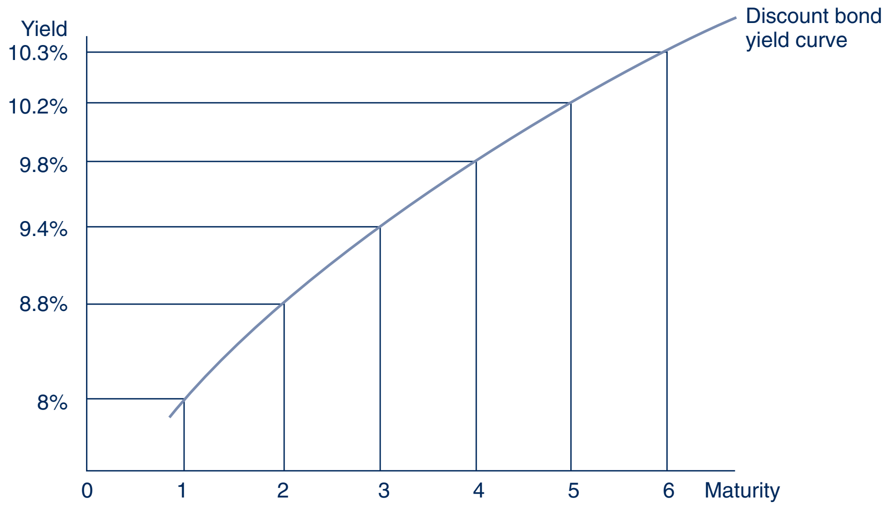


FIGURE 9B–7
Non-Flat Yield Curve



Suppose the yield on one-year discount bonds rises. Assume also that the discounted changes in longer-maturity discount bonds yields are just proportional to the change in the one-year discount bond yield:

$$\frac{\Delta R_1}{1 + R_1} = \frac{\Delta R_2}{1 + R_2} = \cdots = \frac{\Delta R_6}{1 + R_6}$$

Given this quite restrictive assumption, it can be proved that the appropriate duration measure of the bond—call it D^*—can be derived by discounting the coupons and principal value of the bond by the discount rates or yields on appropriate maturity zero-coupon bonds. Given the discount bond yield curve plotted in Figure 9B–7, D^* is calculated in Table 9B–2.[13]

Notice that D^* is 4.916 years, while simple Macaulay duration (with an assumed flat 8 percent yield curve) is 4.993 years. D^* and D differ because, by taking into account the upward-sloping yield curve in Figure 9B–7, the later cash flows are discounted at higher rates than they are under the flat yield curve assumption underlying Macaulay's measure D.

With respect to the FI manager's problem, choosing to use D^* instead of D does not change the basic problem except for a concern with the gap between the D^* on assets and leverage-weighted liabilities:

$$D_A^* - kD_L^*$$

[13] For more details, see G. Hawawini, "Controlling the Interest-Rate Risk of Bonds," *Financial Markets and Portfolio Management,* 1(4), 1987, pp. 8–18 and G. O. Bierwag, G. G. Kaufman, and A. Toevs, "Duration: Its Development and Use in Bond Portfolio Management," *Financial Analysts Journal,* 39, 1983, pp. 15–35.

TABLE 9B–2
Duration with an Upward-Sloping Yield Curve

t	*CF*	*DF*	*CF* × *DF*	*CF* × *DF* × *t*
1	80	$\frac{1}{(1.08)} = 0.9259$	74.07	74.07
2	80	$\frac{1}{(1.088)^2} = 0.8448$	67.58	135.16
3	80	$\frac{1}{(1.094)^3} = 0.7637$	61.10	183.30
4	80	$\frac{1}{(1.098)^4} = 0.6880$	55.04	220.16
5	80	$\frac{1}{(1.102)^5} = 0.6153$	49.22	246.10
6	1,080	$\frac{1}{(1.103)^6} = 0.5553$	599.75	3,598.50
			906.76	4,457.29

$$D^* = \frac{4{,}457.29}{906.76} = 4.91562$$

However, remember that the D^* was calculated under very restrictive assumptions about the yield curve. If we change these assumptions in any way, the measure of D^* changes.[14]

THE PROBLEM OF DEFAULT RISK

The models and the duration calculations we have looked at assume that the issuer of bonds or the borrower of a loan pays the promised interest and principal with a probability of 1; we assume no default or delay in the payment of cash flows. In the real world, problems with principal and interest payments are common and lead to restructuring and workouts on debt contracts as bankers and bond trustees renegotiate with borrowers; that is, the borrower reschedules or recontracts interest and principal payments rather than defaulting outright. If we view default risk as synonymous with the rescheduling of cash flows to a later date, this is quite easy to deal with in duration models.

