



55. Suppose that it costs a company  $C(x) = 0.01x^2 + 40x + 3600$  dollars to manufacture  $x$  units of a product. For this **cost function**, the **average cost function** is  $\bar{C}(x) = \frac{C(x)}{x}$ . Find the value of  $x$  that minimizes the average cost.
56. In exercise 55, the cost function can be related to the efficiency of the production process. Explain why a cost function that is concave down indicates better efficiency than a cost function that is concave up.
57. Show that there is an inflection point at  $(0, 0)$  for any function of the form  $f(x) = x^4 + cx^3$ , where  $c$  is a nonzero constant. What role(s) does  $c$  play in the graph of  $y = f(x)$ ?
58. The following examples show that there is not a perfect match between inflection points and places where  $f''(x) = 0$ . First, for  $f(x) = x^6$ , show that  $f''(0) = 0$ , but there is no inflection point at  $x = 0$ . Then, for  $g(x) = x|x|$ , show that there is an inflection point at  $x = 0$ , but that  $g''(0)$  does not exist.
59. Give an example of a function showing that the following statement is false. If the graph of  $y = f(x)$  is concave down for all  $x$ , the equation  $f(x) = 0$  has at least one solution.
60. Determine if the following statement is true or false. If  $f(0) = 1$ ,  $f''(x)$  exists for all  $x$  and the graph of  $y = f(x)$  is concave down for all  $x$ , the equation  $f(x) = 0$  has at least one solution.
61. One basic principle of physics is that light follows the path of minimum time. Assuming that the speed of light in the earth's atmosphere decreases as altitude decreases, argue that the path that light follows is concave down. Explain why this means that the setting sun appears higher in the sky than it really is.



62. Prove Theorem 5.2 (the Second Derivative Test). (Hint: Think about what the definition of  $f''(c)$  says when  $f''(c) > 0$  or  $f''(c) < 0$ .)
63.  The linear approximation that we defined in section 3.1 is the line having the same location and the same slope as the function being approximated. Since two points determine a line, two requirements (point, slope) are all that a linear function can satisfy. However, a quadratic function can satisfy three requirements since three points determine a parabola (and there are three constants in a general quadratic function  $ax^2 + bx + c$ ). Suppose we want to define a **quadratic approximation** to  $f(x)$  at  $x = a$ . Building on the linear approximation, the general form is  $g(x) = f(a) + f'(a)(x - a) + c(x - a)^2$  for some constant  $c$  to be determined. In this way, show that  $g(a) = f(a)$  and  $g'(a) = f'(a)$ . That is,  $g(x)$  has the right position and slope at  $x = a$ . The third requirement is that  $g(x)$  have the right concavity at  $x = a$ , so that  $g''(a) = f''(a)$ . Find the constant  $c$  that makes this true. Then, find such a quadratic approximation for each of the functions  $\sin x$ ,  $\cos x$  and  $e^x$  at  $x = 0$ . In each case, graph the original function, linear approximation and quadratic approximation and describe how close the approximations are to the original functions.
64.  In this exercise, we explore a basic problem in genetics. Suppose that a species reproduces according to the following probabilities:  $p_0$  is the probability of having no children,  $p_1$  is the probability of having one offspring,  $p_2$  is the probability of having two offspring,  $\dots$ ,  $p_n$  is the probability of having  $n$  offspring and  $n$  is the largest number of offspring possible. Explain why for each  $i$ , we have  $0 \leq p_i \leq 1$  and  $p_0 + p_1 + p_2 + \dots + p_n = 1$ . We define the function  $F(x) = p_0 + p_1x + p_2x^2 + \dots + p_nx^n$ . The smallest non-negative solution of the equation  $F(x) = x$  for  $0 \leq x \leq 1$  represents the probability that the species becomes extinct. Show graphically that if  $p_0 > 0$  and  $F'(1) > 1$ , then there is a solution of  $F(x) = x$  with  $0 < x < 1$ . Thus, there is a positive probability of survival. However, if  $p_0 > 0$  and  $F'(1) < 1$ , show that there are no solutions of  $F(x) = x$  with  $0 < x < 1$ . (Hint: First show that  $F$  is increasing and concave up.)

### 3.6 OVERVIEW OF CURVE SKETCHING

You might be wondering why you need to spend any more time on curve sketching. We have already drawn numerous graphs over the last three sections. Besides, with a graphing calculator or computer algebra system at your disposal, why must you even consider drawing graphs by hand?

Of course, graphing calculators or computer algebra systems are powerful tools today in the study or application of mathematics. As the authors of this text, we admit it. We have



made extensive use of several computer algebra systems (Maple and Mathematica) in preparing the manuscript for this text and even in designing our problems. So then, why should we condemn you to drawing graphs by hand? We're certain that late night talk show hosts would have a list of the top 10 reasons why sadistic mathematics professors would want to inflict such pain on their students, but there's really only one reason. For better or worse, graphing calculators and computer algebra systems do not actually draw graphs. What they do is plot points (albeit lots of them) and then connect the points with a smooth curve. Of course, this works exceptionally well for some functions, but leaves something to be desired for others. The problem boils down to the window in which you draw the graph and how many points you plot in that window. The only way to know how to choose the window or how many points to plot in that window is to use the calculus to determine the properties of the graph that you are interested in seeing. We have already made this point a number of times. The calculus tells us the properties that a representative graph should exhibit. We then try to adjust the window and the number of points plotted in order to produce such a graph.

We begin this section by summarizing the various tests that you should perform on a function when trying to draw a graph of  $y = f(x)$ .

- **Domain:** You should always determine the domain of  $f$  first.
- **Vertical Asymptotes:** For any isolated point not in the domain of  $f$ , check the limiting value of the function as  $x$  approaches that point, to see if there is a vertical asymptote or a jump or removable discontinuity at that point.
- **First Derivative Information:** Determine where  $f$  is increasing and decreasing and find any local extrema.
- **Vertical Tangent Lines:** At any isolated point not in the domain of  $f'$ , **but in the domain of  $f$** , check the limiting values of  $f'(x)$ , to determine if there is a vertical tangent line at that point.
- **Second Derivative Information:** Determine where the graph is concave up and concave down and locate any inflection points.
- **Horizontal Asymptotes:** Check the limit of  $f(x)$  as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ .
- **Intercepts:** Locate  $x$ - and  $y$ -intercepts, if any. If this can't be done exactly, then do so approximately (e.g., using Newton's method).

