CHAPTER

3

STRESS AND STRAIN

Outline

3.1 INTRODUCTION

This chapter provides a review and insight into the stress and strain analyses. Expressions for both stresses and deflections in mechanical elements are developed throughout the text as the subject unfolds, after examining their function and general geometric behavior. With the exception of Sections 3.12 through 3.18, we employ mechanics of materials approach, simplifying the assumptions related to the deformation pattern so that strain distributions for a cross section of a member can be determined. A fundamental assumption is that *plane sections remain plane*. This hypothesis can be shown to be exact for axially loaded elastic prismatic bars and circular torsion members and for slender beams, plates, and shells subjected to pure bending. The assumption is approximate for other stress analysis problems. Note, however, that there are many cases where applications of the *basic formulas of mechanics of materials*, so-called elementary formulas for stress and displacement, lead to useful results for slender members under any type of loading.

Our coverage presumes a knowledge of mechanics of materials procedures for determining stresses and strains in a homogeneous and an isotropic bar, shaft, and beam. In Sections 3.2 through 3.9, we introduce the basic formulas, the main emphasis being on the underlying assumptions used in their derivations. Next to be treated are the transformation of stress and strain at a point. Then attention focuses on stresses arising from various combinations of fundamental loads applied to members and the stress concentrations. The chapter concludes with discussions on contact stresses in typical members referring to the solutions obtained by the methods of the theory of elasticity and the general states of stress and strain.

In the treatment presented here, the study of complex stress patterns at the supports or locations of concentrated load is not included. According to Saint-Venant's Principle (Section 1.4), the actual stress distribution closely approximates that given by the formulas of the mechanics of materials, except near the restraints and geometric discontinuities in the members. For further details, see texts on solid mechanics and theory of elasticity; for example, References 1 through 3.

3.2 STRESSES IN AXIALLY LOADED MEMBERS

Axially loaded members are structural and machine elements having straight longitudinal axes and supporting only axial forces (tensile or compressive). Figure 3.1a shows a homogeneous prismatic bar loaded by tensile forces P at the ends. To determine the normal stress, we make an imaginary cut (section a-a) through the member at right angles to its

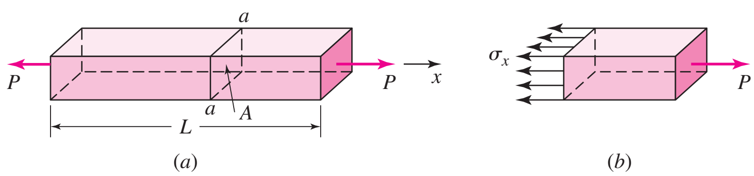


Figure 3.1 Prismatic bar in tension.

axis (x). A free-body diagram of the isolated part is shown in Figure 3.1b. Here the stress is substituted on the cut section as a replacement for the effect of the removed part.

Assuming that the stress has a uniform distribution over the cross section, the equilibrium of the axial forces, the first of Eqs. (1.4), yields $P = \int \sigma_x \, dA$ or $P = A\sigma_x$. The normal stress is therefore

$$\sigma_x = \frac{P}{A} \tag{3.1}$$

where A is the cross-sectional area of the bar. The remaining conditions of Eqs. (1.4) are also satisfied by the stress distribution pattern shown in Figure 3.1b. When the member is being stretched as depicted in the figure, the resulting stress is a uniaxial tensile stress; if the direction of the forces is reversed, the bar is in compression and uniaxial compressive stress occurs. Equation (3.1) is applicable to tension members and chunky, short compression bars. For slender members, the approaches discussed in Chapter 6 must be used.

Stress due to the restriction of thermal expansion or contraction of a body is called *thermal stress,* σ_t. Using Hooke's law and Eq. (1.21), we have

$$\sigma_t = \alpha(\Delta T)E \tag{3.2}$$

The quantity ΔT represents a temperature change. We observe that a high modulus of elasticity E and high coefficient of expansion α for the material increase the stress.

Design of Tension Members

Tension members are found in bridges, roof trusses, bracing systems, and mechanisms. They are used as tie rods, cables, angles, channels, or combinations of these. Of special concern is the design of prismatic tension members for strength under static loading. In this case, a rational design procedure (see Section 1.6) may be briefly described as follows:

1. *Evaluate the mode of possible failure*. Usually the normal stress is taken to be the quantity most closely associated with failure. This assumption applies regardless of the type of failure that may actually occur on a plane of the bar.
2. *Determine the relationships between load and stress*. This important value of the normal stress is defined by $\sigma = P/A$.
3. *Determine the maximum usable value of stress*. The maximum usable value of σ without failure, σ_{max}, is the yield strength S_y or the ultimate strength S_u. Use this value in connection with equation found in step 2, if needed, in any expression of failure criteria, discussed in Chapter 7.
4. *Select the factor of safety*. A safety factor n is applied to σ_{max} to determine the allowable stress $\sigma_{all} = \sigma_{max}/n$. The required cross-sectional area of the member is therefore

$$A = \frac{P}{\sigma_{all}} \tag{3.3}$$

If the bar contains an abrupt change of cross-sectional area, the foregoing procedure is repeated, using a stress concentration factor to find the normal stress (step 2).

EXAMPLE 3.1

Design of a Hoist

A pin-connected two-bar assembly or hoist is supported and loaded as shown in Figure 3.2a. Determine the cross-sectional area of the round aluminum eyebar *AC* and the square wood post *BC*.

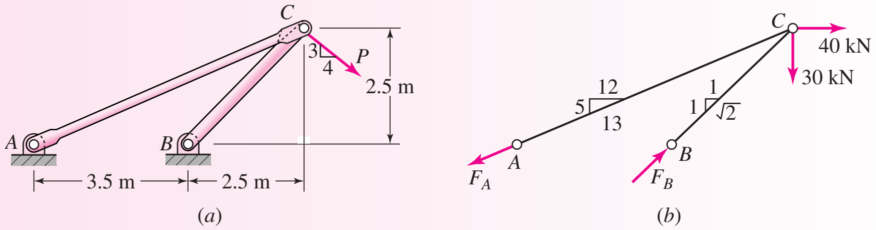


Figure 3.2 Example 3.1.

Given: The required load is $P = 50$ kN. The maximum usable stresses in aluminum and wood are 480 and 60 MPa, respectively.

Assumptions: The load acts in the plane of the hoist. Weights of members are insignificant compared to the applied load and omitted. Friction in pin joints and the possibility of member *BC* buckling are ignored.

Design Decision: Use a factor of safety of $n = 2.4$.

Solution: Members *AC* and *BC* carry axial loading. Applying equations of statics to the free-body diagram of Figure 3.2b, we have

$$\sum M_B = -40(2.5) - 30(2.5) + \frac{5}{13} F_A(3.5) = 0 \qquad F_A = 130 \text{ kN}$$

$$\sum M_A = -40(2.5) - 30(6) + \frac{1}{\sqrt{2}} F_B(3.5) = 0 \qquad F_B = 113.1 \text{ kN}$$

Note, as a check, that $\sum F_x = 0$.

