## 3.8 RATES OF CHANGE IN APPLICATIONS

In this section, we round out our exposition of the derivative by presenting a collection of applications selected from a variety of fields. It has often been said that mathematics is the language of nature. Today, the concepts of calculus are being applied in virtually every field of human endeavor. The applications in this section represent but a small sampling of some elementary uses of the derivative. These are not all of the uses of the derivative nor are they necessarily the most important uses. Our intent is simply to present some interesting applications in a variety of settings.

Recall that the derivative of a function gives the instantaneous rate of change of that function. So, when you see the word *rate,* you should be thinking of a *derivative*. You can hardly pick up a newspaper without finding reference to some rates (e.g., inflation rate or interest rate). These can be thought of as derivatives. There are also many quantities with which you are familiar, but that you might not recognize as rates of change. Our first example is of this type.

Suppose that $Q(t)$ represents the electrical charge in a wire at time $t$. Then, the derivative $Q'(t)$ gives the **current** flowing through the wire. To see this, consider the cross section of a wire as shown in Figure 3.86. Between times $t_1$ and $t_2$, the net charge passing through such a cross section is $Q(t_2) - Q(t_1)$. The **average current** (charge per unit time) over this time interval is then defined as

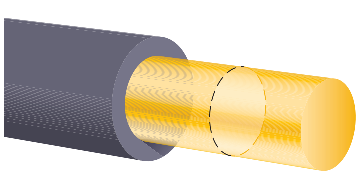


**Figure 3.86**
An electrical wire.

$$\frac{Q(t_2) - Q(t_1)}{t_2 - t_1}.$$

The **instantaneous current** $I(t)$ at any time $t_1$ can then be found by computing the limit

$$I(t_1) = \lim_{t \to t_1} \frac{Q(t) - Q(t_1)}{t - t_1}. \tag{8.1}$$

Notice that (8.1) is simply the alternative definition of derivative discussed in section 2.2. Thus, we have that $I(t) = Q'(t)$.

**Example 8.1** **Modeling Electrical Current in a Wire**

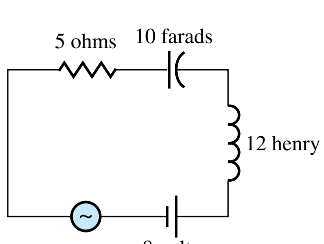



**Figure 3.87**
A simple electrical circuit.

The electrical circuit shown in Figure 3.87 includes a 5-ohm resistor, a 12-henry inductor, a 10-farad capacitor and a battery supplying 8 volts of AC current modeled by the oscillating function $8 \sin 2t$, where $t$ is measured in seconds. Find the current in the circuit at any time $t$.

**Solution** It can be shown (using the elementary laws of electricity) that the charge in this circuit is given by

$$Q(t) = 12 \sin(4t - \pi/3) + 4 \sin 2t \text{ coulombs.}$$

The current is then

$$Q'(t) = 48 \cos(4t - \pi/3) + 8 \cos 2t \text{ amps (coulombs per second).}$$

The next example we offer comes from chemistry. It is very important for chemists to have a handle on the rate at which a given reaction proceeds. Reaction rates give chemists information about the nature of the chemical bonds being formed and broken, as well as information about the type and quantity of product to expect. A simple situation is depicted in the schematic

$$A + B \longrightarrow C,$$

which indicates that chemicals $A$ and $B$ (the *reactants*) combine to form chemical $C$ (the *product*). Let $[C](t)$ denote the concentration (in moles per liter) of the product. The average reaction rate between times $t_1$ and $t_2$ is

$$\frac{[C](t_2) - [C](t_1)}{t_2 - t_1}.$$

The instantaneous reaction rate at any given time $t_1$ is then given by

$$\lim_{t \to t_1} \frac{[C](t) - [C](t_1)}{t - t_1} = \frac{d[C]}{dt}(t_1).$$

Depending on the details of the reaction, it is often possible to write down an equation relating the reaction rate $\frac{d[C]}{dt}$ to the concentrations of the reactants, $[A]$ and $[B]$.

### Example 8.2 Modeling the Rate of a Chemical Reaction

In an **autocatalytic** chemical reaction, the reactant and the product are the same. The reaction continues until some saturation level is reached. From experimental evidence, chemists know that the reaction rate is jointly proportional to the amount of the product present and the difference between the saturation level and the amount of the product. If the initial concentration of the chemical is 0 and the saturation level is 1 (corresponding to 100%), then the concentration $x(t)$ of the chemical satisfies the equation

$$x'(t) = rx(t)[1 - x(t)],$$

where $r > 0$ is a constant.

Find the concentration of chemical for which the reaction rate $x'(t)$ is a maximum.

**Solution** To clarify the problem, we write the reaction rate as

$$f(x) = rx(1 - x).$$

Our aim is then to find $x \geq 0$ that maximizes $f(x)$. We have

$$\begin{aligned} f'(x) &= r(1)(1 - x) + rx(-1) \\ &= r(1 - 2x) \end{aligned}$$

and so, the only critical number is $x = \frac{1}{2}$. Notice that the graph of $y = f(x)$ is a parabola opening downward and hence, the critical number must correspond to the absolute maximum. (Draw your own graph to see for yourself.) Although the mathematical problem here was easy to solve, the result gives a chemist some precise information. At the time the reaction rate reaches a maximum, the concentration of chemical equals exactly half of the saturation level.

