

### III. CONTROL VOLUME RELATIONS FOR FLUID ANALYSIS

From consideration of hydrostatics, we now move to problems involving fluid flow with the addition of effects due to fluid motion, e.g. inertia and convective mass, momentum, and energy terms.

We will present the analysis based on a control volume (not differential element) formulation, e.g. similar to that used in Thermodynamics for the First Law.

#### Basic Conservation Laws:

Each of the following basic conservation laws is presented in its most fundamental, **fixed mass** form. We will subsequently develop an equivalent expression for each law that includes the effects of the flow of mass, momentum, and energy (as appropriate) across a control volume boundary. These transformed equations will be the basis for the control volume analyses developed in this chapter.

#### **Conservation of Mass:**

Defining  $m$  as the mass of a fixed mass system, the mass for a control volume  $V$  is given by:

$$m_{\text{sys}} = \int_{\text{sys}} \rho dV$$

The basic equation for conservation of mass is then expressed as:

$$\left. \frac{dm}{dt} \right)_{\text{sys}} = 0$$

The time rate of change of mass for the control volume is zero since at this point we are still working with a fixed mass system.

#### **Linear Momentum:**

Defining  $\bar{P}_{\text{sys}}$  as the linear momentum of a fixed mass, the linear momentum of a fixed mass control volume is given by:



$$\bar{P}_{\text{sys}} = m\bar{V} = \int_{\text{sys}} \bar{V} \rho dV$$

where:  $\bar{V}$  is the local fluid velocity and  $dV$  is a differential volume element in the control volume.

The basic linear momentum equation is then written as:

$$\sum \bar{F} = \frac{d\bar{P}}{dt} \Bigg)_{\text{sys}} = \frac{d(m\bar{V})}{dt} \Bigg)_{\text{sys}}$$

### Moment of Momentum:

Defining  $\bar{H}$  as the moment of momentum for a fixed mass, the moment of momentum for a fixed mass control volume is given by:

$$\bar{H}_{\text{sys}} = \int_{\text{sys}} \bar{r} \times \bar{V} \rho dV$$

where  $\bar{r}$  is the moment arm from an inertial coordinate system to the differential control volume of interest. The basic equation is then written as:

$$\sum \bar{M}_{\text{sys}} = \sum \bar{r} \times \bar{F} = \frac{d\bar{H}}{dt} \Bigg)_{\text{sys}}$$

### Energy:

Defining  $E_{\text{sys}}$  as the total energy of an element of fixed mass, the energy of a fixed mass control volume is given by:

$$E_{\text{sys}} = \int_{\text{sys}} e \rho dV$$

where  $e$  is the total energy per unit mass ( includes kinetic, potential, and internal energy ) of the differential control volume element of interest.

The basic equation is then written as:

$$\left. \dot{Q} - \dot{W} = \frac{dE}{dt} \right)_{\text{sys}} \quad (\text{Note: written on a rate basis})$$

It is again noted that each of the conservation relations as previously written apply only to fixed, constant mass systems.

However, since most fluid problems of importance are for open systems, we must transform each of these relations to an equivalent expression for a control volume which includes the effect of mass entering and/or leaving the system.

This is accomplished with the Reynolds Transport Theorem.

### **Reynolds Transport Theorem**

We define a general, extensive property (an extensive property depends on the size or extent of the system),  $B_{\text{sys}}$ , where:

$$B_{\text{sys}} = \int_{\text{sys}} \beta \rho dV$$

$B_{\text{sys}}$  could be total mass, total energy, total momentum, etc. of a system.

and  $B_{\text{sys}}$  per unit mass is defined as  $\beta$  or  $\beta = \frac{dB}{dm}$

Thus,  $\beta$  is therefore the intensive equivalent of  $B_{\text{sys}}$

Applying a general control volume formulation to the time rate of change of  $B_{\text{sys}}$ , we obtain the following (see text for detailed development):

$$\left. \frac{dB}{dt} \right)_{\text{sys}} = \frac{\partial}{\partial t} \int_{\text{cv}} \beta \rho dV + \int_{A_e} \beta_e \rho_e V_e dA_e - \int_{A_i} \beta_i \rho_i V_i dA_i$$

$\downarrow$        $\downarrow$                                    $\downarrow$                                    $\downarrow$   
**System rate**              **Rate of**                                  **Rate of B**                                  **Rate of B**  
**of change**                      **change of**                                  **leaving c.v.**                                  **entering c.v.**  
**of B**                                      **B in c.v.**

**transient term    convective terms**

where: B is any conserved quantity, e.g. **mass, linear momentum, moment of momentum, or energy.**

We will now apply this theorem to each of the basic conservation equations to develop their equivalent open system, control volume forms.

