VII. BOUNDARY LAYER FLOWS

The previous chapter considered only viscous internal flows.

Viscous internal flows have the following major boundary layer characteristics:

* An entrance region where the boundary layer grows and $dP/dx \neq$ constant,
* A fully developed region where:
  - The boundary layer fills the entire flow area.
  - The velocity profiles, pressure gradient, and τ_w are constant; i.e. they are not equal to f(x),
  - The flow is either laminar or turbulent over the entire length of the flow, i.e. transition from laminar to turbulent is not considered.

However, viscous flow boundary layer characteristics for external flows are significantly different as shown below for flow over a flat plate:

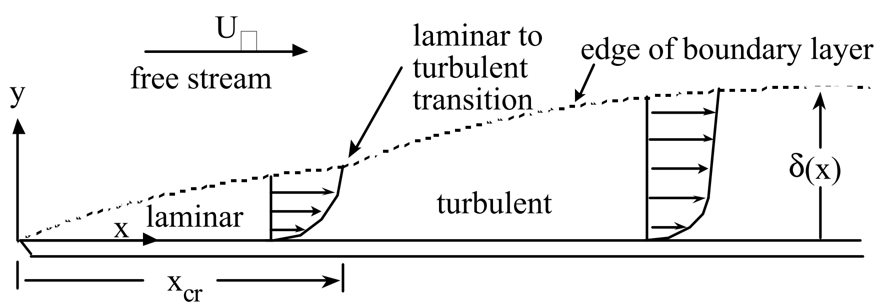


Fig. 7.1 Schematic of boundary layer flow over a flat plate

For these conditions, we note the following characteristics:

- The boundary layer thickness δ grows continuously from the start of the fluid-surface contact, e.g. the leading edge. It is a function of x, not a constant.
- Velocity profiles and shear stress τ are f(x,y).
- The flow will generally be laminar starting from x = 0.
- The flow will undergo laminar-to-turbulent transition if the streamwise dimension is greater than a distance x_{cr} corresponding to the location of the transition Reynolds number Re_{cr}.
- Outside of the boundary layer region, free stream conditions exist where velocity gradients and therefore viscous effects are typically negligible.

As it was for internal flows, the most important fluid flow parameter is the local Reynolds number defined as

$$\mathrm{Re}_x = \frac{\rho U_\infty x}{\mu} = \frac{U_\infty x}{\upsilon}$$

where

ρ = fluid density μ = fluid dynamic viscosity
ν = fluid kinematic viscosity U_∞ = characteristic flow velocity
x = characteristic flow dimension

It should be noted at this point that all external flow applications will not use a distance from the leading edge x as the characteristic flow dimension. For example, for flow over a cylinder, the diameter will be used as the characteristic dimension for the Reynolds number.

Transition from laminar to turbulent flow typically occurs at the local transition Reynolds number, which for flat plate flows can be in the range of

$$500,000 \le \mathrm{Re}_{cr} \le 3,000,00$$

With x_{cr} = the value of x where transition from laminar to turbulent flow occurs, the typical value used for steady, incompressible flow over a flat plate is

$$\mathrm{Re}_{cr} = \frac{\rho U_\infty x_{cr}}{\mu} = 500,000$$

Thus for flat plate flows for which:

$x < x_{cr}$ the flow is laminar

$x \geq x_{cr}$ the flow is turbulent

The solution to boundary layer flows is obtained from the reduced "Navier – Stokes" equations, i.e., Navier-Stokes equations for which boundary layer assumptions and approximations have been applied.

Flat Plate Boundary Layer Theory

<u>Laminar Flow Analysis</u>

For steady, incompressible flow over a flat plate, the laminar boundary layer equations are:

Conservation of mass: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

'X' momentum: $$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

'Y' momentum: $$-\frac{\partial p}{\partial y} = 0$$

The solution to these equations was obtained in 1908 by Blasius, a student of Prandtl's. He showed that the solution to the velocity profile, shown in the table below, could be obtained as a function of a single, non-dimensional variable η defined as

$$\eta = y\left(\frac{U_\infty}{\upsilon x}\right)^{1/2}$$

with the resulting ordinary differential equation:

$$f''' + \frac{1}{2} f\, f'' = 0$$

and $$f'(\eta) = \frac{u}{U_\infty}$$

Table 7.1 the Blasius Velocity Profile

$y[U/(\nu x)]^{1/2}$	u/U	$y[U/(\nu x)]^{1/2}$	u/U
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	∞	1.00000
2.6	0.77246		