Calculus and elementary physics are quite closely connected historically. It should come as no surprise, then, that physics provides us with such a large number of important applications of the calculus. We have already explored the concepts of velocity and acceleration. Another important application in physics where the derivative plays a role involves density. There are many different kinds of densities that we could consider. For example, we could study population density (number of people per unit area) or color density (depth of color per unit area) used in the study of radiographs. However, the most common type

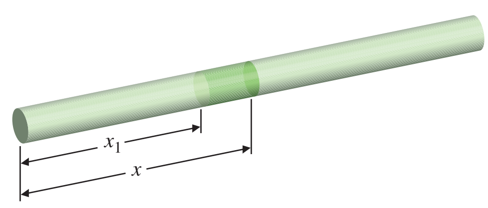



**Figure 3.88**
A thin rod.

of density discussed is **mass density** (mass per unit volume). You probably already have some idea of what we mean by mass density, but how would you define it? If the object of interest is made of some homogeneous material (i.e., the mass of any portion of the object of a given volume is the same), then the mass density is simply

$$\textit{mass density} = \frac{\textit{mass}}{\textit{volume}}$$

and this quantity is constant throughout the object. However, if the mass of a given volume varies in different parts of the object, then this formula only represents the *average density* of the object. In the next example we find a means of computing the mass density at a specific point in a nonhomogeneous object.

Suppose that the function $f(x)$ gives us the mass (in kilograms) of the first $x$ meters of a thin rod (see Figure 3.88).

The total mass between marks $x$ and $x_1$ $(x > x_1)$ is given by $f(x) - f(x_1)$ kg. The **average linear density** (i.e., mass per unit length) between $x$ and $x_1$ is then defined as

$$\frac{f(x) - f(x_1)}{x - x_1}.$$

Finally, the **linear density at** $x = x_1$ is defined as

$$\rho(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} = f'(x_1), \tag{8.2}$$

where we have again recognized the alternative definition of derivative.

### Example 8.3 Density of a Thin Rod

Suppose that the mass in a thin rod is given by $f(x) = \sqrt{2x}$. Compute the linear density at $x = 2$ and at $x = 8$ and compare the densities at the two points.

**Solution** From (8.2), we have

$$\rho(x) = f'(x) = \frac{1}{2\sqrt{2x}}(2) = \frac{1}{\sqrt{2x}}.$$

Thus, $\rho(2) = 1/\sqrt{4} = 1/2$ and $\rho(8) = 1/\sqrt{16} = 1/4$. Notice that this says that the rod is *inhomogeneous* (i.e., the mass density in the rod is not constant). Specifically, we have that the rod is less dense at $x = 8$ than at $x = 2$.

Our next example comes from medicine (cardiology, to be precise). The rate of change in this application is not actually a derivative. However, heart rate is one of the most familiar and

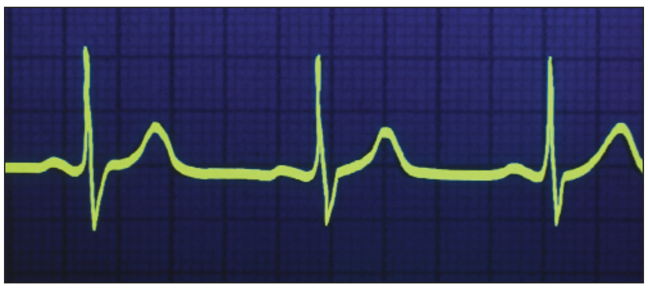


**Figure 3.89**
A normal electrocardiogram (EKG).

important rates of change in our lives. The language and concepts of calculus enable us to understand some of the exciting research currently being conducted in this field.

Cardiologists have long used graphs to help them identify heart problems. You are probably familiar with the term *electrocardiogram* (ECG or more commonly, EKG), a graph depicting some of the electrical activity of the heart. A small section of a normal EKG is shown in Figure 3.89.

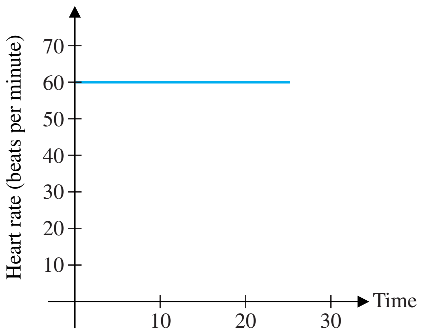
A well-trained cardiologist can determine an amazing amount of information from an EKG strip. Each section of the EKG (containing one peak) corresponds to a primary phase of heart activity, and the particular shape of the curve can indicate various problems with the heart. You should notice that the plot appears to be periodic (that is, the same shape is repeated over and over). It is a simple matter to measure the length of the period from peak to peak (each peak corresponds to a contraction of the ventricles of the heart, and the distance between peaks gives the heart rate). Specifically, if successive peaks occur at times $t_1$ and $t_2$ seconds, the **heart rate** equals

$$\frac{1 \text{ beat}}{t_2 - t_1 \text{ seconds}} = \frac{60}{t_2 - t_1} \text{ beats per minute.}$$

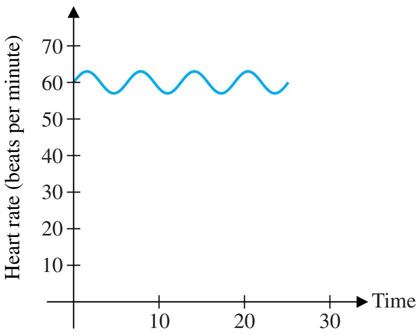
Note that heart rate is **not** a derivative. To get a derivative, we would need to have a function representing heart beats as a function of time and would then take the limit as $t_2 \to t_1$. From an EKG, all we have is a number of observations at specific times and therefore cannot compute a limit. Having defined heart rate, what can we learn about the human body?