We start with a very simple example.

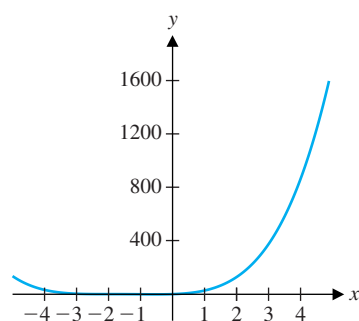


Figure 3.57a

$$y = x^4 + 6x^3 + 12x^2 + 8x + 1$$

(one view).

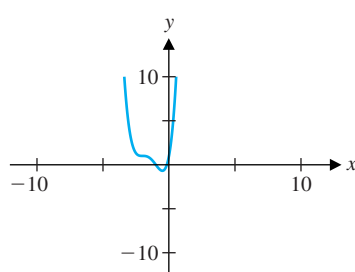


Figure 3.57b

$$y = x^4 + 6x^3 + 12x^2 + 8x + 1$$

(standard calculator view).

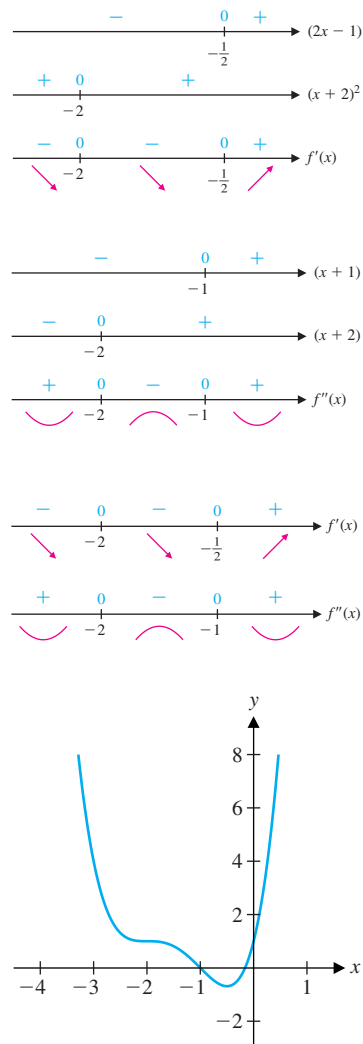
### Example 6.1 Drawing a Graph of a Polynomial

Draw a graph of  $f(x) = x^4 + 6x^3 + 12x^2 + 8x + 1$  showing all significant features.

**Solution** Why shouldn't we simply ask our computer algebra system for a graph? Computer algebra systems and graphing calculators usually do one of two things to determine the window in which they will display a graph. (Some give the user an option as to how to set the default method.) One method is to compute a set number of function values over a given standard range of  $x$ -values. The  $y$ -range is then chosen so that all of the calculated points can be displayed. This might result in a graph that looks like the one in Figure 3.57a, produced using the computer algebra system Maple. Another method is to draw a graph in a fixed, default window. For instance, most graphing calculators use the default window defined by

$$-10 \leq x \leq 10 \quad \text{and} \quad -10 \leq y \leq 10.$$

Using this window, we get the graph shown in Figure 3.57b. Of course, these two graphs are very different. Without the calculus, it's difficult to tell which, if either, of these is



**Figure 3.58**  
 $y = x^4 + 6x^3 + 12x^2 + 8x + 1.$

truly representative of the behavior of  $f$ . Some analysis will help to clear up the situation. First, note that the domain of  $f$  is the entire real line. Further, since  $f(x)$  is a polynomial, it doesn't have any vertical or horizontal asymptotes. (Think about this!) Next, note that

$$f'(x) = 4x^3 + 18x^2 + 24x + 8 = 2(2x + 1)(x + 2)^2.$$

Drawing number lines for the individual factors in  $f'(x)$ , we have that

$$f'(x) \begin{cases} > 0, \text{ on } (-\frac{1}{2}, -\infty) & f \text{ increasing.} \\ < 0, \text{ on } (-\infty, -2) \cup (-2, -\frac{1}{2}). & f \text{ decreasing.} \end{cases}$$

This also tells us that there is a local minimum at  $x = -\frac{1}{2}$  and that there are no local maxima. Next, we have

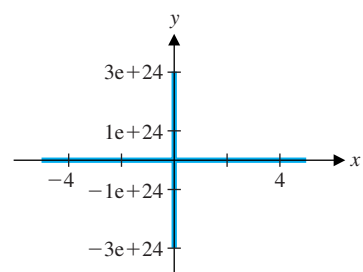
$$f''(x) = 12x^2 + 36x + 24 = 12(x + 2)(x + 1).$$

Drawing number lines for the factors of  $f''(x)$ , we have

$$f''(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \cup (-1, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-2, -1). & \text{Concave down.} \end{cases}$$

From this, we see that there are inflection points at  $x = -2$  and at  $x = -1$ . Finally, to find the  $x$ -intercepts, we need to solve  $f(x) = 0$  approximately. Doing this (we leave the details as an exercise: use Newton's method or your calculator's solver), we find that there are two  $x$ -intercepts:  $x = -1$  (exactly) and  $x \approx -0.160713$ . Notice that the significant  $x$ -values that we have identified are  $x = -2$ ,  $x = -1$  and  $x = -\frac{1}{2}$ . Computing the corresponding  $y$ -values from  $y = f(x)$ , we get the points  $(-2, 1)$ ,  $(-1, 0)$  and  $(-\frac{1}{2}, -\frac{11}{16})$ . We summarize the first and second derivative information in the number lines in the margin. In Figure 3.58, we include all of these important points by setting the  $x$ -range to be  $-3 \leq x \leq 1$  and the  $y$ -range to be  $-2 \leq y \leq 8$ .

In the following example, we examine a function that has local extrema, inflection points and both vertical and horizontal asymptotes.



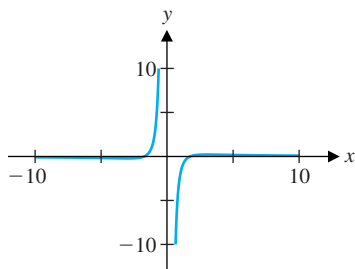
**Figure 3.59a**  
 $y = \frac{x^2 - 3}{x^3}.$

**Example 6.2** Drawing a Graph of a Rational Function

Draw a graph of  $f(x) = \frac{x^2 - 3}{x^3}$  showing all significant features.

