REVIEW AND SUMMARY FOR CHAPTER 9

In the first half of this chapter, we discussed the determination of the resultant **R** of forces **ΔF** distributed over a plane area A when the magnitudes of these forces are proportional to both the areas ΔA of the elements on which they act and the distances y from these elements to a given x axis; we thus had $\Delta F = ky\, \Delta A$. We found that the magnitude of the resultant **R** is proportional to the first moment $Q_x = \int y\, dA$ of the area A, while the moment of **R** about the x axis is proportional to the *second moment,* or *moment of inertia,* $I_x = \int y^2\, dA$ of A with respect to the same axis [Sec. 9.2].

Rectangular moments of inertia

The *rectangular moments of inertia I_x and I_y of an area* [Sec. 9.3] were obtained by evaluating the integrals

$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA \tag{9.1}$$

These computations can be reduced to single integrations by choosing dA to be a thin strip parallel to one of the coordinate axes. We also recall that it is possible to compute I_x and I_y from the same elemental strip (Fig. 9.35) using the formula for the moment of inertia of a rectangular area [Sample Prob. 9.3].

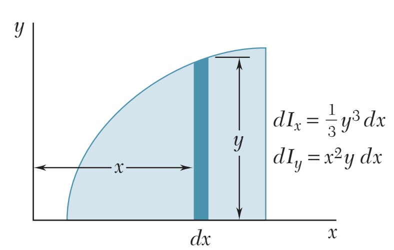

Fig. 9.35

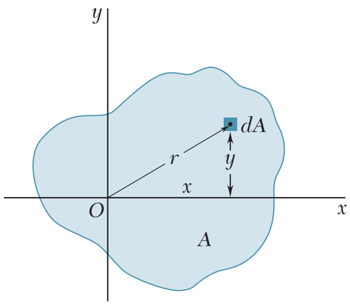

Fig. 9.36

Polar moment of inertia

The *polar moment of inertia of an area* A with respect to the pole O [Sec. 9.4] was defined as

$$J_O = \int r^2\, dA \tag{9.3}$$

where r is the distance from O to the element of area dA (Fig. 9.36). Observing that $r^2 = x^2 + y^2$, we established the relation

$$J_O = I_x + I_y \tag{9.4}$$

Radius of gyration

The *radius of gyration of an area A* with respect to the x axis [Sec. 9.5] was defined as the distance k_x, where $I_x = k_x^2 A$. With similar definitions for the radii of gyration of A with respect to the y axis and with respect to O, we had

$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_O = \sqrt{\frac{J_O}{A}} \tag{9.5–9.7}$$

Parallel-axis theorem

The parallel-axis theorem was presented in Sec. 9.6. It states that the moment of inertia I of an area with respect to any given axis AA' (Fig. 9.37) is equal to the moment of inertia $\bar{I}$ of the area with respect to the centroidal axis BB' that is parallel to AA' *plus* the product of the area A and the square of the distance d between the two axes:

$$I = \bar{I} + Ad^2 \tag{9.9}$$

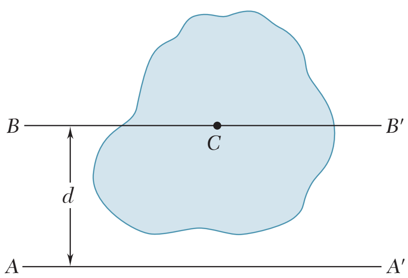

Fig. 9.37

This formula can also be used to determine the moment of inertia $\bar{I}$ of an area with respect to a centroidal axis BB' when its moment of inertia I with respect to a parallel axis AA' is known. In this case, however, the product Ad^2 should be *subtracted* from the known moment of inertia I.

A similar relation holds between the polar moment of inertia J_O of an area about a point O and the polar moment of inertia $\bar{J}_C$ of the same area about its centroid C. Letting d be the distance between O and C, we have

$$J_O = \bar{J}_C + Ad^2 \tag{9.11}$$

Composite areas

The parallel-axis theorem can be used very effectively to compute the *moment of inertia of a composite area* with respect to a given axis [Sec. 9.7]. Considering each component area separately, we first compute the moment of inertia of each area with respect to its centroidal axis, using the data provided in Figs. 9.12 and 9.13 whenever possible. The parallel-axis theorem is then applied to determine the moment of inertia of each component area with respect to the desired axis, and the various values obtained are added [Sample Probs. 9.4 and 9.5].