Boundary conditions for the differential equation are expressed as follows:

at $y = 0$, $v = 0 \rightarrow f(0) = 0$; y component of velocity is zero at $y = 0$

at $y = 0$, $u = 0 \rightarrow f'(0) = 0$; x component of velocity is zero at $y = 0$

The key result of this solution is written as follows:

$$\left.\frac{\partial^2 f}{\partial \eta^2}\right)_{y=0} = 0.332 = \frac{\tau_w}{\mu U_\infty \sqrt{U_\infty / \upsilon x}}$$

With this result and the definition of the boundary layer thickness, the following key results are obtained for the laminar flat plate boundary layer:

Local boundary layer thickness $$\delta(x) = \frac{5x}{\sqrt{Re_x}}$$

Local skin friction coefficient:
(defined below) $$C_{f_x} = \frac{0.664}{\sqrt{Re_x}}$$

Total drag coefficient for length L (integration of τ_w dA over the length of the plate, per unit area, divided by 0.5 ρ U_∞^2) $$C_D = \frac{1.328}{\sqrt{Re_x}}$$

where by definition $C_{f_x} = \dfrac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2}$ and $C_D = \dfrac{F_D / A}{\frac{1}{2}\rho U_\infty^2}$

With these results, we can determine local boundary layer thickness, local wall shear stress, and total drag force for laminar flow over a flat plate.

Example:

Air flows over a sharp edged flat plate with L = 1 m, a width of 3 m and U_∞ = 2 m/s . For one side of the plate, find: δ(L), C_f (L), τ_w(L), C_D, and F_D.

Air: ρ = 1.23 kg/m^3 ν = 1.46 E-5 m^2/s

First check Re: $$Re_L = \frac{U_\infty L}{\upsilon} = \frac{2\, m/s * 2.15\, m}{1.46E-5\, m^2/s} = 294{,}520 < 500{,}000$$

Key Point: Therefore, the flow is laminar over the entire length of the plate and calculations made for any x position from 0 - 1 m must be made using laminar flow equations.

Boundary layer thickness at x = L:

$$\delta(L) = \frac{5\,L}{\sqrt{\mathrm{Re}_L}} = \frac{5 * 2.15\,m}{\sqrt{294,520}} = 0.0198\,m = 1.98\,cm$$

Local skin friction coefficient at x = L:

$$C_f(L) = \frac{0.664}{\sqrt{\mathrm{Re}_L}} = \frac{0.664}{\sqrt{294,520}} = 0.00122$$

Surface shear stress at x = L:

$$\tau_w = 1/2\,\rho U_\infty^2\, C_f = 0.5 * 1.23\,kg/m^3 * 2^2\,m^2/s^2 * 0.00122$$
$$\tau_w = 0.0030\,N/m^2\,(Pa)$$

Drag coefficient over total plate, 0 – L:

$$C_D(L) = \frac{1.328}{\sqrt{\mathrm{Re}_L}} = \frac{1.328}{\sqrt{294,520}} = 0.00245$$

Drag force over plate, 0 – L:

$$F_D = 1/2\,\rho U_\infty^2\, C_D\, A = 0.5 * 1.23\,kg/m^3 * 2^2\,m^2/s^2 * 0.00245 * 2 * 2.15\,m^2$$
$$F_D = 0.0259\,N$$

Two key points regarding this analysis:

1. Each of these calculations can be made for any other location on the plate by simply using the appropriate x location for any $x \le L$.

2. Be careful not to confuse the calculation for C_f and C_D.

   C_f is a local calculation at a particular x location (including x = L) and can only be used to calculate local shear stress at a specific x, not drag force.

   C_D is an integrated average over a specified length (including any $x \le L$) and can only be used to calculate the average shear stress over the entire plate and the integrated force over the total length.

Turbulent Flow Equations

While the previous analysis provides an excellent representation of laminar, flat plate boundary layer flow, a similar analytical solution is not available for turbulent flow due to the complex nature of the turbulent flow structure.

