REVIEW AND SUMMARY FOR CHAPTER 18

This chapter was devoted to the kinetic analysis of the motion of rigid bodies in three dimensions.

Fundamental equations of motion for a rigid body

We first noted [Sec. 18.1] that the two fundamental equations derived in Chap. 14 for the motion of a system of particles

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{18.1}$$

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{18.2}$$

provide the foundation of our analysis, just as they did in Chap. 16 in the case of the plane motion of rigid bodies. The computation of the angular momentum $\mathbf{H}_G$ of the body and of its derivative $\dot{\mathbf{H}}_G$, however, are now considerably more involved.

Angular momentum of a rigid body in three dimensions

In Sec. 18.2, we saw that the rectangular components of the angular momentum $\mathbf{H}_G$ of a rigid body can be expressed as follows in terms of the components of its angular velocity $\boldsymbol{\omega}$ and of its centroidal moments and products of inertia:

$$\begin{aligned} H_x &= +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z \\ H_y &= -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z \\ H_z &= -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z \end{aligned} \tag{18.7}$$

If *principal axes of inertia* $Gx'y'z'$ are used, these relations reduce to

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \qquad H_{y'} = \bar{I}_{y'}\omega_{y'} \qquad H_{z'} = \bar{I}_{z'}\omega_{z'} \tag{18.10}$$

We observed that, in general, *the angular momentum* $\mathbf{H}_G$ *and the angular velocity* $\boldsymbol{\omega}$ *do not have the same direction* (Fig. 18.25). They will, however, have the same direction if $\boldsymbol{\omega}$ is directed along one of the principal axes of inertia of the body.

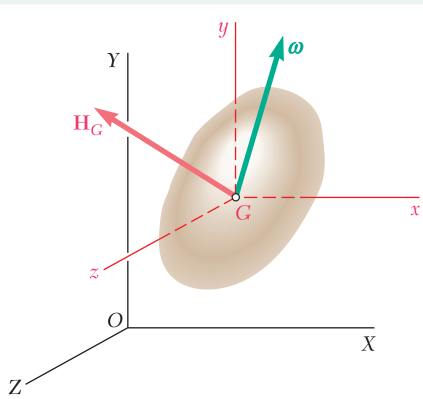

Fig. 18.25

Angular momentum about a given point

Recalling that the system of the momenta of the particles forming a rigid body can be reduced to the vector $m\bar{\mathbf{v}}$ attached at G and the couple $\mathbf{H}_G$ (Fig. 18.26), we noted that, once the linear momentum $m\bar{\mathbf{v}}$ and the angular momentum $\mathbf{H}_G$ of a rigid body have been determined, the angular momentum $\mathbf{H}_O$ of the body about any given point O can be obtained by writing

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G \qquad (18.11)$$

Rigid body with a fixed point

In the particular case of a rigid body *constrained to rotate about a fixed point O*, the components of the angular momentum $\mathbf{H}_O$ of the body about O can be obtained directly from the components of its angular velocity and from its moments and products of inertia with respect to axes through O. We wrote

$$\begin{aligned} H_x &= +I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_y\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_z\omega_z \end{aligned} \qquad (18.13)$$

Principle of impulse and momentum

The *principle of impulse and momentum* for a rigid body in three-dimensional motion [Sec. 18.3] is expressed by the same fundamental formula that was used in Chap. 17 for a rigid body in plane motion,

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\rightarrow 2} = \textbf{Syst Momenta}_2 \qquad (17.4)$$

but the systems of the initial and final momenta should now be represented as shown in Fig. 18.26, and $\mathbf{H}_G$ should be computed from the relations (18.7) or (18.10) [Sample Probs. 18.1 and 18.2].

Kinetic energy of a rigid body in three dimensions

The *kinetic energy* of a rigid body in three-dimensional motion can be divided into two parts [Sec. 18.4], one associated with the motion of its mass center G and the other with its motion about G. Using principal centroidal axes x', y', z', we wrote

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2) \qquad (18.17)$$

where
$\bar{\mathbf{v}}$ = velocity of mass center
$\boldsymbol{\omega}$ = angular velocity
m = mass of rigid body
$\bar{I}_{x'}, \bar{I}_{y'}, \bar{I}_{z'}$ = principal centroidal moments of inertia

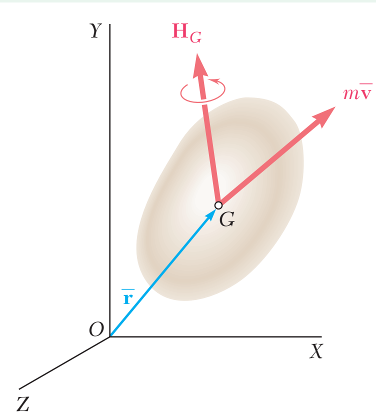

Fig. 18.26

We also noted that, in the case of a rigid body *constrained to rotate about a fixed point O*, the kinetic energy of the body can be expressed as

$$T = \frac{1}{2}(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2) \qquad (18.20)$$

where the x', y', and z' axes are the principal axes of inertia of the body at O. The results obtained in Sec. 18.4 make it possible to extend to the three-dimensional motion of a rigid body the application of the *principle of work and energy* and of the *principle of conservation of energy*.