### Conservation of Mass

For conservation of mass, we have that:

$$B = m \quad \text{and} \quad \beta = 1$$

From the previous statement of conservation of mass and these definitions, Reynolds Transport Theorem becomes:

$$\left. \frac{dm}{dt} \right)_{\text{sys}} = \frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{A_e} \rho_e \bar{V}_e \cdot d\bar{A}_e - \int_{A_i} \rho_i \bar{V}_i \cdot d\bar{A}_i = 0$$

or

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{A_e} \rho_e \bar{V}_e \cdot d\bar{A}_e - \int_{A_i} \rho_i \bar{V}_i \cdot d\bar{A}_i = 0$$

$\downarrow$                        $\downarrow$                                    $\downarrow$   
**Rate of change**              **Rate of mass**                                  **Rate of mass**  
**of mass in c.v.,**              **leaving c.v.,**                                  **entering c.v.,**  
 $\downarrow$                                    $\downarrow$                                    $\downarrow$   
**= 0 for steady-state**               $\dot{m}_e$                                    $\dot{m}_i$

This can be simplified to

$$\left. \frac{d m}{d t} \right)_{cv} + \sum \dot{m}_e - \sum \dot{m}_i = 0$$

Note that the exit and inlet velocities,  $V_e$  and  $V_i$ , are the local components of fluid velocities at the exit and inlet boundaries **relative to an observer standing on the boundary**. Therefore, if the boundary is moving, the velocity is measured relative to the boundary motion. The location and orientation of a coordinate system for the problem is not considered in determining these velocities.

Also, the result of  $\bar{V}_e \cdot d\bar{A}_e$  and  $\bar{V}_i \cdot d\bar{A}_i$  is the product of the normal velocity component times the flow area at the exit or inlet, e.g.

$$V_{e,n} dA_e \quad \text{and} \quad V_{i,n} dA_i$$

**Special Case:** For incompressible flow with a uniform velocity over the flow area, the previous integral expressions simplify to:

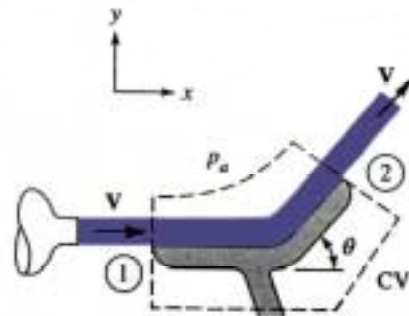
$$\dot{m} = \int \rho V d A = \rho A V$$

### Conservation of Mass Example

Water at a velocity of 7 m/s exits a stationary nozzle with  $D = 4$  cm and is directed toward a turning vane with  $\theta = 40^\circ$ , assume steady-state.

**Determine:**

- Velocity and flow rate entering the c.v.
- Velocity and flow rate leaving the c.v.



a. Find  $V_1$  and  $\dot{m}$

Recall that the mass flow velocity is the normal component of velocity measured **relative to the inlet or exit area**.

Thus, relative to the nozzle,  $V(\text{nozzle}) = 7 \text{ m/s}$  and since there is no relative motion of point 1 relative to the nozzle, we also have  **$V_1 = 7 \text{ m/s}$  ans.**

From the previous equation:

$$\dot{m} = \int \rho V dA = \rho AV = 998 \text{ kg/m}^3 * 7 \text{ m/s} * \pi * .04^2 / 4$$

$$\dot{m}_1 = \underline{\underline{8.78 \text{ kg/s}}} \text{ ans.}$$

b. Find  $V_2$  and  $\dot{m}_2$

Determine the flow rate first.

Since the flow is steady state and no mass accumulates on the vane:

$$\dot{m}_1 = \dot{m}_2, \quad \dot{m}_2 = \underline{\underline{8.78 \text{ kg/s}}} \text{ ans.}$$

Now:  $\dot{m}_2 = 8.78 \text{ kg/s} = \rho A V_2$

Since  $\rho$  and  $A$  are constant,  **$V_2 = 7 \text{ m/s}$  ans.**

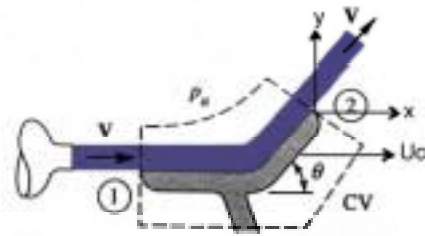
**Key Point:** For steady flow of a constant area, incompressible stream, the flow velocity and total mass flow are the same at the inlet and exit, even though the direction changes.

or alternatively:

**Flexible Hose Concept:** For steady flow of an incompressible fluid, the flow stream can be considered as an incompressible, flexible hose and if it enters a c.v. at a velocity of  $V$ , it exits at a velocity of  $V$ , even if it is redirected.

### Problem Extension:

Let the turning vane (and c.v.) now move to the right at a steady velocity of 2 m/s (other values remain the same), perform the same calculations.



Therefore:

$$\text{Given: } U_c = 2 \text{ m/s} \quad V_J = 7 \text{ m/s}$$

For an observer standing at the c.v. inlet (point 1)

$$V_1 = V_J - U_c = 7 - 2 = 5 \text{ m/s}$$

$$\dot{m}_1 = \rho_1 V_1 A_1 = 998 \text{ kg/m}^3 * 5 \text{ m/s} * \pi * .04^2 / 4 = 6.271 \text{ kg/s}$$

Note: The inlet velocity used to specify the mass flow rate is again measured relative to the inlet boundary, but now is given by  $V_J - U_c$ .