The allowable stress, from design procedure steps 3 and 4,

$$(\sigma_{\text{all}})_{AC} = \frac{480}{2.4} = 200 \text{ MPa}, \qquad (\sigma_{\text{all}})_{BC} = \frac{60}{2.4} = 25 \text{ MPa}$$

By Eq. (3.3), the required cross-sectional areas of the bars,

$$A_{AC} = \frac{130(10^3)}{200} = 650 \text{ mm}^2, \qquad A_{BC} = \frac{113.1(10^3)}{25} = 4524 \text{ mm}^2$$

Comment: A 29-mm diameter aluminum eyebar and a 68 mm × 68 mm wood post should be used.

3.3 DIRECT SHEAR STRESS AND BEARING STRESS

A *shear stress* is produced whenever the applied forces cause one section of a body to tend to slide past its adjacent section. As an example consider the connection shown in Figure 3.3a. This joint consists of a bracket, a clevis, and a pin that passes through holes in the bracket and clevis. The pin resists the shear across the two cross-sectional areas at *b-b* and *c-c*; hence, it is said to be in *double shear*. At each cut section, a shear force V equivalent to $P/2$ (Figure 3.3b) must be developed. Thus, the shear occurs over an area parallel to the applied load. This condition is termed *direct shear*.

The distribution of shear stress τ across a section cannot be taken as uniform. Dividing the total shear force V by the cross-sectional area A over which it acts, we can obtain the average shear stress in the section:

$$\tau_{avg} = \frac{V}{A} \tag{3.4}$$

The average shear stress in the pin of the connection shown in the figure is therefore $\tau_{avg} = (P/2)/(\pi d^2/4) = 2P/\pi d^2$. Direct shear arises in the design of bolts, rivets, welds, glued joints, as well as in pins. In each case, the shear stress is created by a direct action of the forces in trying to cut through the material. Shear stress also arises in an indirect manner when members are subjected to tension, torsion, and bending, as discussed in the following sections.

Note that, under the action of the applied force, the bracket and the clevis press against the pin in bearing and a nonuniform pressure develops against the pin (Figure 3.3b). The

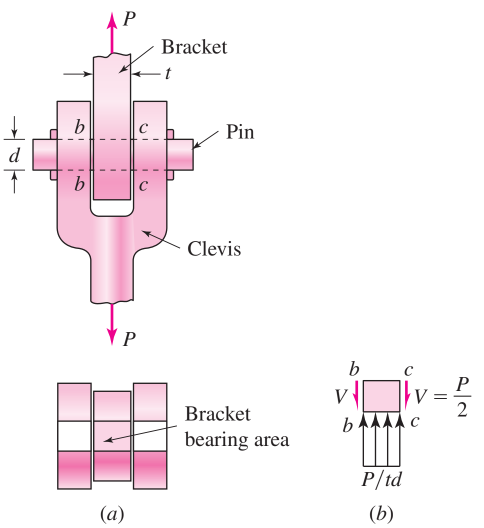


Figure 3.3 (a) A clevis-pin connection, with the bracket bearing area depicted; (b) portion of pin subjected to direct shear stresses and bearing stress.

average value of this pressure is determined by dividing the force P transmitted by the *projected area* A_p of the pin into the bracket (or clevis). This is called the *bearing stress:*

$$\sigma_b = \frac{P}{A_p} \tag{3.5}$$

Therefore, bearing stress in the bracket against the pin is $\sigma_b = P/td$, where t and d represent the thickness of bracket and diameter of the pin, respectively. Similarly, the bearing stress in the clevis against the pin may be obtained.

EXAMPLE 3.2

Design of a Monoplane Wing Rod

The wing of a monoplane is approximated by a pin-connected structure of beam AD and bar BC, as depicted in Figure 3.4a. Determine

(a) The shear stress in the pin at hinge C.

(b) The diameter of the rod BC.

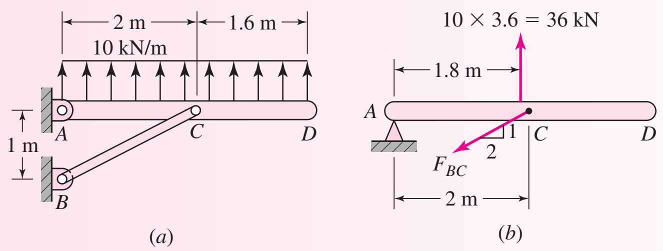


Figure 3.4 Example 3.2.

Given: The pin at C has a diameter of 15 mm and is in double shear.

Assumptions: Friction in pin joints is omitted. The air load is distributed uniformly along the span of the wing. Only rod BC is under tension. A round 2014-T6 aluminum alloy bar (see Table B.1) is used for rod BC with an allowable axial stress of 210 MPa.

Solution: Referring to the free-body diagram of the wing ACD (Figure 3.4b),

$$\sum M_A = 36(1.8) - F_{BC}\frac{1}{\sqrt{5}}(2) = 0 \qquad F_{BC} = 72.45 \text{ kN}$$

(a) Through the use of Eq. (3.4),

$$\tau_{avg} = \frac{F_{BC}}{2A} = \frac{72{,}450}{2[\pi(0.0075)^2]} = 205 \text{ MPa}$$

(b) Applying Eq. (3.1), we have

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}}, \qquad 210(10^6) = \frac{72{,}450}{A_{BC}}$$

Solving

$$A_{BC} = 3.45(10^{-4})\ \text{m}^2 = 345\ \text{mm}^2$$

Hence,

$$345 = \frac{\pi d^2}{4}, \qquad d = 20.96\ \text{mm}$$

Comments: A 21-mm diameter rod should be used. Note that, for steady inverted flight, the rod *BC* would be a compression member.

3.4 THIN-WALLED PRESSURE VESSELS

Pressure vessels are closed structures that contain liquids or gases under pressure. Common examples include tanks for compressed air, steam boilers, and pressurized water storage tanks. Although pressure vessels exist in a variety of different shapes (see Sections 16.10 through 16.14), only thin-walled cylindrical and spherical vessels are considered here. A vessel having a wall thickness less than about $\frac{1}{10}$ of inner radius is called *thin walled*. For this case, we can take $r_i \approx r_o \approx r$, where r_i, r_o, and r refer to inner, outer, and mean radii, respectively. The contents of the pressure vessel exert internal pressure, which produces small stretching deformations in the membranelike walls of an inflated balloon. In some cases external pressures cause contractions of a vessel wall. With either internal or external pressure, stresses termed *membrane stresses* arise in the vessel walls.