Consider the six-year, 8 percent coupon, 8 percent yield Eurobond. Suppose the issuer gets into difficulty and cannot pay the first coupon. Instead, the borrower and the FI agree that the unpaid interest can be paid in year 2. This alleviates part of the cash flow pressure on the borrower while lengthening the duration of the bond from the FI's perspective (see Table 9B–3). The effect of rescheduling the first interest payment is to increase duration from approximately 5 years to 5.08 years.

More commonly, an FI manager unsure of the future cash flows because of future default risk might multiply the promised cash flow (CF_t) by the probability of repayment (p_t) in year t to generate expected cash flows in year t—$E(CF_t)$.[15]

$$E(CF_t) = p_t \times CF_t$$

[14] A number of authors have identified other non-standard measures of duration for more complex yield curve shapes and shifts. See, for example, Bierwag, Kaufman, and Toevs, "Duration: Its Development and Use." *Financial Analysts Journal,* 39, 1983, pp. 15–35. See also I. J. Fooladi and G. S. Roberts, "Macrohedging for Financial Institutions: Beyond Duration," *Journal of Applied Finance,* Spring 2004, 14(1), pp. 11–19, for a discussion of adjustments for convexity as well as adjustments for the default risk of FI assets. K. C. Ahlgrim, S. P. D'Arcy, and R. W. Gorvett, "The Effective Duration and Convexity of Liabilities for Property-Liability Insurers Under Stochastic Interest Rates, *Geneva Papers on Risk and Insurance Theory,* 29, 2004, pp. 75–108 provides modelling techniques applicable to property and casualty insurers.

[15] The probability of repayment is between 0 and 1.

TABLE 9B–3
Duration and Rescheduling

t	CF	DF	$CF \times DF$	$CF \times DF \times t$
1	0	0.9259	0	0
2	160	0.8573	137.17	274.34
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.21
5	80	0.6806	54.45	272.25
6	1,080	0.6302	680.58	4,083.48
			994.51	5,055.81

$$D = \frac{5{,}055.81}{994.51} = 5.0837 \text{ years}$$

Chapter 11 suggests a number of ways to generate these repayment probabilities. Once the cash flows have been adjusted for default risk, a duration measure can be directly calculated in the same manner as the Macaulay formula (or D^*) except that $E(CF_t)$ replaces CF_t.[16]

FLOATING-RATE LOANS AND BONDS

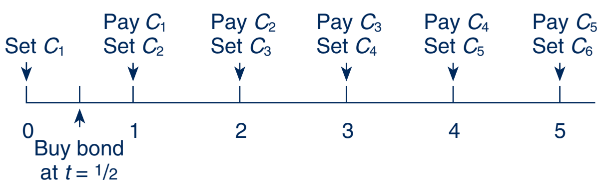
The duration models we have looked at assume that the interest rates on loans or the coupons on bonds are fixed at issue and remain unchanged until maturity. However, many bonds and loans carry floating interest rates. Examples include loan rates indexed to LIBOR (London Interbank Offered Rate) and variable rate mortgages (VRMs) whose rates can be indexed to government bonds or other securities yields. Moreover, in the 1980s, many banks and security firms either issued or underwrote perpetual floating-rate notes (FRNs). These are like consol bonds in that they never mature; unlike consols, their coupons fluctuate with market rates. The FI manager, who wants to analyze overall gap exposure, may ask: What are the durations of such floating-rate securities? The duration of a floating-rate instrument is generally the time interval between the purchase of the security and the time when the next coupon or interest payment is readjusted to reflect current interest rate conditions. We call this the time to repricing of the instrument.

For example, suppose the investor bought a perpetual floating-rate note. These floating-rate notes never mature. At the beginning of each year, the FI sets the coupon rate, which is paid at the end of that year. Suppose the investor buys the bond in the middle of the first year $(t = \frac{1}{2})$ rather than at the beginning (see Figure 9B–8).