**Solution** The default graph drawn by our computer algebra system (Maple) appears in Figure 3.59a. Notice that this doesn't seem to be a particularly useful graph, since very little is visible (or at least distinguishable from the axes). The graph drawn using the most common graphing calculator default window (with a range of  $-10 \leq x \leq 10$  and  $-10 \leq y \leq 10$ ) is seen in Figure 3.59b. This is arguably an improvement over Figure 3.59a, but does this graph convey all that it could about the function (e.g., about local extrema, inflection points, etc.)? We can answer this question only after we do some calculus. We follow the outline given at the beginning of the section.

First, observe that the domain of  $f$  includes all real numbers  $x \neq 0$ . Since  $x = 0$  is an isolated point not in the domain of  $f$ , we scrutinize the limiting behavior of  $f$  as  $x$


**Figure 3.59b**

$$y = \frac{x^2 - 3}{x^3}.$$

approaches 0. We have

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x^2 - 3}{x^3} = -\infty \quad (6.1)$$

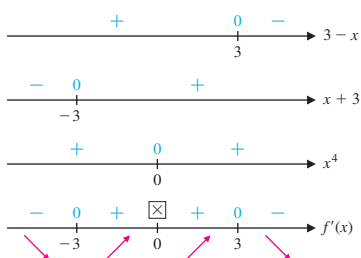
and

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x^2 - 3}{x^3} = \infty. \quad (6.2)$$

From (6.1) and (6.2), we see that the graph has a vertical asymptote at  $x = 0$ .

Next, we look for whatever information the first derivative will yield. We have

$$\begin{aligned} f'(x) &= \frac{2x(x^3) - (x^2 - 3)(3x^2)}{(x^3)^2} && \text{Quotient rule.} \\ &= \frac{x^2[2x^2 - 3(x^2 - 3)]}{x^6} && \text{Factor out an } x^2. \\ &= \frac{9 - x^2}{x^4} && \text{Combine terms.} \\ &= \frac{(3 - x)(3 + x)}{x^4}. && \text{Factor difference of two squares.} \end{aligned}$$



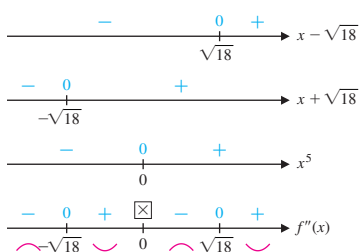
Looking at the individual factors in  $f'(x)$ , we have the number lines shown in the margin. Thus,

$$f'(x) \begin{cases} > 0, \text{ on } (-3, 0) \cup (0, 3) & f \text{ increasing.} \\ < 0, \text{ on } (-\infty, -3) \cup (3, \infty). & f \text{ decreasing.} \end{cases} \quad (6.3)$$

Note that this says that  $f$  has a local minimum at  $x = -3$  and a local maximum at  $x = 3$ .

Next, we look at

$$\begin{aligned} f''(x) &= \frac{-2x(x^4) - (9 - x^2)(4x^3)}{(x^4)^2} && \text{Quotient rule.} \\ &= \frac{-2x^3[x^2 + (9 - x^2)(2)]}{x^8} && \text{Factor out } -2x^3. \\ &= \frac{-2(18 - x^2)}{x^5} && \text{Combine terms.} \\ &= \frac{2(x - \sqrt{18})(x + \sqrt{18})}{x^5}. && \text{Factor difference of two squares.} \end{aligned}$$



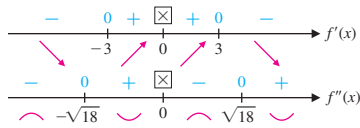
Looking at the individual factors in  $f''(x)$ , we obtain the number lines found in the margin. Thus, we have

$$f''(x) \begin{cases} > 0, \text{ on } (-\sqrt{18}, 0) \cup (\sqrt{18}, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-\infty, -\sqrt{18}) \cup (0, \sqrt{18}). & \text{Concave down.} \end{cases} \quad (6.4)$$

This says that there are inflection points at  $x = \pm\sqrt{18}$ . (Why is there no inflection point at  $x = 0$ ?)

To determine the limiting behavior as  $x \rightarrow \pm\infty$ , we consider

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x^2 - 3}{x^3} \\ &= \lim_{x \rightarrow \infty} \left( \frac{1}{x} - \frac{3}{x^3} \right) = 0. \end{aligned} \quad (6.5)$$



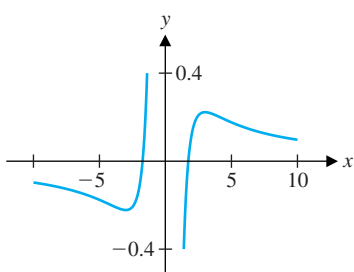
Likewise, we have

$$\lim_{x \rightarrow -\infty} f(x) = 0. \tag{6.6}$$

So, the line  $y = 0$  is a horizontal asymptote both as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ . Finally, the  $x$ -intercepts are where

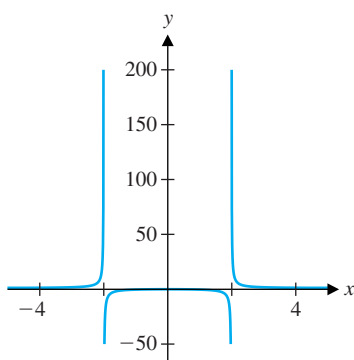
$$0 = f(x) = \frac{x^2 - 3}{x^3},$$

that is, at  $x = \pm\sqrt{3}$ . Notice that there are no  $y$ -intercepts, since  $x = 0$  is not in the domain of the function. We now have all of the information that we need to draw a representative graph. With some experimentation, you can set the  $x$ - and  $y$ -ranges so that most of the significant features of the graph (i.e., vertical and horizontal asymptotes, local extrema, inflection points, etc.) are displayed, as in Figure 3.60. Notice that the graph in Figure 3.60 is consistent with all of the information that we accumulated on the function in (6.1)–(6.6). Although the existence of the inflection points is clearly indicated by the change in concavity, their precise location is as yet a bit fuzzy in this graph. Notice, however, that both vertical and horizontal asymptotes and the local extrema are clearly indicated, something which cannot be said about either of Figures 3.59a or 3.59b.



**Figure 3.60**  
 $y = \frac{x^2 - 3}{x^3}$ .

In the following example, there are multiple vertical asymptotes, only one extremum and no inflection points.