Section 16.11 shows that application of the equilibrium conditions to an appropriate portion of a thin-walled tank suffices to determine membrane stresses. Consider a thin-walled *cylindrical vessel* with closed ends and internal pressure p (Figure 3.5a). The longitudinal or *axial stress* σ_a and circumferential or *tangential stress* σ_θ acting on the side faces

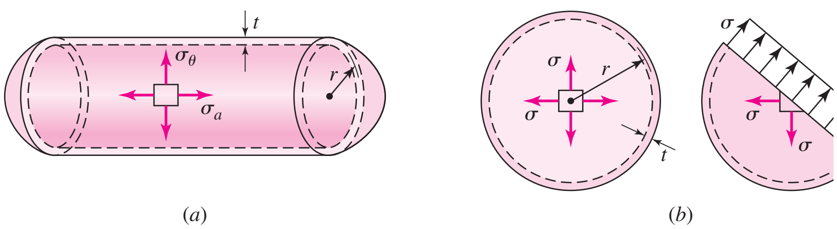


Figure 3.5 Thin-walled pressure vessels: (a) cylindrical; and (b) spherical.

of a stress element shown in the figure are principal stresses from Eqs. (16.74):

$$\sigma_a = \frac{pr}{2t} \tag{3.6a}$$

$$\sigma_\theta = \frac{pr}{t} \tag{3.6b}$$

The circumferential strain as a function of the change in radius δ_c is $\varepsilon_\theta = [2\pi(r + \delta_c) - 2\pi r]/2\pi r = \delta_c/r$. Using Hooke's law, we have $\varepsilon_\theta = (\sigma_\theta - \nu\sigma_a)/E$, where ν and E represent Poisson's ratio and modulus of elasticity, respectively. The extension of the radius of the cylinder, $\delta_c = \varepsilon_\theta r$, under the action of the stresses given by Eqs. (3.6) is therefore

$$\delta_c = \frac{pr^2}{2Et}(2 - \nu) \tag{3.7}$$

The tangential stresses σ act in the plane of the wall of a *spherical vessel* and are the same in any section that passes through the center under internal pressure p (Figure 3.5b). *Sphere stress* is given by Eq. (16.71):

$$\sigma = \frac{pr}{2t} \tag{3.8}$$

They are half the magnitude of the tangential stresses of the cylinder. Thus, sphere is an optimum shape for an internally pressurized closed vessel. The radial extension of the sphere, $\delta_s = \varepsilon r$, applying Hooke's law $\varepsilon = (\sigma - \nu\sigma)/E$ is then

$$\delta_s = \frac{pr^2}{2Et}(1 - \nu) \tag{3.9}$$

Note that the stress acting in the radial direction on the wall of a cylinder or sphere varies from $-p$ at the inner surface of the vessel to 0 at the outer surface. For thin-walled vessels, radial stress σ_r is much smaller than the membrane stresses and is usually omitted. The state of stress in the wall of a vessel is therefore considered biaxial. To conclude, we mention that a pressure vessel design is essentially governed by ASME Pressure Vessel Design Codes, discussed in Section 16.13.

Thick-walled cylinders are often used as vessels or pipe lines. Some applications involve air or hydraulic cylinders, gun barrels, and various mechanical components. Equations for exact elastic and plastic stresses and displacements for these members are developed in Chapter 16.* Composite thick-walled cylinders under pressure, thermal, and dynamic loading are discussed in detail. Numerous illustrative examples also are given.

*Within this chapter, some readers may prefer to study Section 16.3.

Design of Spherical Pressure Vessel

EXAMPLE 3.3

A spherical vessel of radius r is subjected to an internal pressure p. Determine the critical wall thickness t and the corresponding diametral extension.

Assumption: A safety factor n against bursting is used.

Given: $r = 2.5$ ft, $p = 1.5$ ksi, $S_u = 60$ ksi, $E = 30 \times 10^6$ psi, $\nu = 0.3$, $n = 3$.

Solution: We have $r = 2.5 \times 12 = 30$ in. and $\sigma = S_u/n$. Applying Eq. (3.8),

$$t = \frac{pr}{2S_u/n} = \frac{1.5(30)}{2(60/3)} = 1.125 \text{ in.}$$

Then, Eq. (3.9) results in

$$\delta_s = \frac{pr^2(1-\nu)}{2Et} = \frac{1500(30)^2(0.7)}{2(30 \times 10^6)(1.125)} = 0.014 \text{ in.}$$

The diametral extension is therefore $2\delta_s = 0.028$ in.

3.5 STRESS IN MEMBERS IN TORSION

In this section, attention is directed toward stress in prismatic bars subject to equal and opposite end torques. These members are assumed free of end constraints. Both circular and rectangular bars are treated. *Torsion* refers to twisting a structural member when it is loaded by couples that cause rotation about its longitudinal axis. Recall from Section 1.8 that, for convenience, we often show the moment of a couple or torque by a vector in the form of a double-headed arrow.

Circular Cross Sections

Torsion of circular bars or shafts produced by a torque T results in a shear stress τ and an angle of twist or angular deformation ϕ, as shown in Figure 3.6a. The *basic assumptions* of the formulations on the torsional loading of a circular prismatic bar are as follows:

1. A plane section perpendicular to the axis of the bar remains plane and undisturbed after the torques are applied.
2. Shear strain γ varies linearly from 0 at the center to a maximum on the outer surface.
3. The material is homogeneous and obeys Hooke's law; hence, the magnitude of the maximum shear angle $\gamma_{\max}$ must be less than the yield angle.

The maximum shear stress occurs at the points most remote from the center of the bar and is designated $\tau_{\max}$. For a linear stress variation, at any point at a distance r from center, the shear stress is $\tau = (r/c)\tau_{\max}$, where c represents the radius of the bar. On a cross

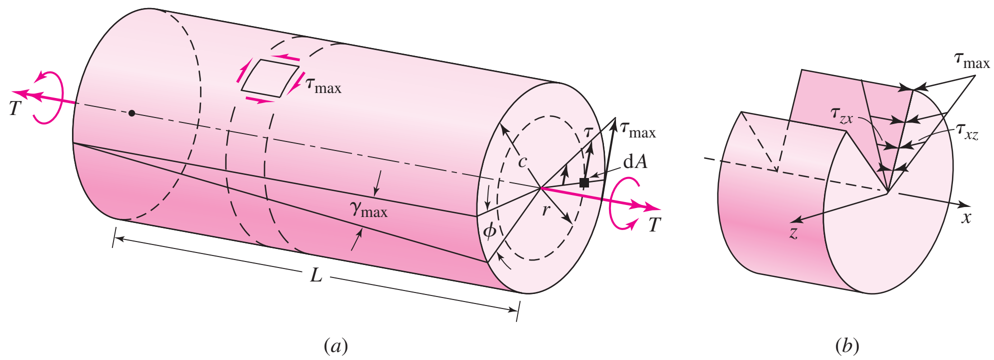


Figure 3.6 (a) Circular bar in pure torsion. (b) Shear stresses on transverse (*xz*) and axial (*zx*) planes in a circular shaft segment in torsion.

section of the shaft the resisting torque caused by the stress, distribution must be equal to the applied torque T. Hence,

$$T = \int r\left(\frac{r}{c}\tau_{\max}\right)\mathrm{d}A$$

The preceding relationship may be written in the form

$$T = \frac{\tau_{\max}}{c}\int r^2\,\mathrm{d}A$$

By definition, the polar moment of inertia J of the cross-sectional area is

$$J = \int r^2\,\mathrm{d}A \tag{a}$$

For a solid shaft, $J = \pi c^4/2$. In the case of a circular tube of inner radius b and outer radius c, $J = \pi(c^4 - b^4)/2$.