## Example 8.4 Heart Rates and Electrocardiograms

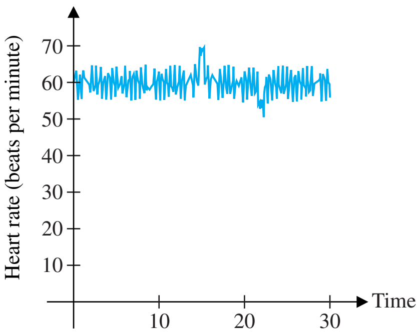
The following graphs depict heart **rate** over time. Which one would you say corresponds to a healthy individual at rest?




Constant rate




Smooth change




Erratic rate

**Solution** Before answering, we should clarify a couple of points. First, since heart rate is only measured at discrete time intervals (the intervals separating the peaks of the EKG), a graph of heart rate over time is like a computer graph: a finite set of points. If the points are connected and there are many points, the plot may appear to be smooth like the idealized plots we show. Second, be sure to read the graphs carefully! Each graph shows a person resting with a heart rate of about 60 beats per minute (bpm). The difference is in the variation from this average. The first person has a constant heart rate of exactly 60 bpm. The graph depicts a steady 1 beat per second, and is not the "flat-line" of a person whose heart has stopped beating. The second person shows a smooth rise and fall in heart rate, and the third person has a somewhat erratic heart rate.

Which heart rate do you think is healthy? Research by cardiologists in the 1980s and 1990s indicate that healthy hearts generally have a very erratic heart rate, even at rest (You're right if you picked plot 3!). Highly periodic heart rates like those in plot 2 have been observed in patients who were resting comfortably prior to experiencing a heart attack. The constant heart rate in plot 1 is typical of a person experiencing a heart attack. We should point out that the variations in the healthy third plot are too small to discern without sophisticated equipment. These observations have led to the design of a new type of heart monitor that can be worn by at-risk patients. Current research is investigating the possibility of designing "smart" heart pacemakers that can identify and avoid an impending crisis.

The following example comes from economics. Much as with heart rate, the rate of change discussed here is not precisely a derivative, but the derivative has proved to be a useful tool in economic modeling. In economics, the term **marginal** is used to indicate a rate. Thus, **marginal cost** is the derivative of the cost function, **marginal profit** is the derivative of the profit function, and so on.

### Example 8.5 Analyzing the Marginal Cost of Producing a Commercial Product

Suppose that

$$C(x) = 0.02x^2 + 2x + 4000$$

is the total cost (in dollars) for a company to produce $x$ units of a certain product. Compute the marginal cost at $x = 100$ and compare this to the actual cost of producing the 100th unit.

**Solution** You might think that it is an unfair assumption to start with a function that purports to represent cost. After all, cost is determined by accountants after a product is produced. That's true, but in order to project what your cost would be for quantities you haven't actually produced, it is helpful to develop a mathematical model of cost. In practice, this means that you make some observations of the cost of producing a number of different quantities and then try to fit that data to the graph of a known function, which you can then analyze using the tools of calculus. (This is one way in which the calculus is brought to bear on real-world problems.) The marginal cost function is

$$C'(x) = 0.04x + 2$$

and so, the marginal cost at $x = 100$ is $C'(100) = 4 + 2 = 6$ dollars per unit. On the other hand, the actual cost of producing item number 100 would be $C(100) - C(99)$.

(Why?) We have

$$C(100) - C(99) = 200 + 200 + 4000 - (196.02 + 198 + 4000)$$
$$= 4400 - 4394.02 = 5.98 \text{ dollars}.$$

Note that this is very close to the marginal cost of \$6.

Our final example comes from psychology. You have probably heard references to the "learning curve" for a piece of computer software or other technical product. The phrase comes from research attempting to quantify and understand the dynamics of learning something new.

**Example 8.6** Analyzing a Learning Curve

Suppose that the percentage of problems a person works correctly on a test is approximately

$$f(t) = \frac{80}{1 + 3e^{-0.4t}}$$

after $t$ hours of training. Graph the learning curve $y = f(t)$ and compute and interpret $f'(2)$ and $f'(10)$.