Exit:

$$\dot{m}_1 = \dot{m}_2 = 6.271 \text{ kg/s, } \text{Again, since } \rho \text{ and } A \text{ are cons., } V_2 = 5 \text{ m/s}$$

Again, the exit flow is most easily specified by conservation of mass concepts.

**Note:** The coordinate system could have been placed either on the moving cart or have been left off the cart with no change in the results.

**Key Point:** The location of the coordinate system does not affect the calculation of mass flow rate which is calculated relative to the flow boundary. It could have been placed at **Georgia Tech** with no change in the results.

Review material and work examples in the text on conservation of mass.



## Linear Momentum

For linear momentum, we have that:

$$\bar{B} = \bar{P} = m\bar{V} \quad \text{and} \quad \bar{\beta} = \bar{V}$$

From the previous statement of linear momentum and these definitions, Reynolds Transport Theorem becomes:

$$\sum \bar{F} = \left. \frac{dm\bar{V}}{dt} \right)_{\text{sys}} = \frac{\partial}{\partial t} \int_{\text{cv}} \bar{V} \rho dV + \int_{A_e} \bar{V} \rho_e \bar{V}_e \cdot d\bar{A}_e - \int_{A_i} \bar{V} \rho_i \bar{V}_i \cdot d\bar{A}_i$$

or

$$\sum \bar{F} = \frac{\partial}{\partial t} \int_{\text{cv}} \bar{V} \rho dV + \int_{A_e} \bar{V} d\dot{m}_e - \int_{A_i} \bar{V} d\dot{m}_i$$

↓  
= the  $\Sigma$  of the external forces acting on the c.v.

↓  
= the rate of change of momentum in the c.v.

↓  
= the rate of momentum leaving the c.v.

↓  
= the rate of momentum entering the c.v.

↓  
= body + point + distributed, e.g. (pressure) forces

↓  
= 0 for steady-state

and where  $\bar{V}$  is the vector momentum velocity relative to an inertial reference frame.

**Key Point:** Thus, the **momentum velocity has magnitude and direction** and is measured relative to the reference frame (coordinate system) being used for the problem. The **velocity in the mass flow terms,  $\dot{m}_i$  and  $\dot{m}_e$ , is a scalar**, as noted previously, and is **measured relative to the inlet or exit boundary**.

Always clearly define a coordinate system and use it to specify the value of all inlet and exit momentum velocities when working linear momentum problems.

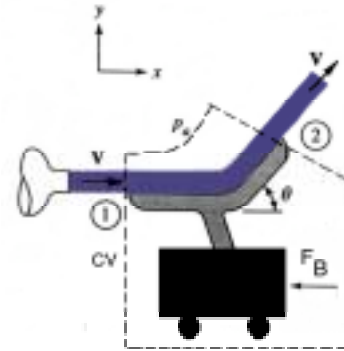
For the 'x' direction, the previous equation becomes:

$$\sum \bar{F}_x = \frac{\partial}{\partial t} \int_{cv} \bar{V}_x \rho dV + \int_{A_e} \bar{V}_{x,e} d\dot{m}_e - \int_{A_i} \bar{V}_{x,i} d\dot{m}_i$$

Note that the above equation is also valid for control volumes moving at constant velocity with the coordinate system placed on the moving control volume. This is because an inertial coordinate system is a non - accelerating coordinate system which is still valid for a c.s. moving at constant velocity.

**Example:**

A water jet 4 cm in diameter with a velocity of 7 m/s is directed to a stationary turning vane with  $\theta = 40^\circ$ . Determine the force,  $F$ , necessary to hold the vane stationary.



Governing Equation:

$$\sum \bar{F}_x = \frac{\partial}{\partial t} \int_{cv} \bar{V}_x \rho dV + \int_{A_e} \bar{V}_{x,e} d\dot{m}_e - \int_{A_i} \bar{V}_{x,i} d\dot{m}_i$$

Since the flow is steady and the c.v. is stationary, the time rate of change of momentum within the c.v. is zero. Also, with uniform velocity at each inlet and exit and a constant flow rate, the momentum equation becomes:

$$-F_b = \dot{m}_e V_e - \dot{m}_i V_i$$

Note that the braking force,  $F_b$ , is written as negative since it is **assumed** to be in the negative “x” direction relative to positive ‘x’ from the coordinate system.