However, experimental results are available to provide equations for key flow field parameters.

A summary of the results for boundary layer thickness and local and average skin friction coefficient for a laminar flat plate and a comparison with experimental results for a smooth, turbulent flat plate are shown below.

Laminar	Turbulent
$\delta(x) = \frac{5x}{\sqrt{\mathrm{Re}_x}}$	$\delta(x) = \frac{0.16\,x}{\mathrm{Re}_x^{1/7}}$
$C_{f_x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f_x} = \frac{0.027}{\mathrm{Re}_x^{1/7}}$
$C_D = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$	$C_D = \frac{0.031}{\mathrm{Re}_L^{1/7}}$ for turbulent flow over entire plate, 0 – L, i.e. assumes turbulent flow in the laminar region

where

$$C_{f_x} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

local drag coefficient based on local wall shear stress (laminar or turbulent flow region)

and

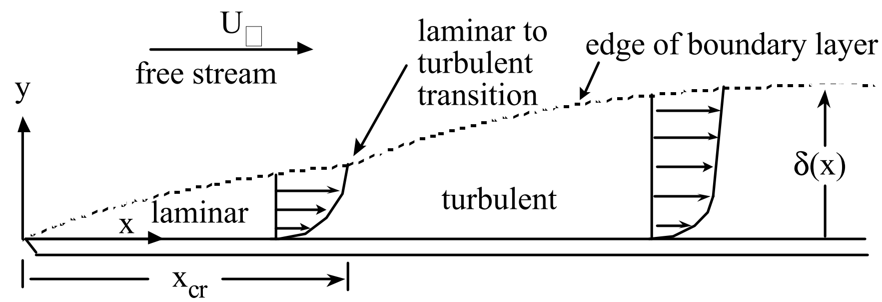
C_D = total drag coefficient based on the integrated force over the length 0 to L

$$C_D = \frac{F/A}{\frac{1}{2}\rho U_\infty^2} = \left(\frac{1}{2}\rho U_\infty^2 A\right)^{-1} \int_0^L \tau_w(x)\, w\, dx$$

A careful study of these results will show that, in general, boundary layer thickness grows faster for turbulent flow, and wall shear and total friction drag are greater for turbulent flow than for laminar flow given the same Reynolds number.

It is noted that the expressions for turbulent flow are valid only for a flat plate with a smooth surface. Expressions including the effects of surface roughness are available in the text.

Combined Laminar and Turbulent Flow



Flat plate with both laminar and turbulent flow sections

For conditions (as shown above) where the length of the plate is sufficiently long that we have both laminar and turbulent sections:

* Local values for boundary layer thickness and wall shear stress for either the laminar or turbulent sections are obtained from the expressions for $\delta(x)$ and Cf_x for laminar or turbulent flow as appropriate for the given region.
* The result for average drag coefficient C_D and thus total frictional force over the combined laminar and turbulent portions of the plate is given by (assuming a transition Re of 500,000)

$$C_D = \frac{0.031}{Re_L^{1/7}} = \frac{0.031}{\left(5x10^6\right)^{1/7}}$$

* Calculations assuming only turbulent flow can typically be made for two cases:
  1. when some physical situation (a trip wire) has caused the flow to be turbulent from the leading edge or
  2. if the total length L of the plate is much greater than the length x_{cr} of the laminar section such that the total length of plate can be considered turbulent from x = 0 to L. Note that this will over predict the friction drag force since turbulent drag is greater than laminar.

With these results, a detailed analysis can be obtained for laminar and/or turbulent flow over flat plates and surfaces that can be approximated as a flat plate.

Figure 7.6 in the text shows results for laminar, turbulent and transition regimes. Equations 7.48a & b can be used to calculate skin friction and drag results for the fully rough regime.

$$c_f \approx \left(2.87 + 1.58 \log \frac{x}{\varepsilon}\right)^{-2.5} \qquad (7.48a)$$

$$C_D \approx \left(1.89 + 1.62 \log \frac{L}{\varepsilon}\right)^{-2.5} \qquad (7.48b)$$

Equations 7.49a & b can be used to calculate total C_D for combined laminar and turbulent flow for transition Reynolds numbers of $5x10^5$ and $3x10^6$ respectively.

$$C_D \approx \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L} \qquad Re_{trans} = 5x10^5$$

$$C_D \approx \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L} \qquad Re_{trans} = 3x10^6$$

Example:

Water flows over a sharp flat plate 2.55 m long, 1 m wide, with $U_\infty = 2$ m/s. Estimate the error in F_D if it is assumed that the entire plate is turbulent.