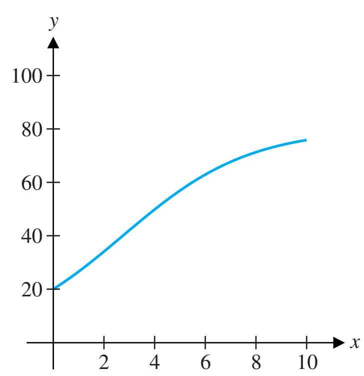


**Figure 3.90**
$y = 80/(1 + 3e^{-0.4t})$.

**Solution** The graph of $y = f(t)$ looks like the one shown in Figure 3.90.

Note that with no training the person gets 20% correct and after some steady improvement seems to have trouble getting beyond the 80% mark. To compute the derivative, note that we do not need to use the quotient rule, since the numerator is simply a constant. We rewrite the function as

$$f(t) = 80(1 + 3e^{-0.4t})^{-1}$$

and obtain

$$f'(t) = -80(1 + 3e^{-0.4t})^{-2}(-1.2e^{-0.4t})$$
$$= 96e^{-0.4t}(1 + 3e^{-0.4t})^{-2}.$$

After 2 hours, we then have

$$f'(2) = 96e^{-0.8}(1 + 3e^{-0.8})^{-2} \approx 7.8243 \text{ percent per hour.}$$

That is, after 2 hours, a person can expect to add about 7.8 points to the test score by training for an extra hour. At the 10-hour mark,

$$f'(10) = 96e^{-4}(1 + 3e^{-4})^{-2} \approx 1.5799 \text{ percent per hour.}$$

Thus, the next hour of training only increases the test score by about 1.6 points.

We have now discussed examples of six rates of change drawn from engineering and the sciences. Add these to the examples we have seen in previous sections (velocity, population growth, etc.) and we have an impressive list of applications of the derivative. Even so, we have barely begun to scratch the surface. In any field where it is possible to quantify and analyze the properties of a function, calculus and the derivative are powerful tools. This list includes at least some portion of every major field of study. The continued study of calculus will give you the ability to read (and understand) technical studies in a wide variety of fields and to see (as we have in this section) the underlying unity that mathematics brings to human endeavors.

# EXERCISES 3.8

1. A variable that is defined only at isolated points is called a **discrete** variable. For example, the size of a calculus class is discrete, since it must be an integer. A variable that can assume an interval of values is called **continuous.** In the six applications discussed in the text, in which cases is time a discrete variable and in which cases is it continuous? Which functions are discrete and which are continuous? Among the six examples given in the text, in which cases were the rates of change truly derivatives? Explain this answer in terms of discrete and continuous variables.

2. An important model of population growth is the so-called **logistic equation** $x'(t) = x(t)[1 - x(t)]$. Here, $x(t)$ represents not the actual population size but the proportion of sustainable capacity: for instance, $x(t) = 0.5$ means that the population is half of the total number of organisms that the environment can support and $x(t) = 1.1$ means that there are 10% more organisms than the available resources can support. Note that the differential equation here is the same as was used to describe an autocatalytic chemical reaction. The equation has two competing contributions to the rate of change $x'(t)$. The term $x(t)$ by itself would mean that the larger $x(t)$ is, the faster the population (or concentration of chemical) grows. This is balanced by the term $1 - x(t)$, which indicates that the closer $x(t)$ gets to 1, the slower the population growth is. With these two terms together, the model has the property that for small $x(t)$, slightly larger $x(t)$ means greater growth, but as $x(t)$ approaches 1, the growth tails off. Explain in terms of population growth and the concentration of chemical why the model is reasonable.

3. Corporate deficits and debt are frequently in the news, but the terms are often confused with each other. To take an example, suppose a company finishes a fiscal year owing \$5,000. That is their **debt.** Suppose that in the following year the company has revenues of \$106,000 and expenses of \$109,000. The company's **deficit** for the year is \$3,000 and the company's debt has increased to \$8,000. Briefly explain why deficit is the derivative of debt.

4. Many people find the healthy heart example (example 8.4) surprising. To make it more believable, explain how your level of activity affects your heart rate. Also, explain how breathing rate and emotional status can affect heart rate. Given these and many other factors, discuss whether you should expect your heart rate to be exactly constant.

5. Suppose that the charge in an electrical circuit is $Q(t) = e^{-2t}(\cos 3t - 2\sin 3t)$ coulombs. Find the current.

6. Suppose that the charge in an electrical circuit is $Q(t) = e^{t}(3\cos 2t + \sin 2t)$ coulombs. Find the current.

7. Suppose that the charge at a particular location in an electrical circuit is $Q(t) = e^{-3t}\cos 2t + 4\sin 3t$ coulombs. What happens to this function as $t \to \infty$? Explain why the term $e^{-3t}\cos 2t$ is called a **transient** term and $4\sin 3t$ is known as the **steady-state** or **asymptotic** value of the charge function. Find the transient and steady-state values of the current function.

8. As in exercise 7, find the steady-state and transient values of the current function if the charge function is given by $Q(t) = e^{-2t}(\cos t - 2\sin t) + te^{-3t} + 2\cos 4t$.

9. If the concentration of a chemical changes according to the equation $x'(t) = 2x(t)[4 - x(t)]$, find the concentration $x(t)$ for which the reaction rate is a maximum.

10. If the concentration of a chemical changes according to the equation $x'(t) = 0.5x(t)[5 - x(t)]$, find the concentration $x(t)$ for which the reaction rate is a maximum.

11. Show that in exercise 9, the maximum concentration is 4 if $0 < x(0) < 4$. Find the maximum concentration in exercise 10.

12. Find the equation for an autocatalytic reaction in which the maximum concentration is $x(t) = 16$ and the reaction rate equals 12 when $x(t) = 8$.