From the previous example for conservation of mass, we can again write:

$$\dot{m} = \int \rho V dA = \rho AV = 998 \text{ kg/m}^3 * 7 \text{ m/s} * \pi * .04^2 / 4$$

$$\dot{m}_1 = \underline{8.78 \text{ kg/s}} \quad \text{and} \quad V_1 = \underline{7 \text{ m/s}}$$

and for the exit:

$$\dot{m}_2 = \underline{8.78 \text{ kg/s}} \quad \text{and} \quad V_2 = \underline{7 \text{ m/s}} \quad \text{inclined } 40^\circ \text{ above the horizontal}$$

Substituting in the momentum equation, we obtain:

$$-F_b = 8.78 \text{ kg/s} * 7 \text{ m/s} * \cos 40^\circ - 8.78 \text{ kg/s} * 7 \text{ m/s}$$

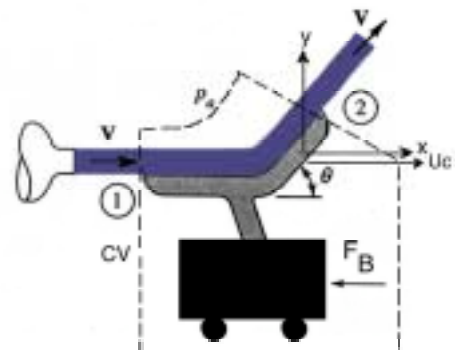
$$\text{and } -F_b = -14.4 \text{ kg m/s}^2 \quad \text{or} \quad F_b = \underline{14.4 \text{ N}} \leftarrow \text{ans.}$$

**Note:** Since our final answer is positive, our original assumption of the applied force being to the left was correct. Had we assumed that the applied force was to the right, our answer would be negative, meaning that the direction of the applied force is opposite of what was assumed.

### Modified Problem:

Now consider the same problem but with the cart moving to the right with a velocity,  $U_c = 2 \text{ m/s}$ . Again solve for the value of braking force,  $F_b$ , necessary to maintain a constant cart velocity of  $2 \text{ m/s}$ .

Note: The c.s. for the problem has now been placed on the moving cart. The momentum velocities will be evaluated relative to this coordinate system located on the moving cart.



The transient term in the momentum equation is still zero. With the coordinate system on the cart, the momentum of the cart relative to the coordinate system is still zero. The fluid stream is still moving relative to the coordinate system, however, the flow is steady with constant velocity and the time rate of change of momentum of the fluid stream is therefore also zero. Thus:

**The momentum equation has the same form as for the previous problem** (however, the value of individual terms will be different).

$$-F_b = \dot{m}_e V_e - \dot{m}_i V_i$$

$$\dot{m}_1 = \rho_1 V_1 A_1 = 998 \text{ kg/m}^3 * 5 \text{ m/s} * \pi * .04^2 / 4 = 6.271 \text{ kg/s} = \dot{m}_2$$

Now determine the momentum velocity at the inlet and exit. With the c.s. on the moving c.v., the values of momentum velocity are

$$V_1 = V_J - U_c = 7 - 2 = 5 \text{ m/s} \quad \text{and} \quad V_2 = 5 \text{ m/s inclined } 40^\circ .$$

The momentum equation ( “x” direction ) now becomes:

$$-F_b = 6.271 \text{ kg/s} * 5 \text{ m/s} * \cos 40^\circ - 6.271 \text{ kg/s} * 5 \text{ m/s}$$

$$\text{and } -F_b = -7.34 \text{ kg m/s}^2 \quad \text{or} \quad F_b = \underline{7.34 \text{ N}} \leftarrow \text{ans.}$$

**Question:** What would happen to the braking force,  $F_b$ , if the turning angle had been  $> 90^\circ$ , e.g.  $130^\circ$ ? Can you explain based on your understanding of change in momentum for the fluid stream?

Review and work examples for linear momentum with fixed and non - accelerating (moving at constant velocity) control volumes.

### Accelerating Control Volume

The previous formulation applies only to inertial coordinate systems, i.e. fixed or moving at constant velocity (non-accelerating).

We will now consider problems with accelerating control volumes. For these problems we will again place the coordinate system on the accelerating control volume thus making it a non-inertial coordinate system.

For coordinate systems placed on an accelerating control volume, we must account for the acceleration of the c.s. by correcting the momentum equation for this acceleration. This is accomplished by including the term as shown below:

$$\sum \bar{F} - \int_{cv} \bar{a}_{cv} dm_{cv} = \frac{\partial}{\partial t} \int_{cv} \bar{V} \rho dV + \int_{A_e} \bar{V} d\dot{m}_e - \int_{A_i} \bar{V} d\dot{m}_i$$

integral sum of  
the local c.v. (c.s.)  
accel. \* the c.v. mass

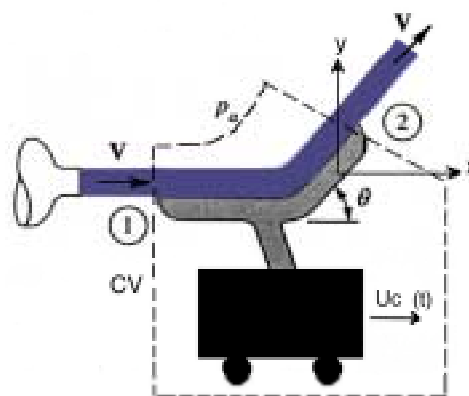
The added term accounts for the acceleration of the control volume and allows the problem to be worked with the coordinate system placed on the accelerating c.v.