Water: $\rho = 1000\ kg/m^3$ $\nu = 1.02\ E\text{-}\ m^2/s$

Reynolds number: $$Re_L = \frac{U_\infty L}{\upsilon} = \frac{2\,m/s * 2.55\,m}{1.02E-6\,m^2/s} = 5E6 > 500,000$$

with $Re_{cr} = 500,000 \Rightarrow x_{cr} = 0.255m$ (or 10% laminar)

a. Assume that the entire plate is turbulent

$$C_D \approx \frac{0.031}{Re_L^{1/7}} = \frac{0.031}{\left(5x10^6\right)^{1/7}} = 0.003423$$

$$F_D = 0.5\rho U_\infty^2 C_D A = 0.5 \cdot 1000 \frac{kg}{m^3} \cdot 2^2 \frac{m^2}{s^2} \cdot 0.003423 \cdot 2.55\,m^2$$

$F_D = 17.46\,N$ **This should be high since we have assumed that the entire plate is turbulent and the first 10% is actually laminar.**

b. Now consider the actual combined laminar and turbulent flow:

$$C_D \approx \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L} = \frac{0.031}{\left(5x10^6\right)^{1/7}} - \frac{1440}{5x10^6} = 0.003135$$

Note that the C_D has decreased when both the laminar and turbulent sections are considered.

$$F_D = 0.5\rho U_\infty^2 C_D A = 0.5 \cdot 1000 \frac{kg}{m^3} \cdot 2^2 \frac{m^2}{s^2} \cdot 0.003135 \cdot 2.55\,m^2$$

$F_D = 15.99\,N$ { Lower than the fully turbulent value}

$$Error = \frac{17.46 - 15.99}{15.99} \cdot 100 = 9.2\%\ high$$

Thus, the effect of neglecting the laminar region and assuming the entire plate is turbulent is as expected.

Question: Since x_{cr} = 0.255 m, what would your answers represent if you had calculated the Re, C_D, and F_D using $x = x_{cr}$ = 0.255 m?

Answer: You would have the value of the transition Reynolds number and the drag coefficient and drag force over the laminar portion of the plate (assuming you used laminar equations).

If you had used turbulent equations you would have red marks on your paper.

Von Karman Integral Momentum Analysis

While the previous results provide an excellent basis for the analysis of flat plate flows, complex geometries and boundary conditions make analytical solutions to most problems difficult.

An alternative procedure provides the basis for an approximate solution which in many cases can provide excellent results.

The key to practical results is to use a reasonable approximation to the boundary layer profile, u(x,y). This is used to obtain the following:

a. Boundary layer mass flow: $$\dot{m} = \int_0^\delta \rho\, u\, b \; dy$$

where b is the width of the area for which the flow rate is being obtained.

b. Wall shear stress: $$\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0}$$

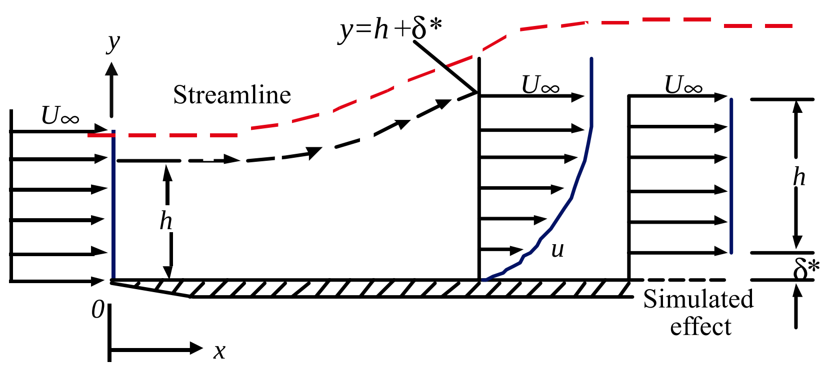
You will also need the streamwise pressure gradient $\dfrac{dP}{dx}$ for many problems.