Shear stress varies with the radius and is largest at the points most remote from the shaft center. This stress distribution leaves the external cylindrical surface of the bar free of stress distribution, as it should. Note that the representation shown in Figure 3.6a is purely schematic. The maximum shear stress on a cross section of a circular shaft, either solid or hollow, is given by the *torsion formula:*

$$\boxed{\tau_{\max} = \frac{Tc}{J}} \tag{3.10}$$

The shear stress at distance r from the center of a section is

$$\boxed{\tau = \frac{Tr}{J}} \tag{3.11}$$

The *transverse* shear stress found by Eq. (3.10) or (3.11) is accompanied by an axial shear stress of equal value, that is, $\tau = \tau_{xz} = \tau_{zx}$ (Figure 3.6b), to satisfy the conditions of static equilibrium of an element. Since the shear stress in a solid circular bar is maximum at the outer boundary of the cross section and 0 at the center, most of the material in a solid shaft is stressed significantly below the maximum shear stress level. When weight reduction and savings of material are important, it is advisable to use hollow shafts (see also Example 3.4).

Noncircular Cross Sections

In treating torsion of noncircular prismatic bars, cross sections initially plane experience out-of-plane deformation or *warping,* and the first two assumptions stated previously are no longer appropriate. Figure 3.7 depicts the nature of distortion occurring in a rectangular section. The mathematical solution of the problem is complicated. For cases that cannot be conveniently solved by applying the theory of elasticity, the governing equations are used in conjunction with the experimental techniques. The finite element analysis is also very efficient for this purpose. Torsional stress (and displacement) equations for a number of noncircular sections are summarized in references such as [2, 4]. Table 3.1 lists the "exact" solutions of the maximum shear stress and the angle of twist ϕ for a few common cross sections. Note that the values of coefficients α and β depend on the ratio of the side lengths a and b of a rectangular section. For thin sections ($a \gg b$), the values of α and β approach $\frac{1}{3}$.

The following approximate formula for the maximum shear stress in a *rectangular member* is of interest:

$$\tau_{\max} = \frac{T}{ab^2}\left(3 + 1.8\frac{b}{a}\right) \tag{3.12}$$

As in Table 3.1, a and b represent the lengths of the long and short sides of a rectangular cross section, respectively. The stress occurs along the centerline of the wider face of the bar. For a *thin section,* where a is much greater than b, the second term may be neglected. Equation (3.12) is also valid for equal-leg angles; these can be considered as two rectangles, each of which is capable of carrying half the torque.

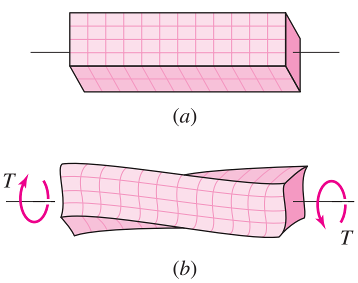
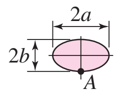
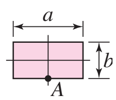


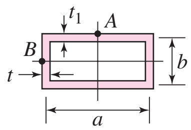
Figure 3.7 Rectangular bar (a) before and (b) after a torque is applied.

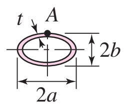
Table 3.1 Expressions for stress and deformation in some cross-section shapes in torsion

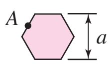
<table>
<tr><th>Cross section</th><th></th><th>Maximum shearing stress</th><th>Angle of twist per unit length</th></tr>
<tr><td>

Ellipse
For circle: $a = b$</td><td></td><td>$\tau_A = \dfrac{2T}{\pi ab^2}$</td><td>$\phi = \dfrac{(a^2 + b^2)T}{\pi a^3 b^3 G}$</td></tr>
<tr><td>

Equilateral triangle</td><td></td><td>$\tau_A = \dfrac{20T}{a^3}$</td><td>$\phi = \dfrac{46.2T}{a^4 G}$</td></tr>
<tr><td rowspan="11">

Rectangle</td><td></td><td>$\tau_A = \dfrac{T}{\alpha ab^2}$</td><td>$\phi = \dfrac{T}{\beta ab^3 G}$</td></tr>
<tr><td>a/b</td><td>α</td><td>β</td></tr>
<tr><td>1.0</td><td>0.208</td><td>0.141</td></tr>
<tr><td>1.5</td><td>0.231</td><td>0.196</td></tr>
<tr><td>2.0</td><td>0.246</td><td>0.229</td></tr>
<tr><td>2.5</td><td>0.256</td><td>0.249</td></tr>
<tr><td>3.0</td><td>0.267</td><td>0.263</td></tr>
<tr><td>4.0</td><td>0.282</td><td>0.281</td></tr>
<tr><td>5.0</td><td>0.292</td><td>0.291</td></tr>
<tr><td>10.0</td><td>0.312</td><td>0.312</td></tr>
<tr><td>∞</td><td>0.333</td><td>0.333</td></tr>
<tr><td>

Hollow rectangle</td><td></td><td>$\tau_A = \dfrac{T}{2abt_1}$
$\tau_B = \dfrac{T}{2abt}$</td><td>$\phi = \dfrac{(at + bt_1)T}{2tt_1 a^2 b^2 G}$</td></tr>
<tr><td>

Hollow ellipse
For hollow circle: $a = b$</td><td></td><td>$\tau_A = \dfrac{T}{2\pi abt}$</td><td>$\phi = \dfrac{\sqrt{2(a^2 + b^2)}T}{4\pi a^2 b^2 tG}$</td></tr>
<tr><td>

Hexagon</td><td></td><td>$\tau_A = \dfrac{5.7T}{a^3}$</td><td>$\phi = \dfrac{8.8T}{a^4 G}$</td></tr>
</table>

EXAMPLE 3.4

Torque Transmission Efficiency of Hollow and Solid Shafts

A hollow shaft and a solid shaft (Figure 3.8) are twisted about their longitudinal axes with torques T_h and T_s, respectively. Determine the ratio of the largest torques that can be applied to the shafts.

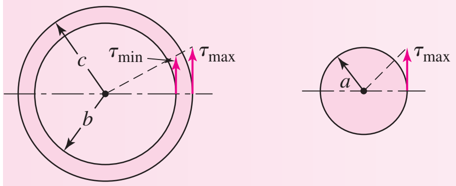


Figure 3.8 Example 3.4.

Given: $c = 1.15b$.

Assumptions: Both shafts are made of the same material with allowable stress and both have the same cross-sectional area.

Solution: The maximum shear stress τ_{max} equals τ_{all}. Since the cross-sectional areas of both shafts are identical, $\pi(c^2 - b^2) = \pi a^2$:

$$a^2 = c^2 - b^2$$

For the hollow shaft, using Eq. (3.10),

$$T_h = \frac{\pi}{2c}(c^4 - b^4)\tau_{all}$$

Likewise, for the solid shaft,

$$T_s = \frac{\pi}{2}a^3\tau_{all}$$

We therefore have

$$\frac{T_h}{T_s} = \frac{c^4 - b^4}{ca^3} = \frac{c^4 - b^4}{c(c^2 - b^2)^{\frac{3}{2}}} \tag{3.13}$$

Substituting $c = 1.15b$, this quotient gives

$$\frac{T_h}{T_s} = 3.56$$

Comments: The result shows that, hollow shafts are more efficient in transmitting torque than solid shafts. Interestingly, thin shafts are also useful for creating an essentially uniform shear (i.e., $\tau_{min} \approx \tau_{max}$). However, to avoid buckling (see Section 6.2), the wall thickness cannot be excessively thin.