Product of inertia

Sections 9.8 through 9.10 were devoted to the transformation of the moments of inertia of an area *under a rotation of the coordinate axes.* First, we defined the *product of inertia of an area A* as

$$I_{xy} = \int xy\, dA \tag{9.12}$$

and showed that $I_{xy} = 0$ if the area A is symmetrical with respect to either or both of the coordinate axes. We also derived the *parallel-axis theorem for products of inertia.* We had

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \tag{9.13}$$

where $\bar{I}_{x'y'}$ is the product of inertia of the area with respect to the centroidal axes x' and y' which are parallel to the x and y axes, respectively, and $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid of the area [Sec. 9.8].

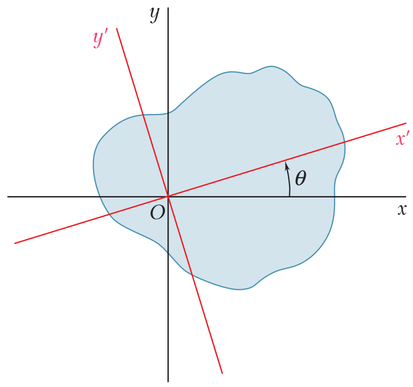

Fig. 9.38

Rotation of axes

In Sec. 9.9 we determined the moments and product of inertia $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of an area with respect to x' and y' axes obtained by rotating the original x and y coordinate axes through an angle θ counterclockwise (Fig. 9.38). We expressed $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ in terms of the moments and product of inertia I_x, I_y, and I_{xy} computed with respect to the original x and y axes. We had

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad (9.18)$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad (9.19)$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \qquad (9.20)$$

Principal axes

The *principal axes of the area about O* were defined as the two axes perpendicular to each other and with respect to which the moments of inertia of the area are maximum and minimum. The corresponding values of θ, denoted by θ_m, were obtained from the formula

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} \qquad (9.25)$$

Principal moments of inertia

The corresponding maximum and minimum values of I are called the *principal moments of inertia* of the area about O; we had

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \qquad (9.27)$$

We also noted that the corresponding value of the product of inertia is zero.

Mohr's circle

The transformation of the moments and product of inertia of an area under a rotation of axes can be represented graphically by drawing *Mohr's circle* [Sec. 9.10]. Given the moments and product of inertia I_x, I_y, and I_{xy} of the area with respect to the x and y

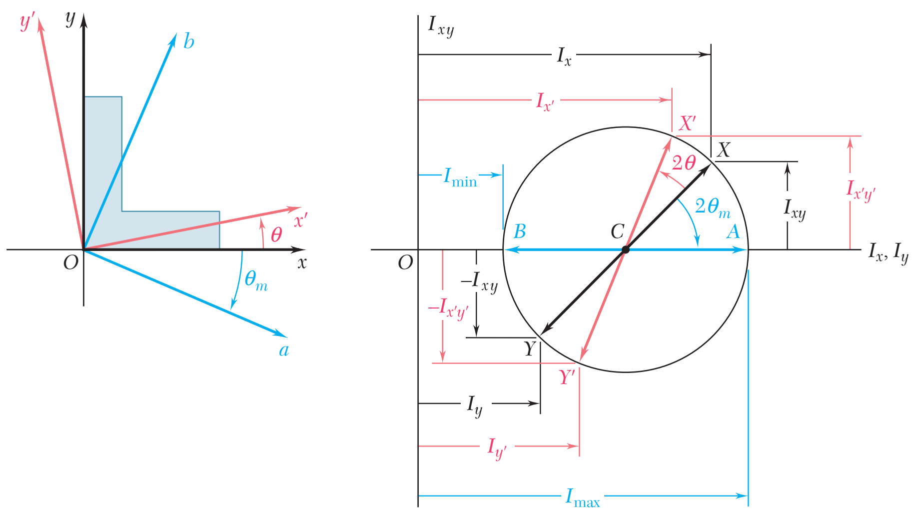

Fig. 9.39

coordinate axes, we plot points X (I_x, I_{xy}) and Y $(I_y, -I_{xy})$ and draw the line joining these two points (Fig. 9.39). This line is a diameter of Mohr's circle and thus defines this circle. As the coordinate axes are rotated through θ, the diameter rotates through *twice that angle*, and the coordinates of X' and Y' yield the new values $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of the moments and product of inertia of the area. Also, the angle θ_m and the coordinates of points A and B define the principal axes a and b and the principal moments of inertia of the area [Sample Prob. 9.8].

