REVIEW AND SUMMARY FOR CHAPTER 19

This chapter was devoted to the study of *mechanical vibrations,* that is, to the analysis of the motion of particles and rigid bodies oscillating about a position of equilibrium. In the first part of the chapter [Secs. 19.2 through 19.7], we considered *vibrations without damping,* while the second part was devoted to *damped vibrations* [Secs. 19.8 through 19.10].

Free vibrations of a particle

In Sec. 19.2, we considered the *free vibrations of a particle,* that is, the motion of a particle P subjected to a restoring force proportional to the displacement of the particle—such as the force exerted by a spring. If the displacement x of the particle P is measured from its equilibrium position O (Fig. 19.17), the resultant $\mathbf{F}$ of the forces acting on P (including its weight) has a magnitude kx and is directed toward O. Applying Newton's second law $F = ma$ and recalling that $a = \ddot{x}$, we wrote the differential equation

$$m\ddot{x} + kx = 0 \qquad (19.2)$$

or, setting $\omega_n^2 = k/m$,

$$\ddot{x} + \omega_n^2 x = 0 \qquad (19.6)$$

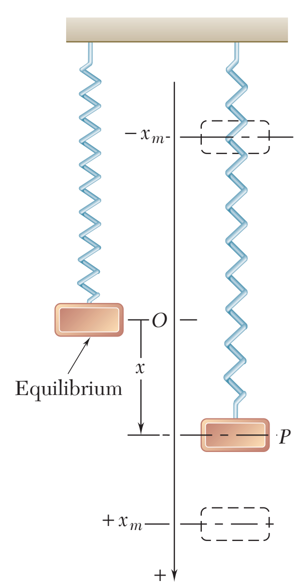

Fig. 19.17

The motion defined by this equation is called a *simple harmonic motion.*

The solution of Eq. (19.6), which represents the displacement of the particle P, was expressed as

$$x = x_m \sin (\omega_n t + \phi) \qquad (19.10)$$

where x_m = amplitude of the vibration
$\omega_n = \sqrt{k/m}$ = natural circular frequency
ϕ = phase angle

The *period of the vibration* (i.e., the time required for a full cycle) and its *natural frequency* (i.e., the number of cycles per second) were expressed as

$$\text{Period} = \tau_n = \frac{2\pi}{\omega_n} \qquad (19.13)$$

$$\text{Natural frequency} = f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} \qquad (19.14)$$

The velocity and acceleration of the particle were obtained by differentiating Eq. (19.10), and their maximum values were found to be

$$v_m = x_m\omega_n \qquad a_m = x_m\omega_n^2 \qquad (19.15)$$

Since all the above parameters depend directly upon the natural circular frequency ω_n and thus upon the ratio k/m, it is essential in any given problem to calculate the value of the constant k; this can be done by determining the relation between the restoring force and the corresponding displacement of the particle [Sample Prob. 19.1].

It was also shown that the oscillatory motion of the particle P can be represented by the projection on the x axis of the motion of a point Q describing an auxiliary circle of radius x_m with the constant angular velocity ω_n (Fig. 19.18). The instantaneous values of the velocity and acceleration of P can then be obtained by projecting on the x axis the vectors $\mathbf{v}_m$ and $\mathbf{a}_m$ representing, respectively, the velocity and acceleration of Q.

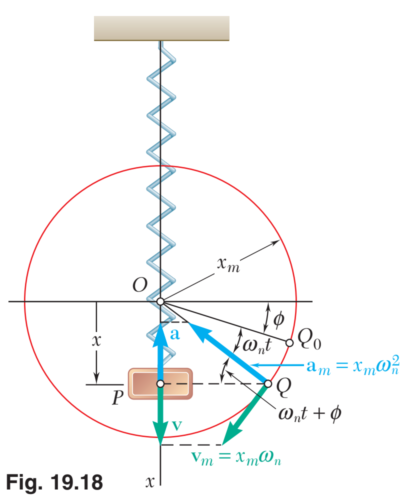

Fig. 19.18

Simple pendulum

While the motion of a *simple pendulum* is not truly a simple harmonic motion, the formulas given above can be used with $\omega_n^2 = g/l$ to calculate the period and natural frequency of the *small oscillations* of a simple pendulum [Sec. 19.3]. Large-amplitude oscillations of a simple pendulum were discussed in Sec. 19.4.

Free vibrations of a rigid body

The *free vibrations of a rigid body* can be analyzed by choosing an appropriate variable, such as a distance x or an angle θ, to define the position of the body, drawing a free-body-diagram equation to express the equivalence of the external and effective forces, and writing an equation relating the selected variable and its second derivative [Sec. 19.5]. If the equation obtained is of the form

$$\ddot{x} + \omega_n^2 x = 0 \qquad \text{or} \qquad \ddot{\theta} + \omega_n^2 \theta = 0 \tag{19.21}$$

the vibration considered is a simple harmonic motion and its period and natural frequency can be obtained *by identifying* ω_n and substituting its value into Eqs. (19.13) and (19.14) [Sample Probs. 19.2 and 19.3].

Using the principle of conservation of energy

The *principle of conservation of energy* can be used as an alternative method for the determination of the period and natural frequency of the simple harmonic motion of a particle or rigid body [Sec. 19.6]. Choosing again an appropriate variable, such as θ, to define the position of the system, we express that the total energy of the system is conserved, $T_1 + V_1 = T_2 + V_2$, between the position of maximum displacement ($\theta_1 = \theta_m$) and the position of maximum velocity ($\dot{\theta}_2 = \dot{\theta}_m$). If the motion considered is simple harmonic, the two members of the equation obtained consist of homogeneous quadratic expressions in θ_m and $\dot{\theta}_m$, respectively.† Substituting $\dot{\theta}_m = \theta_m \omega_n$ in this equation, we can factor out θ_m^2 and solve for the circular frequency ω_n [Sample Prob. 19.4].