13. Mathematicians often study equations of the form $x'(t) = rx(t)[1 - x(t)]$ instead of the more complicated $x'(t) = cx(t)[K - x(t)]$, justifying the simplification with the statement that the second equation "reduces to" the first equation. Starting with $y'(t) = cy(t)[K - y(t)]$, substitute $y(t) = Kx(t)$ and show that the equation reduces to the form $x'(t) = rx(t)[1 - x(t)]$. How does the constant $r$ relate to the constants $c$ and $K$?

14. Suppose a chemical reaction follows the equation $x'(t) = cx(t)[K - x(t)]$. Suppose that at time $t = 4$ the concentration is $x(4) = 2$ and the reaction rate is $x'(4) = 3$. At time $t = 6$, suppose that the concentration is $x(6) = 4$ and the reaction rate is $x'(6) = 4$. Find the values of $c$ and $K$ for this chemical reaction.

15. In a general **second-order chemical reaction,** chemicals $A$ and $B$ (the **reactants**) combine to form chemical $C$ (the **product**). If the initial concentrations of the reactants $A$ and $B$ are $a$ and $b$, respectively, then the concentration $x(t)$ of the product satisfies the equation $x'(t) = [a - x(t)][b - x(t)]$. What is the rate of change of the product when $x(t) = a$? At this value, is the concentration of product increasing, decreasing or staying the same? Assuming that $a < b$ and there is no product present when the reaction starts, explain why the maximum concentration of product is $x(t) = a$.

16. For the second-order reaction defined in exercise 15, find the (mathematical) value of $x(t)$ that minimizes the reaction rate.

Show that the reaction rate for this value of $x(t)$ is negative. Explain why the concentration $x(t)$ would never get this large, so that this mathematical solution is not physically relevant. Explain why $x(t)$ must be between 0 and $a$, and find the maximum and minimum reaction rates on this closed interval.

17. It can be shown that a solution of the equation $x'(t) = [a - x(t)][b - x(t)]$ is given by

$$x(t) = \frac{a[1 - e^{-(b-a)t}]}{1 - (a/b)e^{-(b-a)t}}.$$

Find $x(0)$, the initial concentration of chemical, and $\lim_{t\to\infty} x(t)$, the limiting concentration of chemical (assume $a < b$). Graph $x(t)$ on the interval $[0, \infty)$ and describe in words how the concentration of chemical changes over time.

18. For the solution in exercise 17, find and graph $x'(t)$. Compute $\lim_{t\to\infty} x'(t)$ and describe in words how the reaction rate changes over time.

**In exercises 19–22, the mass of the first $x$ meters of a thin rod is given by the function $m(x)$ on the indicated interval. Find the linear density function for the rod. Based on what you find, briefly describe the composition of the rod.**

19. $m(x) = 4x - \sin x$ grams for $0 \le x \le 6$

20. $m(x) = (x - 1)^3 + 6x$ grams for $0 \le x \le 2$

21. $m(x) = 4x$ grams for $0 \le x \le 2$

22. $m(x) = 4x^2$ grams for $0 \le x \le 2$

**In exercises 23–26, you are given times (in seconds) at which peaks in a heart patient's EKG occurred. Analyze the heart rate and diagnose the patient as resting comfortably or in danger of a heart attack.**

23. 0.0, 0.99, 2.12, 3.19, 4.12, 5.17, 6.08, 6.95, 8.16, 9.26, 10.24, 11.28, 12.22, 13.14, 14.21, 15.18

24. 0.0, 0.99, 2.02, 3.01, 4.02, 5.03, 6.01, 6.99, 8.01, 9.02, 10.02, 11.02, 12.02, 13.01, 14.01, 15.02

25. 0.0, 0.98, 1.96, 2.90, 3.80, 4.74, 5.72, 6.70, 7.70, 8.72, 9.74, 10.80, 11.90, 13.06, 14.08, 15.10

26. 0.0, 0.98, 1.96, 3.02, 3.98, 5.04, 6.22, 7.17, 8.15, 9.17, 10.13, 11.08, 12.11, 13.06, 14.08, 15.10

27. If the cost of manufacturing $x$ items is $C(x) = x^3 + 20x^2 + 90x + 15$, find the marginal cost function and compare the marginal cost at $x = 50$ and the actual cost of manufacturing the 50th item.

28. If the cost of manufacturing $x$ items is $C(x) = x^4 + 14x^2 + 60x + 35$, find the marginal cost function and compare the marginal cost at $x = 50$ and the actual cost of manufacturing the 50th item.

29. If the cost of manufacturing $x$ items is $C(x) = x^3 + 21x^2 + 110x + 20$, find the marginal cost function and compare the marginal cost at $x = 100$ and the actual cost of manufacturing the 100th item.

30. If the cost of manufacturing $x$ items is $C(x) = x^3 + 11x^2 + 40x + 10$, find the marginal cost function and compare the marginal cost at $x = 100$ and the actual cost of manufacturing the 100th item.

31. Suppose the cost of manufacturing $x$ items is $C(x) = x^3 - 30x^2 + 300x + 100$ dollars. Find the inflection point and discuss the significance of this value in terms of the cost of manufacturing.