Moments of inertia of masses

The second half of the chapter was devoted to the determination of *moments of inertia of masses*, which are encountered in dynamics in problems involving the rotation of a rigid body about an axis. The mass moment of inertia of a body with respect to an axis AA' (Fig. 9.40) was defined as

$$I = \int r^2\, dm \qquad (9.28)$$

where r is the distance from AA' to the element of mass [Sec. 9.11]. The *radius of gyration* of the body was defined as

$$k = \sqrt{\frac{I}{m}} \qquad (9.29)$$

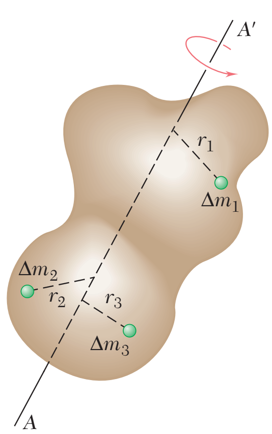

Fig. 9.40

The moments of inertia of a body with respect to the coordinate axes were expressed as

$$I_x = \int (y^2 + z^2)\, dm$$

$$I_y = \int (z^2 + x^2)\, dm \qquad (9.30)$$

$$I_z = \int (x^2 + y^2)\, dm$$

Parallel-axis theorem

We saw that the *parallel-axis theorem* also applies to mass moments of inertia [Sec. 9.12]. Thus, the moment of inertia I of a body with respect to an arbitrary axis AA' (Fig. 9.41) can be expressed as

$$I = \bar{I} + md^2 \tag{9.33}$$

where $\bar{I}$ is the moment of inertia of the body with respect to the centroidal axis BB' which is parallel to the axis AA', m is the mass of the body, and d is the distance between the two axes.

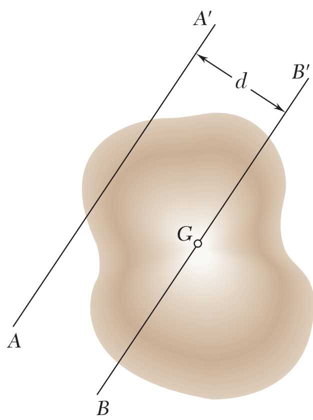

Fig. 9.41

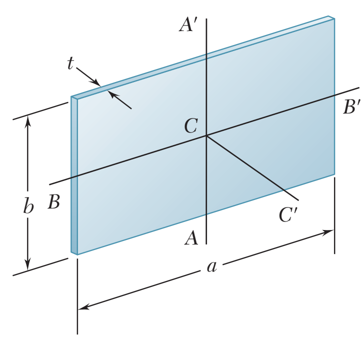

Fig. 9.42

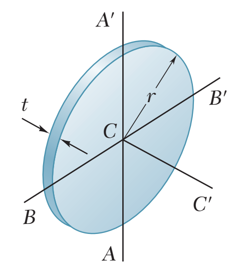

Fig. 9.43

Moments of inertia of thin plates

The moments of inertia of *thin plates* can be readily obtained from the moments of inertia of their areas [Sec. 9.13]. We found that for a *rectangular plate* the moments of inertia with respect to the axes shown (Fig. 9.42) are

$$I_{AA'} = \tfrac{1}{12}ma^2 \qquad I_{BB'} = \tfrac{1}{12}mb^2 \tag{9.39}$$
$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{12}m(a^2 + b^2) \tag{9.40}$$

while for a *circular plate* (Fig. 9.43) they are

$$I_{AA'} = I_{BB'} = \tfrac{1}{4}mr^2 \tag{9.41}$$
$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2}mr^2 \tag{9.42}$$

Composite bodies

When a body possesses *two planes of symmetry,* it is usually possible to use a single integration to determine its moment of inertia with respect to a given axis by selecting the element of mass dm to be a thin plate [Sample Probs. 9.10 and 9.11]. On the other hand, when a body consists of *several common geometric shapes,* its moment of inertia with respect to a given axis can be obtained by using the formulas given in Fig. 9.28 together with the parallel-axis theorem [Sample Probs. 9.12 and 9.13].