**Figure 3.61a**  
 $y = \frac{x^2}{x^2 - 4}$ .

**Example 6.3** A Graph with Two Vertical Asymptotes

Draw a graph of  $f(x) = \frac{x^2}{x^2 - 4}$  showing all significant features.

**Solution** The default graph produced by our computer algebra system is seen in Figure 3.61a, while the default graph drawn by most graphing calculators looks like the graph seen in Figure 3.61b. Notice that the domain of  $f$  includes all  $x$  except  $x = \pm 2$  (since the denominator is zero at  $x = \pm 2$ ). Figure 3.61b suggests that there are vertical asymptotes at  $x = \pm 2$ , but let's establish this carefully. We have

$$\lim_{x \rightarrow 2^+} \frac{x^2}{x^2 - 4} = \lim_{x \rightarrow 2^+} \frac{x^2}{\underset{+}{(x - 2)}\underset{+}{(x + 2)}} = \infty. \tag{6.7}$$

Similarly, we get

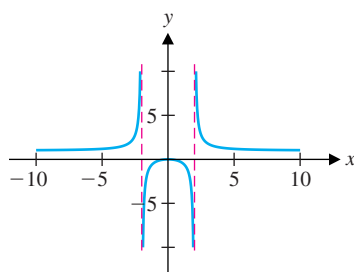
$$\lim_{x \rightarrow 2^-} \frac{x^2}{x^2 - 4} = -\infty, \quad \lim_{x \rightarrow -2^+} \frac{x^2}{x^2 - 4} = -\infty \tag{6.8}$$

and

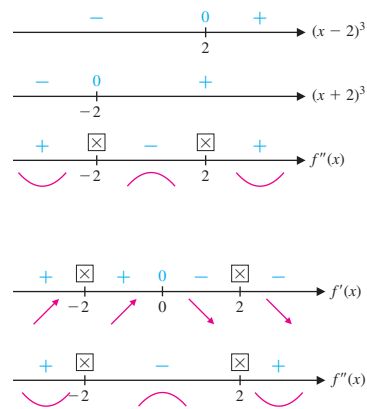
$$\lim_{x \rightarrow -2^-} \frac{x^2}{x^2 - 4} = \infty. \tag{6.9}$$

Thus, there are vertical asymptotes at  $x = \pm 2$ . Next, we have

$$f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{(x^2 - 4)^2} = \frac{-8x}{(x^2 - 4)^2}.$$



**Figure 3.61b**  
 $y = \frac{x^2}{x^2 - 4}$ .



Since the denominator is positive for  $x \neq \pm 2$ , it is a simple matter to see that

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \cup (-2, 0) & f \text{ increasing.} \\ < 0, \text{ on } (0, 2) \cup (2, \infty). & f \text{ decreasing.} \end{cases} \quad (6.10)$$

In particular, notice that the only critical number is  $x = 0$  (since  $x = -2, 2$  are not in the domain of  $f$ ). Thus, the only local extremum is the local maximum located at  $x = 0$ . Next, we have

$$\begin{aligned} f''(x) &= \frac{-8(x^2 - 4)^2 + (8x)2(x^2 - 4)^1(2x)}{(x^2 - 4)^4} && \text{Quotient rule.} \\ &= \frac{8(x^2 - 4)[-(x^2 - 4) + 4x^2]}{(x^2 - 4)^4} && \text{Factor out } 8(x^2 - 4). \\ &= \frac{8(3x^2 + 4)}{(x^2 - 4)^3} && \text{Combine terms.} \\ &= \frac{8(3x^2 + 4)}{(x - 2)^3(x + 2)^3}. && \text{Factor difference of two squares.} \end{aligned}$$

Since the numerator is positive for all  $x$ , we need only consider the terms in the denominator, as seen in the margin. We then have

$$f''(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \cup (2, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-2, 2). & \text{Concave down.} \end{cases} \quad (6.11)$$

However, since  $x = 2, -2$  are not in the domain of  $f$ , there are no inflection points. It is an easy exercise to verify that

$$\lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 4} = 1 \quad (6.12)$$

and

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 - 4} = 1. \quad (6.13)$$

From (6.12) and (6.13), we have that  $y = 1$  is a horizontal asymptote, both as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ . Finally, we observe that the only  $x$ -intercept is at  $x = 0$ . We can now summarize the information in (6.7)–(6.13) in the graph seen in Figure 3.62.

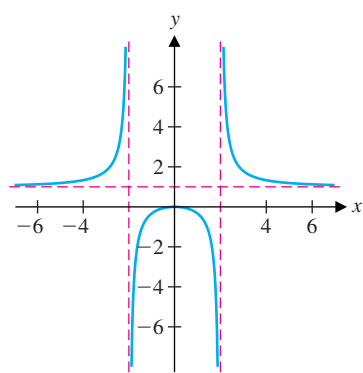


Figure 3.62

$$y = \frac{x^2}{x^2 - 4}.$$

In the following example, we need to use computer-generated graphs, as well as a rootfinding method to determine the behavior of the function.

**Graphing Where the Domain and Extrema Must be Approximated**

**Example 6.4**

Draw a graph of  $f(x) = \frac{1}{x^3 + 3x^2 + 3x + 3}$  showing all significant features.

**Solution** The default graph drawn by most graphing calculators and computer algebra systems looks something like the one shown in Figure 3.63. This seems to reveal more about the function than did its counterparts in example 6.3, but we can only determine all the significant features by doing some calculus.

Since  $f$  is a rational function, it is defined for all  $x$ , except for where the denominator is zero, that is, where

$$x^3 + 3x^2 + 3x + 3 = 0.$$

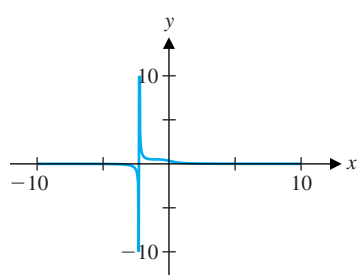


Figure 3.63

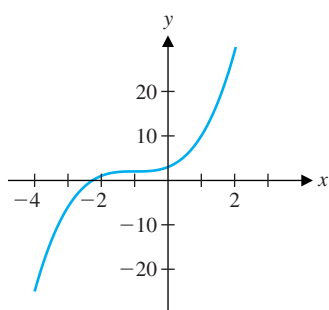
$$y = \frac{1}{x^3 + 3x^2 + 3x + 3}.$$



If you don't see how to factor the expression to find the zeros exactly, you must rely on approximate methods. First, to get an idea of where the zero(s) might be, draw a graph of the cubic (see Figure 3.64). The graph does not need to be elaborate, merely detailed enough to get an idea of where and how many zeros there are. In the present case, we see that there is only one zero, around  $x = -2$ . We can verify that this is the only zero, since

$$\frac{d}{dx}(x^3 + 3x^2 + 3x + 3) = 3x^2 + 6x + 3 = 3(x + 1)^2 \geq 0.$$

Since the derivative is never negative, observe that the function cannot decrease to cross the  $x$ -axis a second time. We can obtain an approximation of this zero, as accurate as needed, using Newton's method or your calculator's solver. Here, we get the approximate zero  $x = a \approx -2.25992$ . Even though we don't know this zero exactly, we can use the graph in Figure 3.64 to help us compute the limits