The Von Karman integral momentum theory provides the basis for such an approximate analysis. The following summarizes this theory.

<u>Displacement thickness</u>:

Consider the problem indicated in the adjacent figure:

A uniform flow field with velocity U_∞ approaches a solid surface. As a result of viscous shear, a boundary layer velocity profile develops.



A viscous boundary layer is created when the flow comes in contact with the solid surface.

Key Point: Compared to the uniform velocity profile approaching the solid surface, the effect of the viscous boundary layer is to displace streamlines of the flow outside the boundary layer away from the wall.

With this concept, we define δ^* = displacement thickness

δ^* = distance the solid surface would have to be displaced to maintain the same mass flow rate as for non-viscous flow.

From the development in the text, we obtain

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

Thus, the displacement thickness varies only with the local non-dimensional velocity profile. Therefore, with an expression for u / U_∞, we can obtain δ^* = $f(\delta)$.

Example:

Given: $\dfrac{u}{U_\infty} = 2\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2$ determine an expression for $\delta^* = f(\delta)$

Note that for this assumed form for the velocity profile:

1. At $y = 0$, $u = 0$ correct for no slip condition
2. At $y = \delta$, $u = U_\infty$ correct for edge of boundary layer
3. The form is quadratic

To simplify the mathematics,

let $\eta = y/\delta$, at $y = 0$, $\eta = 0$; at $y = \delta$, $\eta = 1$; $dy = \delta\, d\eta$

Therefore: $\dfrac{u}{U_\infty} = 2\eta - \eta^2$

Substituting: $$\delta^* = \int_0^1 \left(1 - 2\eta + \eta^2\right)\delta \, d\eta = \delta\left\{\eta - \frac{2\eta^2}{2} + \frac{\eta^3}{3}\right\}_0^1$$

which yields $$\delta^* = \tfrac{1}{3}\delta$$

Therefore, for flows for which the assumed quadratic equation approximates the velocity profile, streamlines outside of the boundary layer are displaced approximately according to the equation

$$\delta^* = \tfrac{1}{3}\delta$$

This closely approximates flow for a flat plate.

Key Point: When assuming a form for a velocity profile to use in the Von Karman analysis, make sure that the resulting equation satisfies both surface and free stream boundary conditions as well as has a form that approximates u(y).

Momentum Thickness:

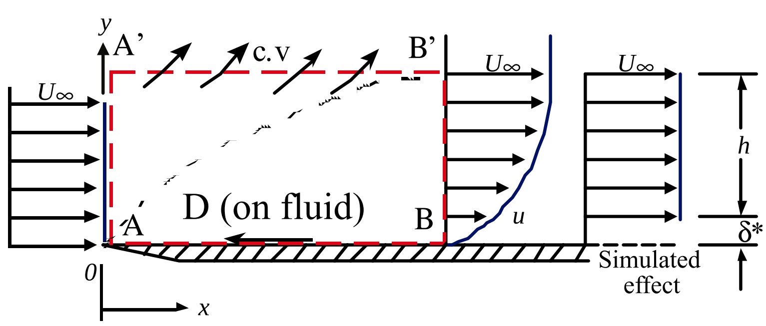
The second concept used in the Von Karman momentum analysis is that of

momentum thickness - θ

The concept is similar to that of displacement thickness in that θ is related to the loss of momentum due to viscous effects in the boundary layer.

Consider the viscous flow regions shown in the adjacent figure.

Define a control volume as shown and integrate around the control volume to obtain the net change in momentum for the control volume.



If D = drag force on the plate due to viscous flow, taking the fluid as the control volume, we can write

$$- D = \sum (\text{momentum leaving c.v.}) - \sum (\text{momentum entering c.v.})$$

Completing an analysis shown in the text, we obtain

$$D = \rho U_\infty^2 \theta \qquad\qquad \theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Using a drag coefficient defined as $\quad C_D = \dfrac{D/A}{\frac{1}{2}\rho U_\infty^2}$

We can also show that $\quad C_D = \dfrac{2\,\theta(L)}{L}$

where: θ(L) is the momentum thickness evaluated over the length L.