Forced vibrations

In Sec. 19.7, we considered the *forced vibrations* of a mechanical system. These vibrations occur when the system is subjected to a periodic force (Fig. 19.19) or when it is elastically connected to a support which has an alternating motion (Fig. 19.20). Denoting by ω_f the forced circular frequency, we found that in the first case, the motion of the system was defined by the differential equation

$$m\ddot{x} + kx = P_m \sin \omega_f t \tag{19.30}$$

and that in the second case it was defined by the differential equation

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t \tag{19.31}$$

The general solution of these equations is obtained by adding a particular solution of the form

$$x_{\text{part}} = x_m \sin \omega_f t \tag{19.32}$$

†If the motion considered can only be *approximated* by a simple harmonic motion, such as for the small oscillations of a body under gravity, the potential energy must be approximated by a quadratic expression in θ_m.

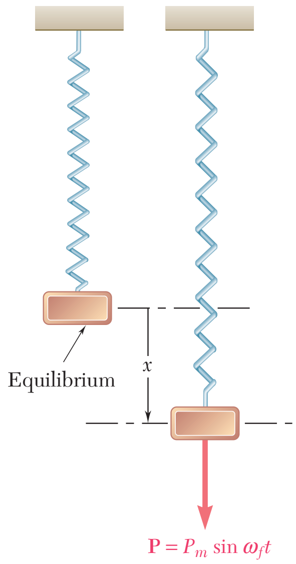

Fig. 19.19

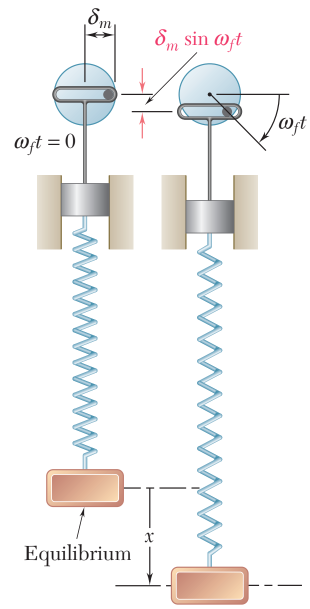

Fig. 19.20

to the general solution of the corresponding homogeneous equation. The particular solution (19.32) represents a *steady-state vibration* of the system, while the solution of the homogeneous equation represents a *transient free vibration* which can generally be neglected.

Dividing the amplitude x_m of the steady-state vibration by P_m/k in the case of a periodic force, or by δ_m in the case of an oscillating support, we defined the *magnification factor* of the vibration and found that

$$\text{Magnification factor} = \frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{1 - (\omega_f/\omega_n)^2} \qquad (19.36)$$

According to Eq. (19.36), the amplitude x_m of the forced vibration *becomes infinite when* $\omega_f = \omega_n$, that is, *when the forced frequency is equal to the natural frequency of the system.* The impressed force or impressed support movement is then said to be in *resonance* with the system [Sample Prob. 19.5]. Actually the amplitude of the vibration remains finite, due to damping forces.

Damped free vibrations

In the last part of the chapter, we considered the *damped vibrations* of a mechanical system. First, we analyzed the *damped free vibrations* of a system with *viscous damping* [Sec. 19.8]. We found that the motion of such a system was defined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (19.38)$$

where c is a constant called the *coefficient of viscous damping.* Defining the *critical damping coefficient* c_c as

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \qquad (19.41)$$

where ω_n is the natural circular frequency of the system in the absence of damping, we distinguished three different cases of damping, namely, (1) *heavy damping,* when $c > c_c$; (2) *critical damping,* when $c = c_c$; and (3) *light damping,* when $c < c_c$. In the first two cases, the system when disturbed tends to regain its equilibrium position without any oscillation. In the third case, the motion is vibratory with diminishing amplitude.

Damped forced vibrations

In Sec. 19.9, we considered the *damped forced vibrations* of a mechanical system. These vibrations occur when a system with viscous damping is subjected to a periodic force **P** of magnitude $P = P_m \sin \omega_f t$ or when it is elastically connected to a support with an alternating motion $\delta = \delta_m \sin \omega_f t$. In the first case, the motion of the system was defined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \qquad (19.47)$$

and in the second case by a similar equation obtained by replacing P_m by $k\delta_m$ in (19.47).

The *steady-state vibration* of the system is represented by a particular solution of Eq. (19.47) of the form

$$x_{\text{part}} = x_m \sin (\omega_f t - \varphi) \qquad (19.48)$$

Dividing the amplitude x_m of the steady-state vibration by P_m/k in the case of a periodic force, or by δ_m in the case of an oscillating support, we obtained the following expression for the magnification factor:

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{\sqrt{[1 - (\omega_f/\omega_n)^2]^2 + [2(c/c_c)(\omega_f/\omega_n)]^2}} \qquad (19.53)$$

where $\omega_n = \sqrt{k/m}$ = natural circular frequency of undamped system
$c_c = 2m\omega_n$ = critical damping coefficient
c/c_c = damping factor

We also found that the *phase difference* φ between the impressed force or support movement and the resulting steady-state vibration of the damped system was defined by the relation

$$\tan \varphi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1 - (\omega_f/\omega_n)^2} \qquad (19.54)$$

Electrical analogues

The chapter ended with a discussion of *electrical analogues* [Sec. 19.10], in which it was shown that the vibrations of mechnical systems and the oscillations of electrical circuits are defined by the same differential equations. Electrical analogues of mechanical systems can therefore be used to study or predict the behavior of these systems.