32. A baseball team owner has determined that if tickets are priced at \$10, the average attendance at a game will be 27,000 and if tickets are priced at \$8, the average attendance will be 33,000. Using a linear model, we would then estimate that tickets priced \$9 would produce an average attendance of 30,000. Discuss whether or not you think the use of a linear model here is reasonable. Then, using the linear model, determine the price at which the revenue is maximized.

33. The learning curve $f(t) = 80/(1 + 3e^{-0.4t})$ was discussed in example 8.6. Show that $\lim_{t\to\infty} f(t) = 80$ and show that $f(t)$ is an increasing function for $t > 0$. Discuss whether you think both results are reasonable properties for a learning curve.

34. Suppose the maximum score that a person will ever score on a test is 90. Further suppose that the person scores 60 after 2 hours of studying. Find a learning curve of the form $f(t) = a/(1 + 3e^{-bt})$ for this person on this test.

35. Suppose that a person scores $f(t) = 180/(2 + 4e^{-0.2t})$ after $t$ hours of studying. What is the person's score after 3 hours of studying? Find $f'(3)$ and estimate how many additional points the person would earn by studying a fourth hour. Find $f'(10)$ and estimate how many additional points the person would earn by studying an eleventh hour.

36. Suppose that a person scores $f(t) = 10/(1 + 4e^{-0.5t})$ after $t$ hours of studying. What is the person's score after 4 hours of studying? Find $f'(4)$ and estimate how many additional points the person would earn by studying a fifth hour. Find $f'(10)$ and estimate how many additional points the person would earn by studying an eleventh hour.

37. The function $f(t) = a/(1 + 3e^{-bt})$ has also been used to model the spread of a rumor. Suppose that $a = 70$ and $b = 0.2$. Compute $f(2)$, the percentage of the population that has heard the rumor after 2 hours. Compute $f'(2)$ and describe what it represents. Compute $\lim_{t\to\infty} f(t)$ and describe what it represents.

38. After an injection, the concentration of drug in a muscle varies according to a function of time, $f(t)$. Suppose that $t$ is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Determine the time when the maximum concentration of drug occurs.

39. Suppose that the size of the pupil of an animal is given by $f(x)$ (mm), where $x$ is the intensity of the light on the pupil. If

$$f(x) = \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15},$$

show that $f(x)$ is a decreasing function. Interpret this result in terms of the response of the pupil to light.

40. Suppose that the body temperature 1 hour after receiving $x$ mg of a drug is given by $T(x) = 102 - \frac{1}{6}x^2(1 - x/9)$ for $0 \le x \le 6$. The absolute value of the derivative, $|T'(x)|$, is defined as the **sensitivity** of the body to the drug dosage. Find the dosage which maximizes sensitivity.

41. Let $C(x)$ be the cost of manufacturing $x$ items and define $\bar{C}(x) = C(x)/x$ as the average cost function. Suppose that $C(x) = 0.01x^2 + 40x + 3600$. Show that $C'(100) < \bar{C}(100)$ and show that increasing the production ($x$) by 1 will decrease the average cost.

42. For the cost function in exercise 41, show that $C'(1000) > \bar{C}(1000)$ and show that increasing the production ($x$) by 1 will increase the average cost.

43. For the cost function in exercise 41, prove that average cost is minimized at the $x$-value where $C'(x) = \bar{C}(x)$.

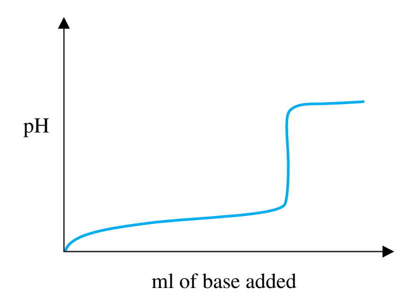
44. If the cost function is linear, $C(x) = a + bx$ with $a$ and $b$ positive, show that there is no minimum average cost and that $C'(x) \ne \bar{C}(x)$ for all $x$.

45. Let $R(x)$ be the revenue and $C(x)$ be the cost from manufacturing $x$ items. **Profit** is defined as $P(x) = R(x) - C(x)$. Show that at the value of $x$ that maximizes profit, marginal revenue equals marginal cost.

46. Find the maximum profit if $R(x) = 10x - 0.001x^2$ dollars and $C(x) = 2x + 5000$ dollars.

47. In the titration of a weak acid and strong base, the pH is given by $c + \ln \dfrac{f}{1 - f}$ where $c$ is a constant (closely related to the **acid dissociation constant**) and $f$ is the fraction $(0 < f < 1)$ of converted acid (see Harris' *Quantitative Chemical Analysis* for more details). Find the value of $f$ at which the rate of change of pH is the smallest. What happens as $f$ approaches 1?

48. In exercise 47, you found the significance of one inflection point of a titration curve. A second inflection point, called the **equivalence point,** corresponds to $f = 1$. In the generalized titration curve below, identify on the graph both inflection points and briefly explain why chemists prefer to measure the equivalence point and not the inflection point of exercise 47. (Note: the horizontal axis of a titration curve indicates the amount of base added to the mixture. This is directly proportional to the amount of converted acid in the region where $0 < f < 1$.)