The present value of the bond from time of purchase is[17]

$$P = \frac{C_1}{(1 + \frac{1}{2}R)} + \frac{C_2}{(1 + \frac{1}{2}R)(1 + R)} + \frac{C_3}{(1 + \frac{1}{2}R)(1 + R)^2} + \frac{C_2}{(1 + \frac{1}{2}R)(1 + R)^3} + \frac{C_5}{(1 + \frac{1}{2}R)(1 + R)^4} + \cdots$$

FIGURE 9B–8
Floating-Rate Note



[16] Alternatively, the promised cash flow could be discounted by the appropriate discount yield on a risk-free government security plus an appropriate credit risk spread; that is, $CF_t /(1 + d_t + S_t)t$, where CF_t is the promised cash flow in year t, d_t is the yield on a t-period zero-coupon government bond, and S_t is a credit risk premium.

[17] This formula follows the Eurobond convention that any cash flows received in less than one full coupon period's time are discounted using simple interest. Thus, we use $1 + \frac{1}{2}R$ rather than $(1 + R)^{1/2}$ for the first coupon's cash flow in the example above. Also see R. A. Grobel, "Understanding the Duration of Floating Rate Notes," *MIMED* (New York: Salomon Brothers, 1986).

Note three important aspects of this present value equation. First, the investor has to wait only a half year to get the first coupon payment—hence, the discount rate is $(1 + \frac{1}{2}R)$. Second, the investor knows with certainty only the size of the first coupon C_1, which was preset at the beginning of the first coupon period to reflect interest rates at that time. The FI set the first coupon rate six months before the investor bought the bond. Third, the other coupons on the bond, $C_2, C_3, C_4, C_5, \ldots$, are unknown at the time the bond is purchased because they depend on the level of interest rates at the time they are reset (see Figure 9B–8).

To derive the duration of the bond, rewrite the cash flows at one-half year onward as

$$P = \frac{C_1}{(1 + \frac{1}{2}R)} + \frac{1}{(1 + \frac{1}{2}R)} \left[\frac{C_2}{(1 + R)} + \frac{C_3}{(1 + R)^2} + \frac{C_4}{(1 + R)^3} + \frac{C_5}{(1 + R)^4} + \ldots\right]$$

where P is the present value of the bond (the bond price) at one-half year, the time of purchase.

The term in brackets is the present value or fair price (P_1) of the bond if it were sold at the end of year 1, the beginning of the second coupon period. As long as the variable coupons exactly match fluctuations in yields or interest rates, the present value of the cash flow in the square brackets is unaffected by interest rate changes. Thus,

$$P = \frac{C_1}{(1 + \frac{1}{2}R)} + \frac{P_1}{(1 + \frac{1}{2}R)}$$

Since C_1 is a fixed cash flow preset before the investor bought the bond and P_1 is a fixed cash flow in present value terms, buying this bond is similar to buying two single-payment, deep-discount bonds, each with a maturity of six months. Because the duration of a deep-discount bond is the same as its maturity, this FRN bond has

$$D = \tfrac{1}{2} \text{ year}$$

As indicated earlier, a half year is exactly the interval between the time when the bond was purchased and the time when it was first repriced.[18]

DEMAND DEPOSITS AND SAVINGS ACCOUNT LIABILITIES

Many banks and other deposit-taking institutions hold large amounts of chequing and savings account liabilities. The problem in assessing the duration of such claims is that their maturities are open-ended and many demand deposit accounts do not turn over very frequently. Although demand deposits allow holders to demand cash immediately—suggesting a very short maturity—many customers tend to retain demand deposit balances for lengthy periods. In the parlance of banking, they behave as if they were a bank's core deposits. A problem arises because defining the duration of a security requires defining its maturity. Yet demand deposits have open-ended maturities. One way for an FI manager to get around this problem is to analyze the runoff, or the turnover characteristics, of the FI's demand and savings account deposits. For example, suppose the manager learned that on average each dollar in demand deposit accounts turned over five times a year. This suggests an average turnover or maturity per dollar of around 73 days.[19]

A second method is to consider demand deposits as bonds that can be instantaneously put back to the bank in return for cash. As instantaneously putable bonds, the duration of demand deposits is approximately zero.