**Figure 3.64**  
 $y = x^3 + 3x^2 + 3x + 3.$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} \frac{1}{x^3 + 3x^2 + 3x + 3} = \infty \quad (6.14)$$

and

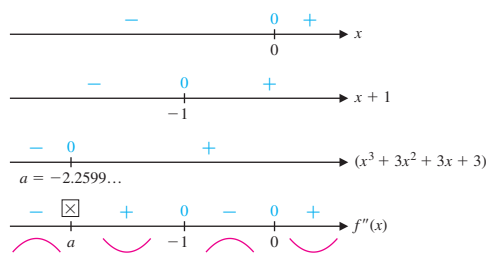
$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} \frac{1}{x^3 + 3x^2 + 3x + 3} = -\infty. \quad (6.15)$$

From (6.14) and (6.15),  $f$  has a vertical asymptote at  $x = a$ . Turning to the derivative information, we have

$$\begin{aligned} f'(x) &= -(x^3 + 3x^2 + 3x + 3)^{-2}(3x^2 + 6x + 3) \\ &= -3 \left[ \frac{(x + 1)^2}{(x^3 + 3x^2 + 3x + 3)^2} \right] \\ &= -3 \left( \frac{x + 1}{x^3 + 3x^2 + 3x + 3} \right)^2 \\ &< 0, \text{ for } x \neq a. \end{aligned} \quad (6.16)$$

Thus,  $f$  is decreasing for all  $x \neq a$ . Also, notice that the only critical number is  $x = -1$ , but since  $f$  is decreasing everywhere except at  $x = a$ , there are no local extrema. Turning to the second derivative, we get

$$\begin{aligned} f''(x) &= -6 \left( \frac{x + 1}{x^3 + 3x^2 + 3x + 3} \right) \frac{1(x^3 + 3x^2 + 3x + 3) - (x + 1)(3x^2 + 6x + 3)}{(x^3 + 3x^2 + 3x + 3)^2} \\ &= \frac{-6(x + 1)}{(x^3 + 3x^2 + 3x + 3)^3} (-2x^3 - 6x^2 - 6x) \\ &= \frac{12x(x + 1)(x^2 + 3x + 3)}{(x^3 + 3x^2 + 3x + 3)^3}. \end{aligned}$$



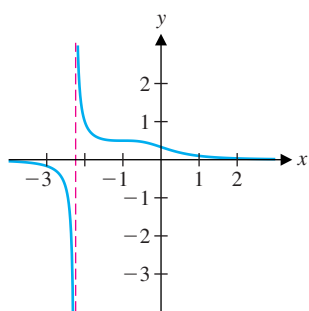


Figure 3.65

$$y = \frac{1}{x^3 + 3x^2 + 3x + 3}.$$

Since  $(x^2 + 3x + 3) > 0$  for all  $x$  (Why is that?), we need not consider this factor. Considering the remaining factors, we have the number lines shown at the bottom of the previous page. Thus, we have that

$$f''(x) \begin{cases} > 0, & \text{on } (a, -1) \cup (0, \infty) & \text{Concave up.} \\ < 0, & \text{on } (-\infty, a) \cup (-1, 0) & \text{Concave down.} \end{cases} \quad (6.17)$$

It now follows that there are inflection points at  $x = 0$  and at  $x = -1$ . Notice that in Figure 3.63, the concavity information is not very clear and the inflection points are difficult to discern.

We note the obvious fact that the function is never zero and hence, there are no  $x$ -intercepts. Finally, we consider the limits

$$\lim_{x \rightarrow \infty} \frac{1}{x^3 + 3x^2 + 3x + 3} = 0 \quad (6.18)$$

and

$$\lim_{x \rightarrow -\infty} \frac{1}{x^3 + 3x^2 + 3x + 3} = 0. \quad (6.19)$$

Using all of the information in (6.14)–(6.19), we draw the graph seen in Figure 3.65. Here, we can clearly see the vertical and horizontal asymptotes, the changes in concavity and the fact that the function is decreasing across its entire domain. Since these are all of the features of the graph that we discovered through the calculus, this graph is representative of the major features in the behavior of the function.



In the following example, we consider the graph of a transcendental function with a vertical asymptote.

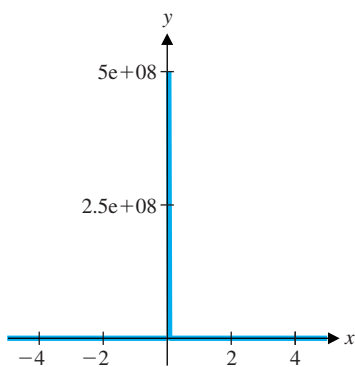


Figure 3.66a

$$y = e^{1/x}.$$

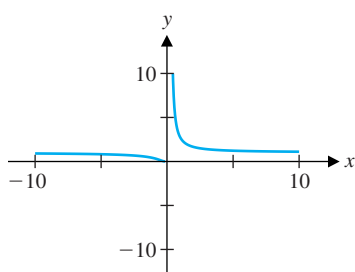


Figure 3.66b

$$y = e^{1/x}.$$

### Example 6.5 Graphing Where Some Features Are Difficult to See

Draw a graph of  $f(x) = e^{1/x}$  showing all significant features.