Thus, knowledge of the boundary layer velocity distribution u = f(y) also allows the drag coefficient to be determined.

Momentum integral:

The final step in the Von Karman theory applies the previous control volume analysis to a differential length of surface. Performing an analysis similar to the previous analysis for drag D we obtain

$$\boxed{\frac{\tau_w}{\rho} = \delta^* U_\infty \frac{dU_\infty}{dx} + \frac{d}{dx}\left(U_\infty^2 \theta\right)}$$

This is the momentum integral for 2-D, incompressible flow and is valid for laminar or turbulent flow.

where $\delta^* U_\infty \dfrac{dU_\infty}{dx} = -\dfrac{\delta^*}{\rho}\dfrac{dP}{dx}$ Therefore, this analysis also accounts for the effect of freestream pressure gradient.

For a flat plate with non-accelerating flow, we can show that

$$P = const., \quad U_\infty = const., \quad \frac{dU_\infty}{dx} = 0$$

Therefore, for a flat plate, non-accelerating flow, the Von Karman momentum integral becomes

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U_\infty^2 \theta\right) = U_\infty^2 \frac{d\theta}{dx}$$

From the previous analysis and the assumed velocity distribution of

$$\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$$

The wall shear stress can be expressed as

$$\tau_w = \mu \left.\frac{du}{dy}\right)_w = 2U_\infty \left\{\frac{2}{\delta} - \frac{2y}{\delta^2}\right\}_{y=0} = \frac{2\mu U_\infty}{\delta} \qquad \text{(A)}$$

Also, with the assumed velocity profile, the momentum thickness θ can be evaluated as

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

or

$$\theta = \int_0^{\delta} \left(2\eta - \eta^2\right)\left(1 - 2\eta + \eta^2\right)\delta\, d\eta = \quad \frac{2\delta}{15}$$

We can now write from the previous equation for τ_w

$$\tau_w = \rho\, U_\infty^2 \frac{d\theta}{dx} = \frac{2}{15}\rho\, U_\infty^2 \frac{d\delta}{dx}$$

Equating this result to Eqn. A we obtain

$$\tau_w = \frac{2}{15}\rho\, U_\infty^2 \frac{d\delta}{dx} = \frac{2\mu U_\infty}{\delta}$$

or

$$\delta\, d\delta = \frac{15\mu}{\rho U_\infty} dx \qquad \text{which after integration yields}$$

$$\delta = \left\{\frac{30\,\mu\, x}{\rho U_\infty}\right\}^{1/2} \qquad \text{or} \qquad \delta = \frac{5.48}{\sqrt{\mathrm{Re}_x}}$$

Note that this result is within 10% of the exact result from Blasius flat plate theory.

Since for a flat plate, we only need to consider friction drag (not pressure drag), we can write

$$\mathrm{C}_{\mathrm{f}_x} = \frac{\tau_{\mathrm{w}}(\mathrm{x})}{\frac{1}{2}\rho \mathrm{U}_\infty^2} = \frac{2\,\mu U_\infty}{\delta}\frac{1}{\frac{1}{2}\rho\mathrm{U}_\infty^2}$$

Substitute for δ to obtain

$$\mathrm{C}_{\mathrm{f}_x} = \frac{2\,\mu\, U_\infty}{5.48}\frac{\sqrt{\mathrm{Re}}}{\frac{1}{2}\rho\mathrm{U}_\infty^2} = \frac{0.73}{\sqrt{\mathrm{Re}_x}}$$

Exact theory has a numerical constant of 0.664 compared with 0.73 for the Von Karman integral analysis.

It is seen that the Von Karman integral theory provides the means to determine approximate expressions for

$$\delta, \tau_w, \text{ and } C_f$$

using only an assumed velocity profile.

Solution summary:

1. Assume an analytical expression for the velocity profile for the problem.
2. Use the assumed velocity profile to determine the solution for the displacement thickness for the problem.
3. Use the assumed velocity profile to determine the solution for the momentum thickness for the problem.
4. Use the previous results and the Von Karman integral momentum equation to determine the solution for the drag/wall shear for the problem.

and 7.3 and does not have to be accounted for separately.