Moment of inertia with respect to an arbitrary axis

In the last portion of the chapter, we learned to determine the moment of inertia of a body *with respect to an arbitrary axis OL* which is drawn through the origin O [Sec. 9.16]. Denoting by λ_x,

λ_y, λ_z the components of the unit vector $\boldsymbol{\lambda}$ along OL (Fig. 9.44) and introducing the *products of inertia*

$$I_{xy} = \int xy\,dm \qquad I_{yz} = \int yz\,dm \qquad I_{zx} = \int zx\,dm \qquad (9.45)$$

we found that the moment of inertia of the body with respect to OL could be expressed as

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x \quad (9.46)$$

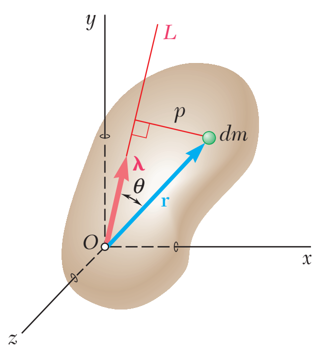

Fig. 9.44

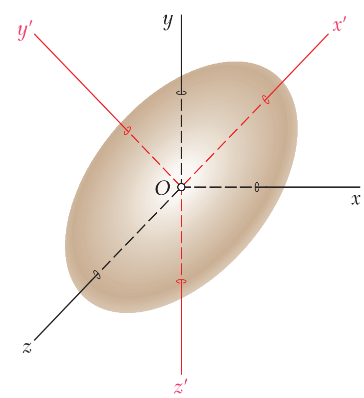

Fig. 9.45

Ellipsoid of inertia

By plotting a point Q along each axis OL at a distance $OQ = 1/\sqrt{I_{OL}}$ from O [Sec. 9.17], we obtained the surface of an ellipsoid, known as the *ellipsoid of inertia* of the body at point O. The principal axes x', y', z' of this ellipsoid (Fig. 9.45) are the *principal axes of inertia* of the body; that is, the products of inertia $I_{x'y'}$, $I_{y'z'}$, $I_{z'x'}$ of the body with respect to these axes are all zero. There are many situations when the principal axes of inertia of a body can be deduced from properties of symmetry of the body. Choosing these axes to be the coordinate axes, we can then express I_{OL} as

Principal axes of inertia
Principal moments of inertia

$$I_{OL} = I_{x'}\lambda_{x'}^2 + I_{y'}\lambda_{y'}^2 + I_{z'}\lambda_{z'}^2 \qquad (9.50)$$

where $I_{x'}$, $I_{y'}$, $I_{z'}$ are the *principal moments of inertia* of the body at O.

When the principal axes of inertia cannot be obtained by observation [Sec. 9.17], it is necessary to solve the cubic equation

$$K^3 - (I_x + I_y + I_z)K^2 + (I_xI_y + I_yI_z + I_zI_x - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)K$$
$$- (I_xI_yI_z - I_xI_{yz}^2 - I_yI_{zx}^2 - I_zI_{xy}^2 - 2I_{xy}I_{yz}I_{zx}) = 0 \qquad (9.56)$$

We found [Sec. 9.18] that the roots K_1, K_2, and K_3 of this equation are the principal moments of inertia of the given body. The direction cosines $(\lambda_x)_1$, $(\lambda_y)_1$, and $(\lambda_z)_1$ of the principal axis corresponding to the principal moment of inertia K_1 are then determined by substituting K_1 into Eqs. (9.54) and solving two of these equations and Eq. (9.57) simultaneously. The same procedure is then repeated using K_2 and K_3 to determine the direction cosines of the other two principal axes [Sample Prob. 9.15].