Note: Thus, all vector (momentum) velocities are then measured relative to an observer (coordinate system) on the accelerating control volume. For example, the velocity of a rocket as seen by an observer (c.s.) standing on the rocket is zero and the time rate of change of momentum is zero in this reference frame even if the rocket is accelerating.

### Accelerating Control Volume Example

A turning vane with  $\theta = 60^\circ$  accelerates from rest due to a jet of water ( $V_J = 35 \text{ m/s}$ ,  $A_J = .003 \text{ m}^2$ ). Assuming the mass of the cart,  $m_c$ , is 75 kg and neglecting drag and friction effects, find:

a. Cart acceleration at  $t = 0$ .



b.  $Uc f(t)$

Starting with the general equation shown above, we can make the following assumptions:

1.  $\sum F_x = 0$ , no friction or body forces.
2. The jet has uniform velocity and constant properties.
3. The entire cart accelerates uniformly over the entire control volume.
4. Neglect the relative momentum change of the jet stream that is within the control volume.

With these assumptions, the governing equation simplifies to:

$$-a_c m_c = \dot{m}_e U_{x,e} - \dot{m}_i U_{x,i}$$

We thus have terms that account for the acceleration of the control volume, for the exit momentum, and for the inlet momentum (both of which change with time).

**mass flow:**

As with the previous example for a moving control volume, the mass flow terms are given by:

$$\dot{m}_i = \dot{m}_e = \dot{m} = \rho A_J (V_J - U_c)$$

Note that since the cart accelerates,  $U_c$  is not a constant, but rather changes with time.

**momentum velocities:**

$$U_{x,i} = V_J - U_c \quad U_{x,e} = (V_J - U_c) \cos \theta$$

Substituting, we now obtain:

$$-a_c m_c = \rho A_J (V_J - U_c)^2 \cos \theta - \rho A_J (V_J - U_c)$$

Solving for the cart acceleration, we obtain:

$$a_c = \frac{\rho A_J (1 - \cos \theta) (V_J - U_c)^2}{m_c}$$

Substituting for the given values at  $t = 0$ , i.e.  $U_c = 0$ , we obtain:

$$a_c (t = 0) = 24.45 \text{ m/s}^2 = 2.49 \text{ g's}$$

Note: The acceleration at any other time can be obtained once the cart velocity,  $U_c$ , at that time is known.

To determine the equation for cart velocity as a function of time, the equation for the acceleration must be written in terms of  $U_c(t)$  and integrated.

$$\frac{dU_c}{dt} = \frac{\rho A_J (1 - \cos\theta) (V_J - U_c)^2}{m_c}$$

Separating variables, we obtain:

$$\int_0^{U_c(t)} \frac{dU_c}{(V_J - U_c)^2} = \int_0^t \frac{\rho A_J (1 - \cos\theta)}{m_c} dt$$

Completing the integration and rearranging the terms, we obtain a final expression of the form:

$$\frac{U_c}{V_J} = \frac{V_J b t}{1 + V_J b t} \quad \text{where} \quad b = \frac{\rho A_J (1 - \cos\theta)}{m_c}$$

Substituting for known values, we obtain:  $V_J b = .699 \text{ s}^{-1}$

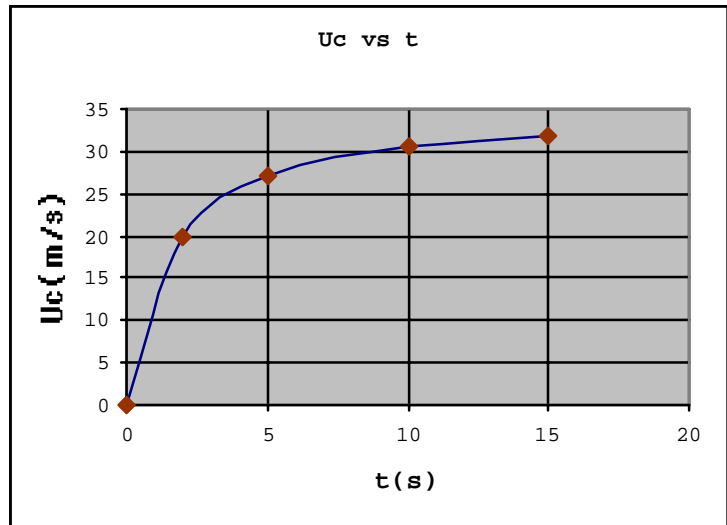
Thus, the final equation for  $U_c$  is give by:

$$\frac{U_c}{V_J} = \frac{0.699 t}{1 + 0.699 t}$$



The final results are now given as shown below:

t (s)	$U_c/V_J$	$U_c$ (m/s)	$a_c$ (m/s <sup>2</sup> )
0	0.0	0.0	24.45
2	.583	20.0	4.49
5	.757	27.2	1.22
10	.875	30.6	0.39
15	.912	31.9	.192
$\infty$	1.0	35	0.0



Note that the limiting case occurs when the cart velocity reaches the jet velocity. At this point, the jet can impart no more momentum to the cart, the acceleration is now zero, and the terminal velocity has been reached.