Using a rotating frame to write the equations of motion of a rigid body in space

The second part of the chapter was devoted to the application of the fundamental equations

$$\Sigma \mathbf{F} = m\overline{\mathbf{a}} \qquad (18.1)$$

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (18.2)$$

to the motion of a rigid body in three dimensions. We first recalled [Sec. 18.5] that $\mathbf{H}_G$ represents the angular momentum of the body relative to a centroidal frame $GX'Y'Z'$ of fixed orientation (Fig. 18.27)

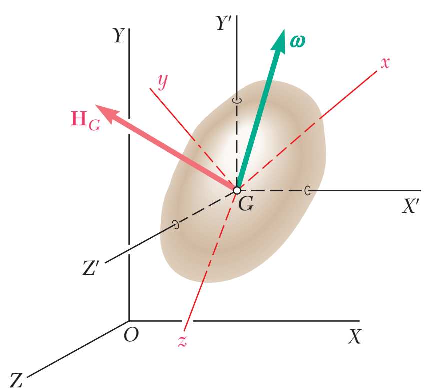

Fig. 18.27

and that $\dot{\mathbf{H}}_G$ in Eq. (18.2) represents the rate of change of $\mathbf{H}_G$ with respect to that frame. We noted that, as the body rotates, its moments and products of inertia with respect to the frame $GX'Y'Z'$ change continually. Therefore, it is more convenient to use a rotating frame $Gxyz$ when resolving $\boldsymbol{\omega}$ into components and computing the moments and products of inertia that will be used to determine $\mathbf{H}_G$ from Eqs. (18.7) or (18.10). However, since $\dot{\mathbf{H}}_G$ in Eq. (18.2) represents the rate of change of $\mathbf{H}_G$ with respect to the frame $GX'Y'Z'$ of fixed orientation, we must use the method of Sec. 15.10 to determine its value. Recalling Eq. (15.31), we wrote

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \qquad (18.22)$$

where $\mathbf{H}_G$ = angular momentum of body with respect to frame $GX'Y'Z'$ of fixed orientation

$(\dot{\mathbf{H}}_G)_{Gxyz}$ = rate of change of $\mathbf{H}_G$ with respect to rotating frame $Gxyz$, to be computed from relations (18.7)

$\boldsymbol{\Omega}$ = angular velocity of the rotating frame $Gxyz$

Substituting for $\dot{\mathbf{H}}_G$ from (18.22) into (18.2), we obtained

$$\Sigma \mathbf{M}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \qquad (18.23)$$

If the rotating frame is actually attached to the body, its angular velocity $\boldsymbol{\Omega}$ is identically equal to the angular velocity $\boldsymbol{\omega}$ of the body. There are many applications, however, where it is advantageous to use a frame of reference which is not attached to the body but rotates in an independent manner [Sample Prob. 18.5].

Euler's equations of motion. d'Alembert's principle

Setting $\boldsymbol{\Omega} = \boldsymbol{\omega}$ in Eq. (18.23), using principal axes and writing this equation in scalar form, we obtained *Euler's equations of motion* [Sec. 18.6]. A discussion of the solution of these equations and of the scalar equations corresponding to Eq. (18.1) led us to extend d'Alembert's principle to the three-dimensional motion of a rigid body and to conclude that the system of the external forces acting on the rigid body is not only equipollent, but actually *equivalent* to the effective forces of the body represented by the vector $m\bar{\mathbf{a}}$ and the couple $\dot{\mathbf{H}}_G$ (Fig. 18.28). Problems involving the three-dimensional motion of a rigid body can be solved by considering the free-body-diagram equation represented in Fig. 18.28 and writing appropriate scalar equations relating the components or moments of the external and effective forces [Sample Probs. 18.3 and 18.5].

Free-body-diagram equation

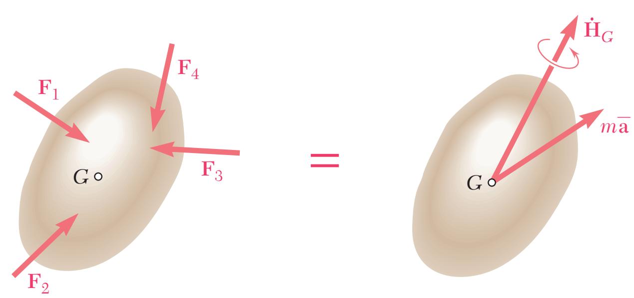

Fig. 18.28

Rigid body with a fixed point

In the case of a rigid body *constrained to rotate about a fixed point O*, an alternative method of solution, involving the moments of the forces and the rate of change of the angular momentum about point O, can be used. We wrote [Sec. 18.7]:

$$\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \qquad (18.28)$$

where $\Sigma \mathbf{M}_O$ = sum of moments about O of forces applied to rigid body
$\mathbf{H}_O$ = angular momentum of body with respect to fixed frame $OXYZ$
$(\dot{\mathbf{H}}_O)_{Oxyz}$ = rate of change of $\mathbf{H}_O$ with respect to rotating frame $Oxyz$, to be computed from relations (18.13)
$\boldsymbol{\Omega}$ = angular velocity of rotating frame $Oxyz$