3.6 SHEAR AND MOMENT IN BEAMS

In beams loaded by transverse loads in their planes, only two components of stress resultants occur: the shear force and bending moment. These loading effects are sometimes referred to as *shear* and *moment in beams*. To determine the magnitude and sense of shearing force and bending moment at any section of a beam, the method of sections is applied. The sign conventions adopted for internal forces and moments (see Section 1.8) are associated with the deformations of a member. To illustrate this, consider the positive and negative shear forces V and bending moments M acting on segments of a beam cut out between two cross sections (Figure 3.9). We see that a positive shear force tends to raise the left-hand face relative to the right-hand face of the segment, and a positive bending moment tends to bend the segment concave upward, so it "retains water." Likewise, a positive moment compresses the upper part of the segment and elongates the lower part.

LOAD, SHEAR, AND MOMENT RELATIONSHIPS

Consider the free-body diagram of an element of length dx, cut from a loaded beam (Figure 3.10a). Note that the distributed load w per unit length, the shears, and the bending moments are shown as positive (Figure 3.10b). The changes in V and M from position x to $x + \mathrm{d}x$ are denoted by dV and dM, respectively. In addition, the resultant of the distributed load (w dx) is indicated by the dashed line in the figure. Although w is not uniform, this is permissible substitution for a very small distance dx.

Equilibrium of the vertical forces acting on the element of Figure 3.10b, $\sum F_x = 0$, results in $V + w\,\mathrm{d}x = V + \mathrm{d}V$. Therefore,

$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}x} = w} \tag{3.14a}$$

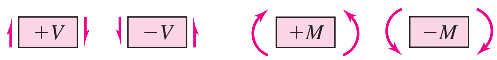


Figure 3.9 Sign convention for beams: definitions of positive and negative shear and moment.

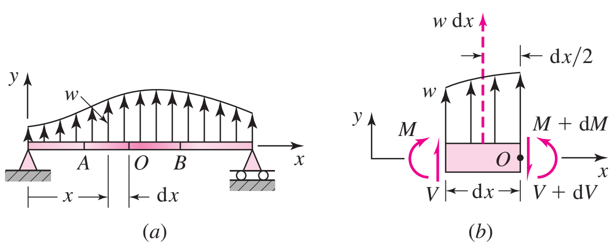


Figure 3.10 Beam and an element isolated from it.

This states that, at any section of the beam, the slope of the shear curve is equal to w. Integration of Eq. (3.14a) between points A and B on the beam axis gives

$$V_B - V_A = \int_A^B w\,\mathrm{d}x = \text{area of load diagram between } A \text{ and } B \tag{3.14b}$$

Clearly, Eq. (3.14a) is not valid at the point of application of a concentrated load. Similarly, Eq. (3.14b) cannot be used when concentrated loads are applied between A and B. For equilibrium, the sum of moments about O must also be 0: $\sum M_O = 0$ or $M + \mathrm{d}M - (V + \mathrm{d}V)\,\mathrm{d}x - M = 0$. If second-order differentials are considered as negligible compared with differentials, this yields

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V \tag{3.15a}$$

The foregoing relationship indicates that the slope of the moment curve is equal to V. Therefore the shear force is inseparably linked with a change in the bending moment along the length of the beam. Note that the maximum value of the moment occurs at the point where V (and hence $\mathrm{d}M/\mathrm{d}x$) is 0. Integrating Eq. (3.15a) between A and B, we have

$$M_B - M_A = \int_A^B V\,\mathrm{d}x = \text{area of shear diagram between } A \text{ and } B \tag{3.15b}$$

The differential equations of equilibrium, Eqs. (3.14a) and (3.15a), show that the shear and moment curves, respectively, always are 1 and 2 degrees higher than the load curve. We note that Eq. (3.15a) is not valid at the point of application of a concentrated load. Equation (3.15b) can be used even when concentrated loads act between A and B, but the relation is not valid if a couple is applied at a point between A and B.

Shear and Moment Diagrams

When designing a beam, it is useful to have a graphical visualization of the shear force and moment variations along the length of a beam. A shear diagram is a graph where the shearing force is plotted against the horizontal distance (x) along a beam. Similarly, a graph showing the bending moment plotted against the x axis is the bending-moment diagram. The signs for shear V and moment M follow the general convention defined in Figure 3.9. It is convenient to place the shear and bending moment diagrams directly below the free-body, or load, diagram of the beam. The maximum and other significant values are generally marked on the diagrams.

We use the so-called summation method of constructing shear and moment diagrams. The procedure of this semigraphical approach is as follows:

1. Determine the reactions from free-body diagram of the entire beam.
2. Determine the value of the shear, successively summing from the *left end* of the beam the vertical external forces or using Eq. (3.14b). Draw the shear diagram obtaining the shape from Eq. (3.14a). Plot a positive V upward and a negative V downward.

3. Determine the values of moment, either continuously summing the external moments from the left end of the beam or using Eq. (3.15b), whichever is more appropriate. Draw the moment diagram. The shape of the diagram is obtained from Eq. (3.15a).

A check on the accuracy of the shear and moment diagrams can be made by noting whether or not they close. Closure of these diagrams demonstrates that the sum of the shear forces and moments acting on the beam are 0, as they must be for equilibrium. When any diagram fails to close, you know that there is a construction error or an error in calculation of the reactions. The following example illustrates the procedure.

EXAMPLE 3.5 Shear and Moment Diagrams for a Simply Supported Beam by Summation Method

Draw the shear and moment diagrams for the beam loaded as shown in Figure 3.11a.

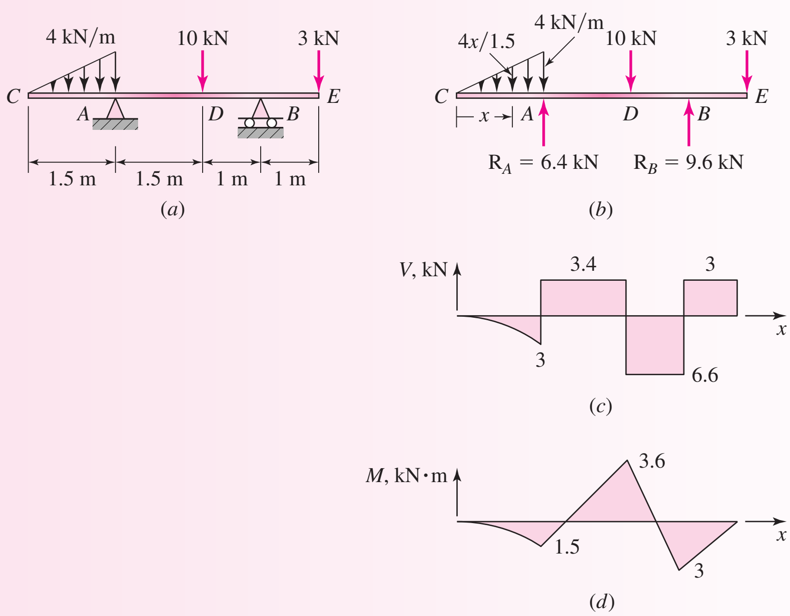


Figure 3.11 Example 3.5: (a) An overhanging beam; (b) free-body or load diagram; (c) shear diagram; and (d) moment diagram.