Review text example on accelerating control volumes.

### **Moment of Momentum (angular momentum)**

For moment of momentum we have that:

$$\bar{B} = \bar{H} = \bar{r}x(m\bar{V}) \quad \text{and} \quad \bar{\beta} = \bar{r}x\bar{V}$$

From the previous equation for moment of momentum and these definitions, Reynolds Transport Theorem becomes:



$R = 160 \text{ mm}$ ,  $D_J = 5 \text{ mm}$ , Therefore, for each of the two jets:

$$Q_J = 0.5 * 9.98 \text{ kg/min} / 998 \text{ kg/m}^3 = .005 \text{ m}^3/\text{min}$$

$$A_J = \pi .0025^2 = 1.963 * 10^{-5} \text{ m}^2$$

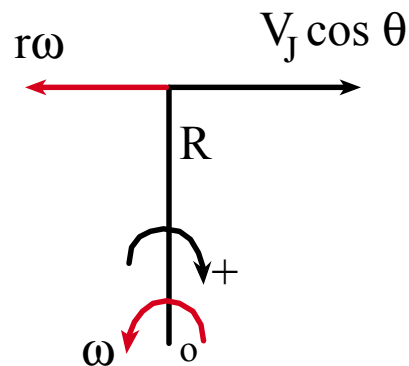
$$V_J = .005 \text{ m}^3/\text{min} / 1.963 * 10^{-5} \text{ m}^2 / 60 \text{ sec/min}$$

$$V_J = 4.24 \text{ m/s} \text{ relative to the nozzle exit } \textbf{ans.}$$

b. torque required to hold the arm stationary

First develop the governing equations and analysis for the general case of the arm rotating.

With the coordinate system at the center of rotation of the arm, a general velocity diagram for the case when the arm is rotating is shown in the adjacent schematic.



Taking the moment about the center of rotation, the moment of the inlet flow is zero since the moment arm is zero for the inlet flow.

The basic equation then becomes:

$$T_o = 2 \dot{m}_e R \{V_J \cos \alpha - R \omega\}$$

Note that the net momentum velocity is the difference between the tangential component of the jet exit velocity and the rotational speed of the arm. Also note that the direction of positive moments was taken as the same as for  $V_J$  and opposite of the direction of rotation.

For a stationary arm:  $R \omega = 0$ . We thus obtain for the stationary torque:

$$T_o = 2 \rho Q_J R V_J \cos \alpha$$

$$T_o = 2 * 998 \frac{kg}{m^3} * .005 \frac{m^3}{min} \frac{1 min}{60 sec} * .16m * 4.24 \frac{m}{s} \cos 40^\circ$$

$$T_o = 0.0864 \text{ N m clockwise. } \quad \mathbf{ans.}$$

A resisting torque of .0864 Nm must be applied in the clockwise direction to keep the arm from rotating in the counterclockwise direction.

c. At  $\omega = 30 \text{ rpm}$ , calculate the friction torque,  $T_f$

$$\omega = 30 \frac{rev}{min} 2\pi \frac{rad}{rev} \frac{1 min}{60 sec} = \pi \frac{rad}{sec}$$

$$T_o = 2 * 998 \frac{kg}{m^3} * .005 \frac{m^3}{min} \frac{1 min}{60 sec} * .16m \left[ 4.24 \frac{m}{s} \cos 40^\circ - .16m * \pi \frac{rad}{sec} \right]$$

$$T_o = .0685 \text{ Nm} \quad \mathbf{ans.}$$

Note: The resisting torque decreases as the speed increases.

d. Find the maximum rotational speed.

The maximum rotational speed occurs when the opposing torque is zero and all the moment of momentum goes to the angular rotation. For this case,

$$V_J \cos \theta - R\omega = 0$$

$$\omega = \frac{V_J \cos \theta}{R} = \frac{4.24m/s \cdot \cos 40}{.16 m} = 20.3 \frac{rad}{sec} = 193.8 \text{ rpm} \quad \mathbf{ans.}$$

Review material and examples on moment of momentum.

## Energy Equation (Extended Bernoulli Equation)

For energy, we have that

$$B = E = \int_{cv} e \rho dV \quad \text{and} \quad \beta = e = u + \frac{1}{2} V^2 + g z$$

From the previous statement of conservation of energy and these definitions, Reynolds Transport Theorem becomes

$$\left( \dot{Q} - \dot{W} = \frac{dE}{dt} \right)_{sys} = \frac{\partial}{\partial t} \int_{cv} e \rho dV + \int_{cv} e_e \rho_e \bar{V}_e \cdot d\bar{A}_e - \int_{cv} e_i \rho_i \bar{V}_i \cdot d\bar{A}_i$$

After extensive algebra and simplification (see text for detailed development), we obtain

$$\frac{P_1 - P_2}{\rho g} = \frac{V_2^2 - V_1^2}{2g} + z_2 - z_1 + h_{f,1-2} + h_s$$