**Solution** The default graph produced by our computer algebra system is not particularly helpful (see Figure 3.66a). The default graph produced by most graphing calculators (see Figure 3.66b) is certainly better, but we can't be sure if this is adequate without further analysis. First, notice that the domain of  $f$  is  $(-\infty, 0) \cup (0, \infty)$ . Thus, we consider

$$\lim_{x \rightarrow 0^+} e^{1/x} = \infty, \quad (6.20)$$

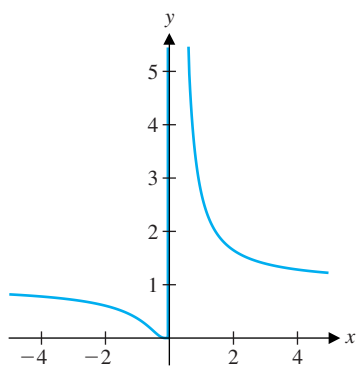
since  $1/x \rightarrow \infty$  as  $x \rightarrow 0^+$ . Also, since  $1/x \rightarrow -\infty$  as  $x \rightarrow 0^-$  (and  $e^t \rightarrow 0$ , as  $t \rightarrow -\infty$ ),

$$\lim_{x \rightarrow 0^-} e^{1/x} = 0. \quad (6.21)$$

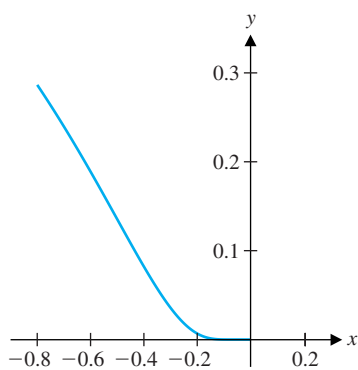
From (6.20) and (6.21), there is a vertical asymptote at  $x = 0$ , but an unusual one, in that  $f(x) \rightarrow \infty$  on one side of 0 and  $f(x) \rightarrow 0$  on the other side. Next,

$$\begin{aligned} f'(x) &= e^{1/x} \frac{d}{dx} \left( \frac{1}{x} \right) \\ &= e^{1/x} \left( \frac{-1}{x^2} \right) < 0, \text{ for all } x \neq 0, \end{aligned} \quad (6.22)$$





**Figure 3.67**  
 $y = e^{1/x}$ .



**Figure 3.68**  
 $y = e^{1/x}$ .

since  $e^{1/x} > 0$ , for all  $x$ . From (6.22), we have that  $f$  is decreasing for all  $x \neq 0$ . We also have

$$\begin{aligned} f''(x) &= e^{1/x} \left( \frac{-1}{x^2} \right) \left( \frac{-1}{x^2} \right) + e^{1/x} \left( \frac{2}{x^3} \right) \\ &= e^{1/x} \left( \frac{1}{x^4} + \frac{2}{x^3} \right) = e^{1/x} \left( \frac{1+2x}{x^4} \right) \\ &\times \begin{cases} < 0, & \text{on } (-\infty, -\frac{1}{2}) & \text{Concave down.} \\ > 0, & \text{on } (-\frac{1}{2}, 0) \cup (0, \infty). & \text{Concave up.} \end{cases} \end{aligned} \quad (6.23)$$

Since  $x = 0$  is not in the domain of  $f$ , the only inflection point is at  $x = -\frac{1}{2}$ . Next, note that

$$\lim_{x \rightarrow \infty} e^{1/x} = 1, \quad (6.24)$$

since  $1/x \rightarrow 0$  as  $x \rightarrow \infty$  and  $e^t \rightarrow 1$  as  $t \rightarrow 0$ . Likewise,

$$\lim_{x \rightarrow -\infty} e^{1/x} = 1. \quad (6.25)$$

From (6.24) and (6.25),  $y = 1$  is a horizontal asymptote, both as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ . Finally, since

$$e^{1/x} > 0,$$

for all  $x \neq 0$ , there are no  $x$ -intercepts. It is worthwhile noting that all of these features of the graph were discernible from Figure 3.66b, except for the inflection point at  $x = -\frac{1}{2}$ . Note that in almost any graph you draw, it is difficult to see all of the features of the function. This happens because the inflection point  $(-\frac{1}{2}, e^{-2})$  or  $(-0.5, 0.135335\dots)$  is so close to the  $x$ -axis. Since the horizontal asymptote is the line  $y = 1$ , it is difficult to see both of these features on the same graph (without drawing the graph on a very large piece of paper). We settle for the graph seen in Figure 3.67, which shows all of the features except the inflection point and the concavity on the interval  $(-\frac{1}{2}, 0)$ . To clearly see the behavior near the inflection point, we draw a graph that is zoomed-in on the area of the inflection point (see Figure 3.68). Here, while we have resolved the problem of the concavity near  $x = 0$  and the inflection point, we have lost the details of the “big picture.”

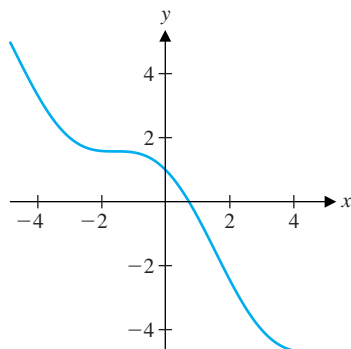
In our final example, we consider the graph of a function that is the sum of a trigonometric function and a polynomial.

### Graphing the Sum of a Polynomial and a Trigonometric Function

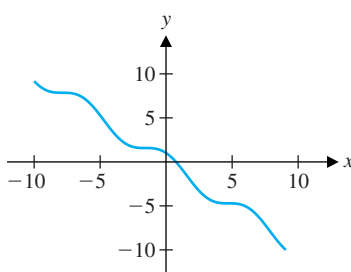
#### Example 6.6

Draw a graph of  $f(x) = \cos x - x$ , showing all significant features.