Assumptions: All forces are coplanar and two dimensional.

Solution: Applying the equations of statics to the free-body diagram of the entire beam, we have (Figure 3.11b):

$$R_A = 6.4 \text{ kN}, \qquad R_B = 9.6 \text{ kN}$$

In the *shear diagram* (Figure 3.11c), the shear at end C is $V_C = 0$. Equation (3.14b) yields

$$V_A - V_C = \frac{1}{2}w(1.5) = \frac{1}{2}(-4)(1.5) = -3, \qquad V_A = -3 \text{ kN}$$

the upward force near to the left of A. From C to A, the load increases linearly, hence the shear curve is parabolic, which has a negative and increasing slope. In the regions AD, DB, and BE, the slope of the shear curve is 0 or the shear is constant. At A, the 6.4 kN upward reaction force increases the shear to 3.4 kN. The shear remains constant up to D where it decreases by a 10 kN downward force to -6.6 kN. Likewise, the value of the shear rises to 3 kN at B. No change in the shear occurs until point E, where the downward 3 kN force closes the diagram. The maximum shear $V_{max} = -6.6$ kN occurs in region BD.

In the *moment diagram* (Figure 3.11d), the moment at end C is $M_C = 0$. Equation (3.15b) gives

$$M_A - M_C = -\int_0^{1.5}\left(\frac{1}{2}\frac{4x}{1.5}x\right)\mathrm{d}x \qquad M_A = -1.5 \text{ N}\cdot\text{m}$$

$$M_D - M_A = 3.4(1.5) \qquad M_D = 3.6 \text{ kN}\cdot\text{m}$$

$$M_B - M_D = -6.6(1) \qquad M_B = -3 \text{ kN}\cdot\text{m}$$

$$M_E - M_B = 3(1) \qquad M_E = 0$$

Since M_E is known to be 0, a check on the calculations is provided. We find that, from C to A, the diagram takes the shape of a cubic curve concave downward with 0 slope at C. This is in accordance with $\mathrm{d}M/\mathrm{d}x = V$. Here V, prescribing the slope of the moment diagram, is negative and increases to the right. In the regions *AD, DB*, and *BE*, the diagram forms straight lines. The maximum moment, $M_{max} = 3.6$ kN · m, occurs at D.

A procedure identical to the preceding one applies to axially loaded bars and twisted shafts. The applied axial forces and torques are positive if their vectors are in the direction of a positive coordinate axis. When a bar is subjected to loads at several points along its length, the internal axial forces and twisting moments vary from section to section. A graph showing the variation of the axial force along the bar axis is called an *axial-force diagram*. A similar graph for the torque is referred to as a *torque diagram*. We note that the axial force and torque diagrams are *not* used as commonly as shear and moment diagrams.

3.7 STRESSES IN BEAMS

A *beam* is a bar supporting loads applied laterally or transversely to its (longitudinal) axis. This flexure member is commonly used in structures and machines. Examples include the main members supporting floors of buildings, automobile axles, and leaf springs. We see in Sections 4.10 and 4.11 that the following formulas for stresses and deflections of beams can readily be reduced from those of rectangular plates.

Assumptions of Beam Theory

The basic assumptions of the technical or engineering *theory for slender beams* are based on geometry of deformation. They can be summarized as follows [1]:

1. The deflection of the beam axis is *small* compared with the depth and span of the beam.
2. The slope of the deflection curve is very small and its square is negligible in comparison with unity.
3. Plane sections through a beam taken normal to its axis remain plane after the beam is subjected to bending. This is the fundamental hypothesis of the flexure theory.
4. The effect of shear stress τ_{xy} on the distribution of bending stress σ_x is omitted. The stress normal to the neutral surface, σ_y, may be disregarded.

A generalization of the preceding presuppositions forms the basis for the theories of plates and shells [5].

When treating the bending problem of beams, it is frequently necessary to distinguish between pure bending and nonuniform bending. The former is the flexure of a beam subjected to a constant bending moment; the latter refers to flexure in the presence of shear forces. We discuss the stresses in beams in both cases of bending.

Normal Stress

Consider a linearly elastic beam having the y axis as a vertical axis of symmetry (Figure 3.12a). Based on assumptions 3 and 4, the normal stress σ_x over the cross section (such as A-B, Figure 3.12b) varies linearly with y and the remaining stress components are 0:

$$\sigma_x = ky \qquad \sigma_y = \tau_{xy} = 0 \tag{a}$$

Here k is a constant, and $y = 0$ contains the neutral surface. The intersection of the neutral surface and the cross section locates the *neutral axis* (abbreviated N.A.). Figure 3.12c depicts the linear stress field in the section A-B.

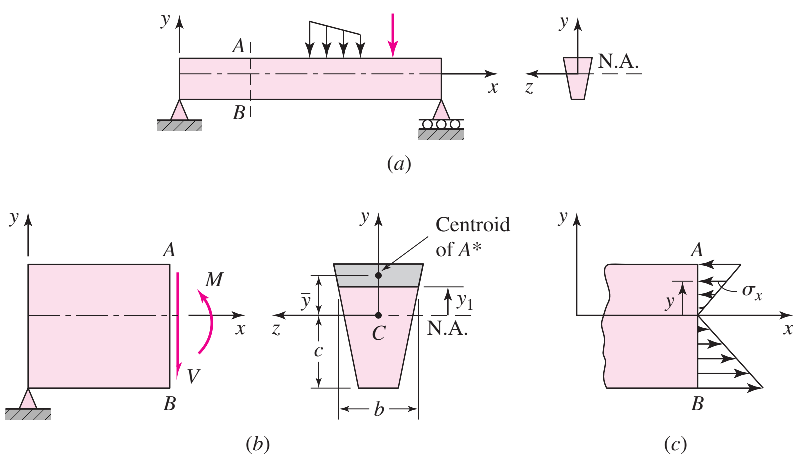


Figure 3.12 (a) A beam subjected to transverse loading; (b) segment of beam; (c) distribution of bending stress in a beam.

Conditions of equilibrium require that the resultant normal force produced by the stresses σ_x be 0 and the moments of the stresses about the axis be equal to the bending moment acting on the section. Hence,

$$\int_A \sigma_x \, dA = 0, \qquad -\int_A (\sigma_x \, dA)y = M \tag{b}$$

in which A represents the cross-sectional area. The negative sign in the second expression indicates that a positive moment M is one that produces compressive (negative) stress at points of positive y. Carrying Eq. (a) into Eqs. (b),

$$k \int_A y \, dA = 0 \tag{c}$$

$$-k \int_A y^2 \, dA = M \tag{d}$$

Since $k = 0$, Eq. (c) shows that the first moment of cross-sectional area about the neutral axis is 0. This requires that the neutral and centroidal axes of the cross section coincide. It should be mentioned that the symmetry of the cross section about the y axis means that the y and z axes are principal centroidal axes. The integral in Eq. (d) defines the moment of inertia, $I = \int y^2 \, dA$, of the cross section about the z axis of the beam cross section. It follows that

$$k = -\frac{M}{I} \tag{e}$$

An expression for the normal stress, known as the elastic *flexure formula* applicable to initially straight beams, can now be written by combining Eqs. (a) and (e):

$$\boxed{\sigma_x = -\frac{My}{I}} \tag{3.16}$$

Here y represents the distance from the neutral axis to the point at which the stress is calculated. It is common practice to recast the flexure formula to yield the maximum normal stress σ_{max} and denote the value of $|y_{max}|$ by c, where c represents the distance from the neutral axis to the outermost fiber of the beam. On this basis, the flexure formula becomes

$$\boxed{\sigma_{max} = \frac{Mc}{I} = \frac{M}{S}} \tag{3.17}$$

The quantity $S = I/c$ is known as the section modulus of the cross-sectional area. Note that the flexure formula also applies to a beam of *unsymmetrical* cross-sectional area, provided I is a principal moment of inertia and M is a moment around a principal axis [1].