$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
<b>Press. drop from 1 – 2, in the flow direction</b>	<b>Press. drop due to acceleration of the fluid</b>	<b>Press. drop due to elevation change</b>	<b>Press. drop due to frictional head loss</b>	<b>Press. drop due to mechanical work by fluid</b>

Note: this formulation must be written in the flow direction from 1 - 2 to be consistent with the sign of the mechanical work term and so that  $h_{f,1-2}$  is always a positive term. Also note the following:

- The points 1 and 2 must be specific points along the flow path,
- Each term has units of linear dimension, e.g. ft. or meters and  $z_2 - z_1$  is positive for  $z_2$  above  $z_1$ .
- The term  $h_{f,1-2}$  is always positive when written in the flow direction and for internal, pipe flow includes pipe or duct friction losses and fitting or piping component (valves, elbows, etc.) losses.

- The term  $h_s$  is negative for pumps and fans,  $-h_p$  ( i.e. pumps increase the pressure in the flow direction) and positive for turbines,  $+h_t$  (turbines decrease the pressure in the flow direction).

- For pumps:

$$h_p = \frac{w_s}{g} \quad \text{Where: } w_s = \text{the useful work per unit mass to the fluid}$$

$$\text{Therefore: } w_s = g h_p \quad \text{and} \quad \dot{W}_f = \dot{m} w_s = \rho Q g h_p$$

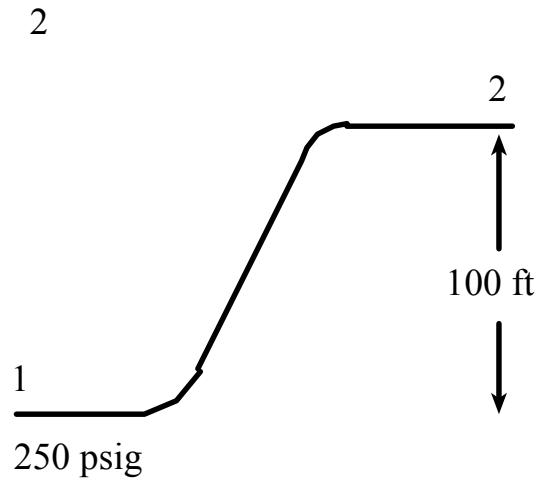
$$\text{Where: } \dot{W}_f = \text{the useful power delivered to the fluid}$$

$$\text{and: } \dot{W}_p = \frac{\dot{W}_f}{\eta_p} \quad \text{and} \quad \eta_p \text{ is the pump efficiency}$$

Water flows at 30 ft/s through a 1000 ft length of 2 in. diameter pipe. The inlet pressure is 250 psig and the exit is 100 ft higher than the inlet.

Assuming that the frictional loss is given by  $18 V^2/2g$ :

Determine the exit pressure.



$$\text{Given: } V_1 = V_2 = 30 \text{ ft/s, } L = 1000 \text{ ft, } Z_2 - Z_1 = 100 \text{ ft, } P_1 = 250 \text{ psig}$$

Also, since there is no mechanical work in the process, the energy equation simplifies to:

$$\frac{P_1 - P_2}{\rho g} = Z_2 - Z_1 + h_f$$

$$\frac{P_1 - P_2}{\rho g} = 100 \text{ ft} + 18 \frac{30^2 \text{ ft}^2 / \text{s}^2}{64.4 \text{ ft} / \text{s}^2} = 351.8 \text{ ft}$$

$$P_1 - P_2 = 62.4 \text{ lbf/ft}^3 \cdot 351.8 \text{ ft} = 21,949 \text{ psf} = 152.4 \text{ psi}$$

$$P_2 = 250 - 152.4 = 97.6 \text{ psig} \quad \mathbf{ans.}$$

### Problem Extension

A pump driven by an electric motor is now added to the system. The motor delivers 10.5 hp. The flow rate and inlet pressure remain constant and the pump efficiency is 71.4 %, determine the new exit pressure.

$$Q = AV = \pi (1/12)^2 \text{ ft}^2 \cdot 30 \text{ ft/s} = 0.6545 \text{ ft}^3/\text{s}$$

$$W_f = \eta_p W_p = \rho Q g h_p$$

$$h_p = \frac{.714 \cdot 10.5 \text{ hp} \cdot 550 \text{ ft} \cdot \text{lbf} / \text{s} / \text{hp}}{62.4 \text{ lbf} / \text{ft}^3 \cdot .6545 \text{ ft}^3 / \text{s}} = 101 \text{ ft}$$

The pump adds a head increase equal to 101 ft to the system and the exit pressure should increase.