A third approach is more directly in line with the idea of duration as a measure of interest rate sensitivity. It looks at the percentage change of

[18] In another case an FI manager might buy a bond whose coupon floated but repaid fixed principal (many loans are priced like this). Calculating the duration on this bond or loan is straightforward. First, we have to think of it as two bonds: a floating-rate bond that pays a variable coupon (C) every year and a deep-discount bond that pays a fixed amount (F) on maturity. The duration of the first bond is the time between purchase and the first coupon reset date; $D = \frac{1}{2}$ year in the preceding example. While the duration of the deep-discount bond equals its maturity, D = three years for a three-year bond. The duration of the bond as a whole is the weighted average of a half year and three years, where the weights (w_1) and $(1 - w_1)$ reflect the present values of, respectively, the coupon cash flows and face value cash flow to the present value of the total cash flows (the sum of the two present values). Thus,

$$D = w_1(\tfrac{1}{2}) + (1 - w_1)(3)$$

[19] That is, 365 days/5 = 73 days.

demand deposits ($\Delta DD/DD$) to interest rate changes (ΔR). Because demand deposits and, to a lesser extent, savings deposits pay either low explicit or implicit interest—where implicit interest takes forms such as subsidized chequing fees—there tend to be enhanced withdrawals and switching into higher-yielding instruments as rates rise. You can use a number of quantitative techniques to test this sensitivity, including linear and non-linear time series regression analysis.

A fourth approach is to use simulation analysis. This is based on forecasts of future interest rates and the net withdrawals by depositors from their accounts over some future time period. Taking the discounted present values of these cash flows allows a duration measure to be calculated.[20]

MORTGAGES AND MORTGAGE-BACKED SECURITIES

Calculating the durations of mortgages and mortgage-backed securities is difficult because of prepayment risk. Essentially, as the level of interest rates falls, mortgage holders have the option to prepay their old mortgages and refinance with a new mortgage at a lower interest rate. In the terminology of finance, fixed-rate mortgages and mortgage-backed securities contain an embedded option. Calculating duration requires projecting the future cash flows of an asset. Consequently, to calculate the duration of mortgages, we need to model the prepayment behaviour of mortgage holders. Possible ways to do this are left to Chapter 27 on mortgage asset securitization.

FUTURES, OPTIONS, SWAPS, CAPS, AND OTHER CONTINGENT CLAIMS

When interest rates change, so do the values of (off-balance-sheet) derivative instruments such as futures, options, swaps, and caps (see Chapter 13). Market value gains and losses on these instruments can also have an impact on the net worth (E) of an FI. The calculation of the durations of these instruments is left to Chapters 23 to 25. However, it should be noted that a fully fledged duration gap model of an FI should take into account the durations of its derivatives portfolio as well as the duration of its on-balance-sheet assets and liabilities. This is especially so today, as more and more FIs take positions in derivative contracts.

[20] For a very sophisticated model along these lines, see E. W. Irmler, *The OTS Net Portfolio Value Model* (Washington, DC: OTS, 1994).

Questions and Problems

1. MLK Bank has an asset portfolio that consists of \$100 million of 30-year, 8 percent coupon \$1,000 bonds that sell at par.
   a. What will be the bonds' new prices if market yields change immediately by $+/-$ 0.10 percent? What will be the new prices if market yields change immediately by $+/-$ 2.00 percent?
   b. The duration of these bonds is 12.1608 years. What are the predicted bond prices in each of the four cases using the duration rule? What is the amount of error between the duration prediction and the actual market values?
   c. Given that convexity is 212.4, what are the bond price predictions in each of the four cases using the duration plus convexity relationship? What is the amount of error in these predictions?
   d. Diagram and label clearly the results in parts (a), (b), and (c).
2. Estimate the convexity for each of the following three bonds, all of which trade at a yield to maturity of 8 percent and have face values of \$1,000.
   - A 7-year, zero-coupon bond.
   - A 7-year, 10 percent annual coupon bond.
   - A 10-year, 10 percent annual coupon bond that has a duration value of 6.994 years (i.e., approximately 7 years).

   Rank the bonds in terms of convexity, and express the convexity relationship between zeros and coupon bonds in terms of maturity and duration equivalencies.
3. A 10-year, 10 percent annual coupon \$1,000 bond trades at a yield to maturity of 8 percent. The bond has a duration of 6.994 years. What is the modified duration of this bond? What is the practical value of calculating modified duration? Does modified duration change the result of using the duration relationship to estimate price sensitivity?