Curved Beam of a Rectangular Cross Section

Many machine and structural components loaded as beams, however, are not straight. When beams with initial curvature are subjected to bending moments, the stress distribution is not linear on either side of the neutral axis but increases more rapidly on the inner side. The

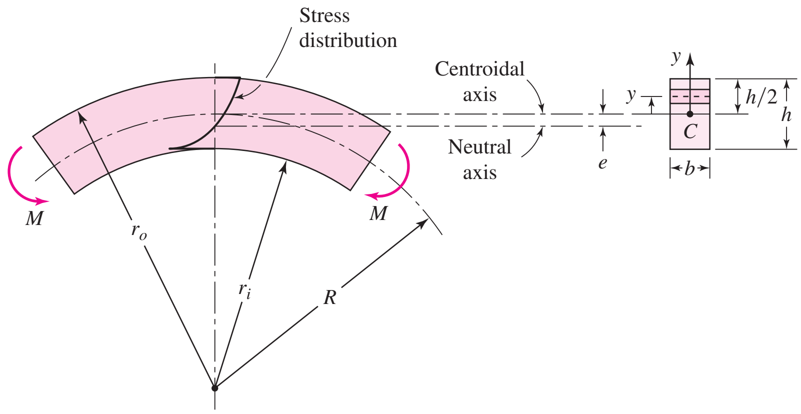


Figure 3.13 Curved bar in pure bending.

flexure and displacement formulas for these axisymmetrically loaded members are developed in the later chapters, using energy, elasticity, or exact, approximate technical theories.*

Here, the general equation for stress in curved members is adapted to the rectangular cross section shown in Figure 3.13. Therefore, for pure bending loads, the normal stress σ in a curved beam of a rectangular cross section, from Eq. (16.55):

$$\sigma = \frac{M}{AR}\left[1 + \frac{y}{Z(R+y)}\right] \tag{3.18}$$

The curved beam factor Z by Table 16.1 is

$$Z = -1 + \frac{R}{h}\ln\frac{r_o}{r_i} \tag{3.19}$$

In the foregoing expressions, we have

A = cross-sectional area
h = depth of beam
R = radius of curvature to the neutral axis
M = bending moment, positive when directed toward the concave side, as shown in the figure
y = distance measured from the neutral axis to the point at which stress is calculated, positive toward the convex side, as indicated in the figure
r_i, r_o = radii of the curvature of the inner and outer surfaces, respectively.

Accordingly, a positive value obtained from Eq. (3.18) means tensile stress.

*Some readers may prefer to study Section 16.8.

The neutral axis shifts toward the center of curvature by distance e from the centroidal axis ($y = 0$), as shown in Figure 3.13. By Eq. (16.57), we have $e = -ZR/(Z+1)$. Expression for Z and e for many common cross-sectional shapes can be found referring to Table 16.1. Combined stresses in curved beams is presented in Chapter 16. A detailed comparison of the results obtained by various methods is illustrated in Example 16.7. Deflections of curved members due to bending, shear, and normal loads are discussed in Section 5.6.

SHEAR STRESS

We now consider the distribution of shear stress τ in a beam associated with the shear force V. The vertical shear stress τ_{xy} at any point on the cross section is numerically equal to the horizontal shear stress at the same point (see Section 1.13). Shear stresses as well as the normal stresses are taken to be uniform across the width of the beam. The shear stress $\tau_{xy} = \tau_{yx}$ at any point of a cross section (Figure 3.12b) is given by the shear formula:

$$\tau_{xy} = \frac{VQ}{Ib} \tag{3.20}$$

Here

V = the shearing force at the section
b = the width of the section measured at the point in question
Q = the first moment with respect to the neutral axis of the area A^* beyond the point at which the shear stress is required; that is,

$$Q = \int_{A^*} y\,\mathrm{d}A = \overline{y}A^* \tag{3.21}$$

By definition, the area A^* represents the area of the part of the section below the point in question and $\overline{y}$ is the distance from the neutral axis to the centroid of A^*. Clearly, if $\overline{y}$ is measured above the neutral axis, Q represents the *first moment of the area* above the level where the shear stress is to be found. Obviously, shear stress varies in accordance with the shape of the cross section.

Rectangular Cross Section

To ascertain how the shear stress varies, we must examine how Q varies, because V, I, and b are constants for a rectangular cross section. In so doing, we find that the distribution of the shear stress on a cross section of a rectangular beam is parabolic. The stress is 0 at the top and bottom of the section ($y_1 = \pm h/2$) and has its maximum value at the neutral axis ($y_1 = 0$) as shown in Figure 3.14. Therefore,

$$\tau_{\max} = \frac{V}{Ib}A^*\,\overline{y} = \frac{V}{(bh^3/12)b}\,\frac{bh}{2}\,\frac{h}{4} = \frac{3}{2}\,\frac{V}{A} \tag{3.22}$$

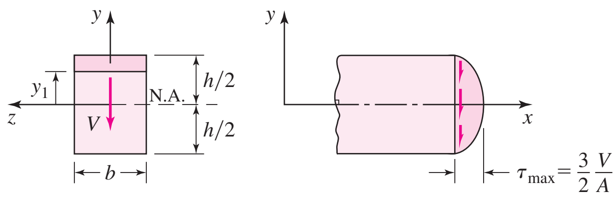


Figure 3.14 Shear stresses in a beam of rectangular cross section.

where $A = bh$ is the cross-sectional area of a beam having depth h and width b. For *narrow beams* with sides parallel to the y axis, Eq. (3.20) gives solutions in good agreement with the "exact" stress distribution obtained by the methods of the theory of elasticity. Equation (3.22) is particularly useful, since beams of rectangular-sectional form are often employed in practice. Stresses in a wide beam and plate are discussed in Section 4.10 after derivation of the strain-curvature relations.

The shear force acting across the width of the beam per unit length along the beam axis may be found by multiplying τ_{xy} in Eq. (3.22) by b (Figure 3.12b). This quantity is denoted by q, known as the *shear flow,*

$$q = \frac{VQ}{I} \tag{3.23}$$

The foregoing equation is valid for any beam having a cross section that is symmetrical about the y axis. It is very useful in the analysis of *built-up beams*. A beam of this type is fabricated by joining two or more pieces of material. Built-up beams are generally designed on the basis of the assumption that the parts are adequately connected so that the beam acts as a single member. Structural connections are taken up in Chapter 15.