49. Epidemiology is the study of the spread of infectious diseases. A simple model for the spread of fatal diseases such as AIDS divides people into the categories of susceptible (but not exposed), exposed (but not infected) and infected. The proportions of people in each category at time $t$ are denoted $S(t)$, $E(t)$ and $I(t)$, respectively. The general equations for this model are

$$\begin{aligned} S'(t) &= mI(t) - bS(t)I(t), \\ E'(t) &= bS(t)I(t) - aE(t), \\ I'(t) &= aE(t) - mI(t), \end{aligned}$$

where $m$, $b$ and $a$ are positive constants. Notice that each equation gives the rate of change of one of the categories. Each rate of change has both a positive and negative term. Explain why the positive term represents people who are entering the category and the negative term represents people who are leaving the category. In the first equation, the term $mI(t)$ represents people who have died from the disease (the constant $m$ is the reciprocal of the life expectancy of someone with the disease). This term is slightly artificial: the assumption is that the population is constant, so that when one person dies, a baby is born who is not exposed or infected. The dynamics of the disease is that susceptible (healthy) people get infected by contact with infected people. Explain why the number of contacts between susceptible people and infected people is proportional to $S(t)$ and $I(t)$. The term $bS(t)I(t)$, then, represents susceptible people who have been exposed by contact with infected people. Explain why this same term shows up as a positive in the second equation. Explain the rest of the remaining two equations in this fashion. (Hint: The constant $a$ represents the reciprocal of the average latency period. In the case of AIDS, this would be how long it takes an HIV-positive person to actually develop AIDS.)

50. Without knowing how to solve differential equations (we hope you will go far enough in your study of mathematics to learn to do so!), we can nonetheless deduce some

important properties of the solutions of differential equations. For example, consider the equation for an autocatalytic reaction $x'(t) = x(t)[1 - x(t)]$. Suppose $x(0)$ lies between 0 and 1. Show that $x'(0)$ is positive by determining the possible values of $x(0)[1 - x(0)]$. Explain why this indicates that the value of $x(t)$ will increase from $x(0)$, and will continue to increase as long as $0 < x(t) < 1$. Explain why if $x(0) < 1$ and $x(t) > 1$ for some $t > 0$, then it must be true that $x(t) = 1$ for some $t > 0$. However, if $x(t) = 1$, then $x'(t) = 0$ and the solution $x(t)$ stays constant (equal to 1). Therefore, we can conjecture that $\lim_{t\to\infty} x(t) = 1$. Similarly, show that if $x(0) > 1$, then $x(t)$ decreases and we could again conjecture that $\lim_{t\to\infty} x(t) = 1$. Changing equations, suppose that $x'(t) = -0.05x(t) + 2$. This is a model of an experiment in which a radioactive substance is decaying at the rate of 5% but the substance is being replenished at the constant rate of 2. Find the value of $x(t)$ for which $x'(t) = 0$. Pick various starting values of $x(0)$ less than and greater than the constant solution and determine if the solution $x(t)$ will increase or decrease. Based on these conclusions, conjecture the value of $\lim_{t\to\infty} x(t)$, the limiting amount of radioactive substance in the experiment.

# CHAPTER REVIEW EXERCISES

**In exercises 1 and 2, find the linear approximation to $f(x)$ at $x_0$.**

1. $f(x) = e^{3x}$, $x_0 = 0$
2. $f(x) = \sqrt{x^2+3}$, $x_0 = 1$

**In exercises 3 and 4, use a linear approximation to estimate the quantity.**

3. $\sqrt[3]{7.96}$
4. $\sin 3$

**In exercises 5 and 6, use Newton's method to find an approximate root.**

5. $x^3 + 5x - 1 = 0$
6. $x^3 = e^{-x}$
7. Explain why Newton's method fails on $x^3 - 3x + 2 = 0$ with $x_0 = 1$.
8. Show that the approximation $\frac{1}{(1-x)} \approx 1 + x$ is valid for "small" $x$.

**In exercises 9–18, do the following by hand. (a) Find all critical numbers, (b) identify all intervals of increase and decrease, (c) determine whether each critical number represents a local maximum, local minimum or neither, (d) determine all intervals of concavity and (e) find all inflection points.**

9. $f(x) = x^3 + 3x^2 - 9x$
10. $f(x) = x^4 - 4x + 1$
11. $f(x) = x^4 - 4x^3 + 2$
12. $f(x) = x^3 - 3x^2 - 24x$
13. $f(x) = xe^{-4x}$
14. $f(x) = x^2 \ln x$
15. $f(x) = x\sqrt{x^2-4}$
16. $f(x) = (x^2-1)^{2/3}$
17. $f(x) = \frac{x}{x^2+4}$
18. $f(x) = \frac{x}{\sqrt{x^2+2}}$

**In exercises 19–22, find the absolute extrema of the function on the interval.**

19. $f(x) = x^3 + 3x^2 - 9x$ on $[0, 4]$
20. $f(x) = x^3 + 3x^2 - 9x$ on $[-4, 0]$
21. $f(x) = x^{4/5}$ on $[-2, 3]$
22. $f(x) = x^2 e^{-x}$ on $[-1, 4]$

**In exercises 23–26, find the $x$-coordinates of all local extrema.**

23. $f(x) = x^3 + 4x^2 + 2x$
24. $f(x) = x^4 - 3x^2 + 2x$
25. $f(x) = x^5 - 2x^2 + x$
26. $f(x) = x^5 + 4x^2 - 4x$
27. Sketch a graph of a function with $f(-1) = 2$, $f(1) = -2$, $f'(x) < 0$ for $-2 < x < 2$, $f'(x) > 0$ for $x < -2$ and $x > 2$.
28. Sketch a graph of a function with $f'(x) > 0$ for $x \neq 0$, $f'(0)$ undefined, $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$.