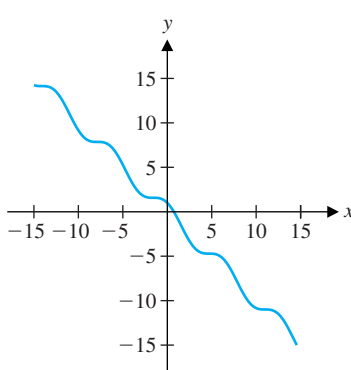
**Solution** The default graph provided by our computer algebra system can be seen in Figure 3.69a. The graph produced by most graphing calculators looks like that in Figure 3.69b. As always, we will use the calculus to determine the behavior of the



**Figure 3.69a**  
 $y = \cos x - x$ .



**Figure 3.69b**  
 $y = \cos x - x$ .



**Figure 3.70**  
 $y = \cos x - x$ .

function more precisely. First, notice that the domain of  $f$  is the entire real line. Consequently, there are no vertical asymptotes. Next, we have

$$f'(x) = -\sin x - 1 \leq 0, \quad \text{for all } x. \quad (6.26)$$

Further,  $f'(x) = 0$  if and only if  $\sin x = -1$ . So, there are critical numbers (here, these are all locations of horizontal tangent lines), but since  $f'(x)$  does not change sign, there are **no** local extrema. Even so, it is still of interest to find the locations of the horizontal tangent lines. Recall that

$$\sin x = -1 \quad \text{for } x = \frac{3\pi}{2}$$

and more generally, for

$$x = \frac{3\pi}{2} + 2n\pi,$$

for any integer  $n$ . Next, we see that

$$f''(x) = -\cos x$$

and on the interval  $[0, 2\pi]$ , we have

$$\cos x \begin{cases} > 0, \text{ on } \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] \\ < 0, \text{ on } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}.$$

So,

$$f''(x) = -\cos x \begin{cases} < 0, \text{ on } \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] & \text{Concave down.} \\ > 0, \text{ on } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). & \text{Concave up.} \end{cases} \quad (6.27)$$

Outside of  $[0, 2\pi]$ ,  $f''(x)$  simply repeats this pattern. In particular, this says that the graph has infinitely many inflection points, located at odd multiples of  $\pi/2$ .

To determine the behavior as  $x \rightarrow \pm\infty$ , we examine the limits

$$\lim_{x \rightarrow \infty} (\cos x - x) = -\infty \quad (6.28)$$

and

$$\lim_{x \rightarrow -\infty} (\cos x - x) = \infty, \quad (6.29)$$


since  $-1 \leq \cos x \leq 1$ , for all  $x$  and since  $\lim_{x \rightarrow \infty} x = \infty$ .

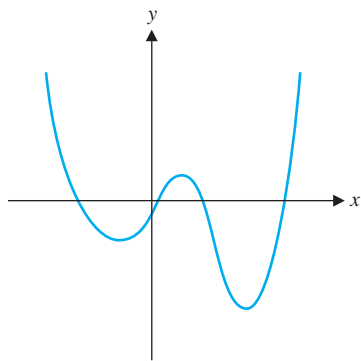
Finally, to determine the  $x$ -intercept(s), we need to solve

$$f(x) = \cos x - x = 0.$$




This can't be solved exactly, however. Since  $f'(x) \leq 0$  for all  $x$  and Figures 3.69a and 3.69b show a zero around  $x = 1$ , there is only one zero and we must approximate this (use Newton's method or your calculator's solver). We get  $x \approx 0.739085$  as an approximation to the only  $x$ -intercept. Assembling all of the information in (6.26)–(6.29), we can draw the graph seen in Figure 3.70. Notice that Figure 3.69b shows the behavior just as clearly as Figure 3.70, but for a smaller range of  $x$ -values. Which of these is more “representative” is open to discussion.

## EXERCISES 3.6

1.  We have talked about sketching representative graphs, but it is often impossible to draw a graph correctly to scale that shows all of the properties we might be interested in. For example, try to generate a computer or calculator graph that shows all three local extrema of  $x^4 - 25x^3 - 2x^2 + 80x - 3$ . When two extrema have  $y$ -coordinates of approximately  $-60$  and  $50$ , it takes a very large graph to also show a point with  $y = -40,000$ ! If an accurate graph cannot show all the points of interest, perhaps a freehand sketch like the one shown below is needed.



There is no scale shown on the graph because we have distorted different portions of the graph in an attempt to show all of the interesting points. Discuss the relative merits of an “honest” graph with a consistent scale but not showing all the points of interest versus a caricature graph which distorts the scale but does show all the points of interest.

2.  While studying for a test, a friend of yours says that a graph is not allowed to intersect an asymptote. While it is often the case that graphs don't intersect asymptotes, there is definitely not any rule against it. Explain why graphs can intersect a horizontal asymptote any number of times (Hint: Look at the graph of  $e^{-x} \sin x$ ) but can't pass through a vertical asymptote.
3.  Explain why polynomials never have vertical or horizontal asymptotes.
4.  Explain how the graph of  $f(x) = \cos x - x$  in example 6.6 relates to the graphs of  $y = \cos x$  and  $y = -x$ . Based on this discussion, explain how to sketch the graph of  $y = x + \sin x$ .

In exercises 5–44, graph the function and completely discuss the graph as in example 6.2.

5.  $f(x) = x^3 - 3x^2 + 3x$       6.  $f(x) = x^3 - 9x + 1$   
7.  $f(x) = x^4 - 3x^2 + 2x$       8.  $f(x) = x^4 + 8x - 2$

9.  $f(x) = x^5 - 2x^3 + 1$   
10.  $f(x) = x^6 - 10x^5 - 7x^4 + 80x^3 + 12x^2 - 192x$   
11.  $f(x) = x\sqrt{x^2 - 4}$       12.  $f(x) = \frac{x}{\sqrt{x^2 - 4}}$   
13.  $f(x) = x + \frac{4}{x}$       14.  $f(x) = \frac{x^2 - 1}{x}$   
15.  $f(x) = \sin x - \cos x$       16.  $f(x) = \cos^3 x$   
17.  $f(x) = e^{-x^2/4}$       18.  $f(x) = xe^{-4x}$   
19.  $f(x) = x \ln x$       20.  $f(x) = x \ln x^2$   
21.  $f(x) = \sqrt{x^2 + 1}$       22.  $f(x) = \sqrt{2x - 1}$   
23.  $f(x) = \frac{x^2 + 1}{3x^2 - 1}$       24.  $f(x) = \frac{2x^2}{x^2 + 2}$   
25.  $f(x) = \frac{2x^2}{x^3 + 1}$       26.  $f(x) = \frac{x^2}{x^4 + 1}$   
27.  $f(x) = \frac{4x}{x^2 - x + 1}$       28.  $f(x) = \frac{4x^2}{x^2 - x + 1}$   
29.  $f(x) = \frac{x - 1}{x^2 - x - 2}$       30.  $f(x) = \frac{x^3}{x^2 - 2x - 3}$   
31.  $f(x) = x + \sin x$       32.  $f(x) = 2x + \sin 2x$   
33.  $f(x) = x^5 - 5x$       34.  $f(x) = x^3 - \frac{3}{400}x$   
35.  $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$       36.  $f(x) = \sqrt{3x^4 + x^3}$   
37.  $f(x) = x^{1/5}(x + 1)$       38.  $f(x) = \sqrt[3]{x^3 + 1}$   
39.  $f(x) = \sin x - \frac{1}{2} \sin 2x$       40.  $f(x) = \sin x + \frac{1}{3} \sin 3x$   
41.  $f(x) = e^{-2/x}$       42.  $f(x) = e^{1/x^2}$   
43.  $f(x) = xe^{-x^2/2}$       44.  $f(x) = x^{10}e^{-x}$

In exercises 45–50, the “family of functions” contains a parameter  $c$ . The value of  $c$  affects the properties of the functions. Determine what differences, if any, there are for  $c$  being zero, positive or negative. Then determine what the graph would look like for very large positive  $c$ 's and for very negative  $c$ 's.