This approach can be used to solve certain problems involving the rotation of a rigid body about a fixed axis [Sec. 18.8], for example, an unbalanced rotating shaft [Sample Prob. 18.4].

Motion of a gyroscope

In the last part of the chapter, we considered the motion of *gyroscopes* and other *axisymmetrical bodies.* Introducing the *Eulerian angles* ϕ, θ, and ψ to define the position of a gyroscope (Fig. 18.29), we observed that their derivatives $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ represent, respectively, the rates of *precession, nutation,* and *spin* of the gyroscope [Sec. 18.9]. Expressing the angular velocity $\boldsymbol{\omega}$ in terms of these derivatives, we wrote

$$\boldsymbol{\omega} = -\dot{\phi} \sin \theta \, \mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta)\mathbf{k} \qquad (18.35)$$

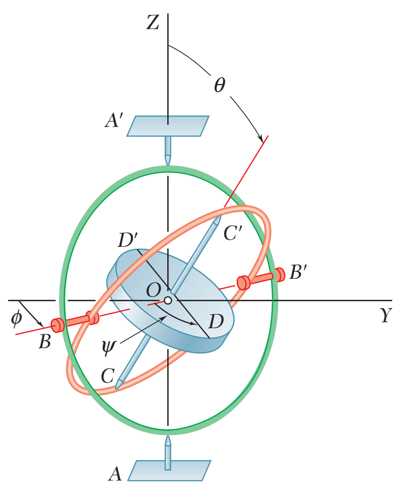

Fig. 18.29

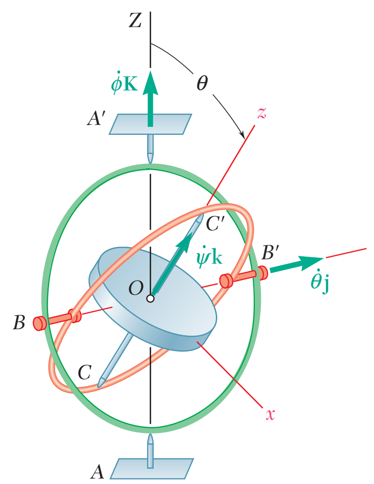

Fig. 18.30

where the unit vectors are associated with a frame $Oxyz$ attached to the inner gimbal of the gyroscope (Fig. 18.30) and rotate, therefore, with the angular velocity

$$\boldsymbol{\Omega} = -\dot{\phi} \sin \theta \, \mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi} \cos \theta \, \mathbf{k} \qquad (18.38)$$

Denoting by I the moment of inertia of the gyroscope with respect to its spin axis z and by I' its moment of inertia with respect to a transverse axis through O, we wrote

$$\mathbf{H}_O = -I'\dot{\phi} \sin \theta \, \mathbf{i} + I'\dot{\theta}\mathbf{j} + I(\dot{\psi} + \dot{\phi} \cos \theta)\mathbf{k} \qquad (18.36)$$

Substituting for $\mathbf{H}_O$ and $\boldsymbol{\Omega}$ into Eq. (18.28) led us to the differential equations defining the motion of the gyroscope.

Steady precession

In the particular case of the *steady precession* of a gyroscope [Sec. 18.10], the angle θ, the rate of precession $\dot{\phi}$, and the rate of spin $\dot{\psi}$ remain constant. We saw that such a motion is possible only if the moments of the external forces about O satisfy the relation

$$\Sigma \mathbf{M}_O = (I\omega_z - I'\dot{\phi} \cos \theta)\dot{\phi} \sin \theta \mathbf{j} \qquad (18.44)$$

that is, if the external forces reduce to a couple of moment equal to the right-hand member of Eq. (18.44) and applied *about an axis perpendicular to the precession axis and to the spin axis* (Fig. 18.31). The chapter ended with a discussion of the motion of an axisymmetrical body spinning and precessing *under no force* [Sec. 18.11; Sample Prob. 18.6].

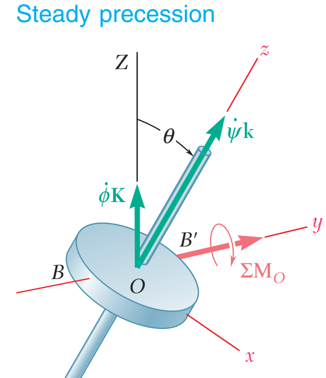

Fig. 18.31