**In exercises 29–38, sketch a graph of the function and completely discuss the graph.**

29. $f(x) = x^4 + 4x^3$
30. $f(x) = x^4 + 4x^2$
31. $f(x) = x^4 + 4x$
32. $f(x) = x^4 - 4x^2$
33. $f(x) = \frac{x}{x^2+1}$
34. $f(x) = \frac{x}{x^2-1}$
35. $f(x) = \frac{x^2}{x^2+1}$
36. $f(x) = \frac{x^2}{x^2-1}$
37. $f(x) = \frac{x^3}{x^2-1}$
38. $f(x) = \frac{4}{x^2-1}$

39. Find the point on the graph of $y = 2x^2$ that is closest to (2, 1).

40. Show that the line through the two points of exercise 39 is perpendicular to the tangent line to $y = 2x^2$ at (2, 1).

41. A city is building a highway from point $A$ to point $B$, which is 4 miles east and 6 miles south of point $A$. The first 4 miles south of point $A$ is swamp land, where the cost of building the highway is \$6 million per mile. On dry land, the cost is \$2 million per mile. Find the point on the boundary of swamp land and dry land to which the highway should be built to minimize the total cost.

42. Repeat exercise 41 with a cost of \$16 million per mile on swamp land. Explain why the optimal point in this exercise is west of the optimal point found in exercise 41.

43. A soda can in the shape of a cylinder is to hold 16 ounces. Find the dimensions of the can that minimizes the surface area of the can.

44. Suppose that $C(x) = 0.02x^2 + 4x + 1200$ is the cost of manufacturing $x$ items. Show that $C'(x) > 0$ and explain in business terms why this has to be true. Show that $C''(x) > 0$ and explain why this indicates that the manufacturing process is not very efficient.

45. The charge in an electrical circuit at time $t$ is given by $Q(t) = e^{-3t} \sin 2t$ coulombs. Find the current.

46. If the concentration $x(t)$ of a chemical in a reaction changes according to the equation $x'(t) = 0.3x(t)[4 - x(t)]$, find the concentration at which the reaction rate is a maximum.

47. Suppose that the mass of the first $x$ meters of a thin rod is given by $m(x) = 20 + x^2$ for $0 \le x \le 4$. Find the density of the rod and briefly describe the composition of the rod.

48. A person scores $f(t) = 90/(1 + 4e^{-0.4t})$ points on a test after $t$ hours of studying. What does the person score without studying at all? Compute $f'(0)$ and estimate how many points one hour of studying will add to the score.

49. The cost of manufacturing $x$ items is given by $C(x) = 0.02x^2 + 20x + 1800$. Find the marginal cost function. Compare the marginal cost at $x = 20$ to the actual cost of producing the 20th item.

50. For the cost function in exercise 49, find the value of $x$ which minimizes the average cost $\bar{C}(x) = C(x)/x$.

51. Let $n(t)$ be the number of photons in a laser field. One model of the laser action is $n'(t) = an(t) - b[n(t)]^2$, where $a$ and $b$ are positive constants. If $n(0) = a/b$, what is $n'(0)$? Based on this calculation, would $n(t)$ increase, decrease or neither? If $n(0) > a/b$, is $n'(0)$ positive or negative? Based on this calculation, would $n(t)$ increase, decrease or neither? If $n(0) < a/b$, is $n'(0)$ positive or negative? Based on this calculation, would $n(t)$ increase, decrease or neither? Putting this information together, conjecture the limit of $n(t)$ as $t \to \infty$. Repeat this analysis under the assumption that $a < 0$. [Hint: Because of its definition, $n(t)$ is positive, so ignore any negative values of $n(t)$.]

52. One way of numerically approximating a derivative is by computing the slope of a secant line. For example, $f'(a) \approx \dfrac{f(b) - f(a)}{b - a}$, if $b$ is close enough to $a$. In this exercise, we will develop an analogous approximation to the second derivative. Graphically, we can think of the secant line as an approximation of the tangent line. Similarly, we can match the second derivative behavior (concavity) with a parabola. Instead of finding the secant line through two points on the curve, we find the parabola through three points on the curve. The second derivative of this approximating parabola will serve as an approximation of the second derivative of the curve. The first step is messy, so we recommend using a CAS if one is available. Find a function of the form $g(x) = ax^2 + bx + c$ such that $g(x_1) = y_1$, $g(x_2) = y_2$ and $g(x_3) = y_3$. Since $g''(x) = 2a$, you actually only need to find the constant $a$. The so-called **second difference** approximation to $f''(x)$ is the value of $g''(x) = 2a$ using the three points $x_1 = x - \Delta x$ $[y_1 = f(x_1)]$, $x_2 = x$ $[y_2 = f(x_2)]$ and $x_3 = x + \Delta x$ $[y_3 = f(x_3)]$. Find the second difference for $f(x) = \sqrt{x + 4}$ at $x = 0$ with $\Delta x = 0.5$, $\Delta x = 0.1$ and $\Delta x = 0.01$. Compare to the exact value of the second derivative, $f''(0)$.