45.  $f(x) = x^4 + cx^2$       46.  $f(x) = x^4 + cx^2 + x$   
47.  $f(x) = \frac{x^2}{x^2 + c^2}$       48.  $f(x) = e^{-x^2/c}$

49.  $f(x) = \sin(cx)$

50.  $f(x) = x^2\sqrt{c^2 - x^2}$

51. In a variety of applications, researchers model a phenomenon whose graph starts at the origin, rises to a single maximum and then drops off to a horizontal asymptote of  $y = 0$ . For example, the probability density function of events such as the time from conception to birth of an animal and the amount of time surviving after contracting a fatal disease might have these properties. Show that the family of functions  $xe^{-bx}$  has these properties for all positive constants  $b$ . What effect does  $b$  have on the location of the maximum? In the case of the time since conception, what would  $b$  represent? In the case of survival time, what would  $b$  represent?

52. The “FM” in FM radio stands for **frequency modulation**, a method of transmitting information encoded in a radio wave by modulating (or varying) the frequency. A basic example of such a modulated wave is  $f(x) = \sin(x + \cos x)$ . Use computer-generated graphs of  $f(x)$ ,  $f'(x)$  and  $f''(x)$  to try to locate all relative extrema of  $f(x)$ .

53. A rational function is a function of the form  $\frac{p(x)}{q(x)}$ , where  $p(x)$  and  $q(x)$  are polynomials. Is it true that all rational functions have vertical asymptotes? Is it true that all rational functions have horizontal asymptotes?

54. It can be useful to identify asymptotes other than vertical and horizontal. For example, the parabola  $x^2$  is an asymptote of  $f(x)$  if  $\lim_{x \rightarrow \infty} [f(x) - x^2] = 0$  and/or  $\lim_{x \rightarrow -\infty} [f(x) - x^2] = 0$ . Show that  $x^2$  is an asymptote of  $f(x) = \frac{x^4 - x^2 + 1}{x^2 - 1}$ . Graph  $y = f(x)$  and zoom out until the graph looks like a parabola. (Note: the effect of zooming out is to emphasize large values of  $x$ .)

**A function  $f(x)$  has a slant asymptote  $y = mx + b$  ( $m \neq 0$ ) if  $\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0$  and/or  $\lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0$ . In exercises 55–60, find the slant asymptote (use long division to rewrite the function). Then, graph the function and its asymptote on the same axes.**

55.  $f(x) = \frac{3x^2 - 1}{x}$

56.  $f(x) = \frac{3x^2 - 1}{x - 1}$

57.  $f(x) = \frac{x^3 - 2x^2 + 1}{x^2}$

58.  $f(x) = \frac{x^3 - 1}{x^2 - 1}$

59.  $f(x) = \frac{x^4}{x^3 + 1}$

60.  $f(x) = \frac{x^4 - 1}{x^3 + x}$


**In exercises 61–64, find a function whose graph has the given asymptotes.**

61.  $x = 1$ ,  $x = 2$  and  $y = 3$

62.  $x = -1$ ,  $x = 1$  and  $y = 0$


63.  $x = -1$ ,  $x = 1$ ,  $y = -2$  and  $y = 2$

64.  $x = 1$ ,  $y = 2$  and  $x = 3$

65.  For each function, find a polynomial  $p(x)$  such that  $\lim_{x \rightarrow \infty} [f(x) - p(x)] = 0$ .

(a)  $\frac{x^4}{x+1}$  (b)  $\frac{x^5 - 1}{x+1}$  (c)  $\frac{x^6 - 2}{x+1}$

Show by zooming out that  $f(x)$  and  $p(x)$  look similar for large  $x$ . The *first term* of a polynomial is the term with the highest power (e.g.,  $x^3$  is the first term of  $x^3 - 3x + 1$ ). Can you zoom out enough to make the graph of  $f(x)$  look like the first term of its polynomial asymptote? State a very quick rule enabling you to look at a rational function and determine the first term of its polynomial asymptote (if one exists).

66.  One of the natural enemies of the balsam fir tree is the spruce budworm, which attacks the leaves of the fir tree in devastating outbreaks. Define  $N(t)$  to be the number of worms on a particular tree at time  $t$ . A mathematical model of the population dynamics of the worm must include a term to indicate the worm's death rate due to its predators (e.g., birds).

The form of this term is often taken to be  $\frac{B[N(t)]^2}{A^2 + [N(t)]^2}$  for positive constants  $A$  and  $B$ . Graph the functions  $\frac{x^2}{4 + x^2}$ ,  $\frac{2x^2}{1 + x^2}$ ,  $\frac{x^2}{9 + x^2}$  and  $\frac{3x^2}{1 + x^2}$  for  $x > 0$ . Based on these graphs, discuss

why  $\frac{B[N(t)]^2}{A^2 + [N(t)]^2}$  is a plausible model for the death rate by predation. What role do the constants  $A$  and  $B$  play? The possible stable population levels for the spruce budworms are determined by intersections of the graphs  $y = r(1 - x/k)$  and  $y = \frac{x}{1 + x^2}$ . Here,  $x = N/A$ ,  $r$  is proportional to the birthrate of the budworms and  $k$  is determined by the amount of food available to the budworms. Note that  $y = r(1 - x/k)$  is a line with  $y$ -intercept  $r$  and  $x$ -intercept  $k$ . How many solutions are there to the equation  $r(1 - x/k) = \frac{x}{1 + x^2}$ ? (Hint: The answer depends on the values of  $r$  and  $k$ .) One current theory is that outbreaks are caused in situations where there are three solutions and the population of budworms jumps from a small population to a large population.