Substituting in the energy equation, we obtain:

$$\frac{P_1 - P_2}{\rho g} = 100 \text{ ft} + 18 \frac{30^2 \text{ ft}^2 / \text{s}^2}{64.4 \text{ ft} / \text{s}^2} - 101 \text{ ft} = 250.8 \text{ ft}$$

$$P_1 - P_2 = 62.4 \text{ lbf/ft}^3 \cdot 250.8 \text{ ft} = 15650 \text{ psf} = 108.7 \text{ psi}$$

$$P_2 = 250 - 108.7 = 141.3 \text{ psig} \quad \mathbf{ans.}$$

Review examples for the use of the energy equation

## Kinetic Energy Correction Factor

Up to this point, the velocity used in the kinetic energy term of the energy equation has been the mass average velocity obtained from the definition of flow rate,

$$\dot{m} = \rho A V$$

However, since  $V$  varies over the flow area and the kinetic energy term varies with the square of the velocity, using this definition of  $V$  may not result in an accurate evaluation of the kinetic energy term for the flow.

This problem can be corrected through the use of the kinetic energy correction factor,  $\alpha$ , defined for incompressible flow from

$$\frac{1}{2} \rho \int u^3 dA = \frac{1}{2} \rho \alpha V_{av}^3 A$$

Solving for  $\alpha$  we obtain

$$\alpha = \frac{1}{A} \int \left( \frac{u}{V_{av}} \right)^3 dA$$

For fully developed laminar flow,  $\alpha = 2$  and for turbulent pipe flow, a value from 1.04 to 1.11. Using the kinetic energy correction factor, the energy equation becomes

$$\frac{P_1 - P_2}{\rho g} = \frac{\alpha V_2^2 - \alpha V_1^2}{2g} + z_2 - z_1 + h_{f,1-2} + h_s$$

While subsequent analyses and examples in these notes will continue to use the energy equation omitting the kinetic energy correction factor ( $\alpha = 1$ ), students are reminded to use this term where appropriate for pipe flow analyses.



## The Bernoulli Equation

The Bernoulli equation is an equation that is closely related to the energy equation and is useful in the analysis of many flows. The most common application is for steady, incompressible, frictionless flow between two points along a stream line. For these conditions, Bernoulli's equation becomes

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g z_2 = \text{const}$$

where the constant is the same along a specified streamline. Different streamlines may have different Bernoulli constants. Bernoulli's equation can also be written as

$$\frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) = 0$$

Students must be careful not to misuse Bernoulli's equation. In particular, do not use the Bernoulli equation for flows with any or all of the following:

1. friction between points 1 and 2
2. shaft work between points 1 and 2
3. heat transfer between points 1 and 2
4. significant compressibility effects between 1 and 2

## Hydraulic and Energy Grade Lines

Hydraulic and energy grade lines are lines that provide a very helpful visual representation of what is happening to key flow and energy equation parameters between points in the flow. They are defined as follows:

energy grade line (EGL) - a line that shows the variation of the height of the total Bernoulli constant,

$$h_o = \frac{P}{\rho g} + \frac{V^2}{2g} + z$$

The EGL has a constant height for steady, frictionless, incompressible flow with no heat transfer or shaft work.

For general flows, the EGL will decrease slowly in the direction of flow due to friction, increase across a pump and decrease across a turbine due to shaft work.

hydraulic grade line (HGL) - a line that shows the variations in the height of the elevation and pressure head,  $z + P/\gamma$  or the EGL minus the velocity head.

Fig. 3.17 shown below shows hydraulic and energy grade lines for frictionless flow in a variable area duct. Note that while the EGL remains constant in the absence of friction, shaft work or heat transfer, the HGL increases in velocity head in the flow direction.

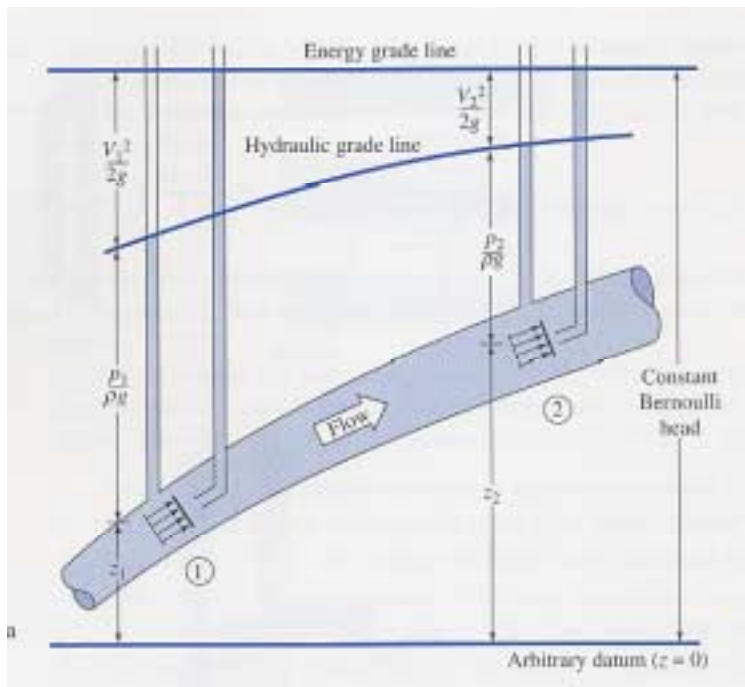


Fig. 3.17 Hydraulic and energy grade lines for frictionless flow in a duct