EXAMPLE 3.6

Determining Stresses in a Simply Supported Beam

A simple beam of T-shaped cross section is loaded as shown in Figure 3.15a. Determine

(a) The maximum shear stress.

(b) The shear flow q_j and the shear stress τ_j in the joint between the flange and the web.

(c) The maximum bending stress.

Given: $P = 4$ kN and $L = 3$ m

Assumptions: All forces are coplanar and two dimensional.

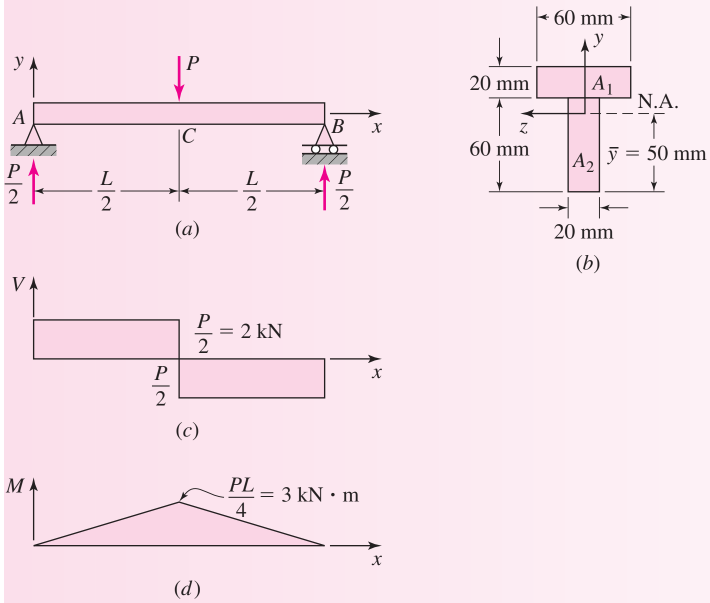


Figure 3.15 Example 3.6.

Solution: The distance $\overline{y}$ to the centroid is determined as follows (Figure 3.15b):

$$\overline{y} = \frac{A_1\overline{y}_1 + A_2\overline{y}_2}{A_1 + A_2} = \frac{20(60)70 + 60(20)30}{20(60) + 60(20)} = 50 \text{ mm}$$

The moment of inertia I about the neutral axis is found using the parallel axis theorem:

$$I = \frac{1}{12}(60)(20)^3 + 20(60)(20)^2 + \frac{1}{12}(20)(60)^3 + 20(60)(20)^2 = 136 \times 10^4 \text{ mm}^4$$

The shear and moment diagrams (Figures 3.15c and 3.15d) are drawn using the method of sections.

(a) The maximum shearing stress in the beam occurs at the neutral axis on the cross section supporting the largest shear force V. Hence,

$$Q_{\text{N.A.}} = 50(20)25 = 25 \times 10^3 \text{ mm}^3$$

Since the shear force equals 2 kN on all cross sections of the beam (Figure 3.12c), we have

$$\tau_{\max} = \frac{V_{\max} Q_{\text{N.A.}}}{Ib} = \frac{2 \times 10^3(25 \times 10^{-6})}{136 \times 10^{-8}(0.02)} = 1.84 \text{ MPa}$$

(b) The first moment of the area of the flange about the neutral axis is

$$Q_f = 20(60)20 = 24 \times 10^3 \text{ mm}^3$$

Applying Eqs. (3.23) and (3.20),

$$q_j = \frac{VQ_f}{I} = \frac{2 \times 10^3(24 \times 10^{-6})}{136 \times 10^{-8}} = 35.3 \text{ kN/m}$$

$$\tau_j = \frac{VQ_f}{Ib} = \frac{35.3(10^3)}{0.02} = 1.76 \text{ MPa}$$

(c) The largest moment occurs at midspan, as shown in Figure 3.15d. Therefore, from Eq. (3.19), we obtain

$$\sigma_{\max} = \frac{Mc}{I} = \frac{3 \times 10^3(0.05)}{136 \times 10^{-8}} = 110.3 \text{ MPa}$$

3.8 DESIGN OF BEAMS

We are here concerned with the elastic design of beams for strength. Beams made of single and two different materials are discussed. We note that some beams must be selected based on allowable deflections. This topic is taken up in Chapters 4 and 5. Occasionally, beam design relies on plastic moment capacity, the so-called limit design [1].

PRISMATIC BEAMS

We select the dimensions of a beam section so that it supports safely applied loads without exceeding the allowable stresses in both flexure and shear. Therefore, the design of the member is controlled by the largest normal and shear stresses developed at the critical section, where the maximum value of the bending moment and shear force occur. Shear and bending-moment diagrams are very helpful for locating these critical sections. In heavily loaded short beams, the design is usually governed by shear stress, while in slender beams, the flexure stress generally predominates. Shearing is more important in wood than steel beams, as wood has relatively low shear strength parallel to the grain.

Application of the rational procedure in design, outlined in Section 3.2, to a beam of ordinary proportions often includes the following steps:

1. It is assumed that failure results from yielding or fracture, and flexure stress is considered to be most closely associated with structural damage.
2. The significant value of bending stress is $\sigma = M_{\max}/S$.
3. The maximum usable value of σ without failure, $\sigma_{\max}$, is the yield strength S_y or the ultimate strength S_u.

4. A factor of safety n is applied to σ_{max} to obtain the allowable stress: $\sigma_{all} = \sigma_{max}/n$. The *required section modulus* of a beam is then

$$S = \frac{M_{max}}{\sigma_{all}} \tag{3.24}$$

There are generally several different beam sizes with the required value of S. We select the one with the lightest weight per unit length or the smallest sectional area from tables of beam properties. When the allowable stress is the same in tension and compression, a doubly symmetric section (i.e., section symmetric about the y and z axes) should be chosen. If σ_{all} is different in tension and compression, a singly symmetric section (for example a T beam) should be selected so that the distances to the extreme fibers in tension and compression are in a ratio nearly the same as the respective σ_{all} ratios.

We now check the *shear-resistance requirement* of beam tentatively selected. After substituting the suitable data for Q, I, b, and V_{max} into Eqs. (3.20), we determine the maximum shear stress in the beam from the formula

$$\tau_{max} = \frac{V_{max} Q}{Ib} \tag{3.25}$$

When the value obtained for τ_{max} is smaller than the allowable shearing stress τ_{all}, the beam is acceptable; otherwise, a stronger beam should be chosen and the process repeated.

Design of a Beam of Doubly Symmetric Section

EXAMPLE 3.7

Select a wide-flange steel beam to support the loads shown in Figure 3.16a.

Given: The allowable bending and shear stresses are 160 and 90 MPa, respectively.

Solution: Shear and bending-moment diagrams (Figures 3.16b and 3.16c) show that $M_{max} = 110\text{ kN}\cdot\text{m}$ and $V_{max} = 40$ kN. Therefore, Eq. (3.24) gives

$$S = \frac{110 \times 10^3}{160(10^6)} = 688 \times 10^3\text{ mm}^3$$

Using Table A.6, we select the lightest member that has a section modulus larger than this value of S: a 200-mm W beam weighing 71 kg/m ($S = 709 \times 10^3\text{ mm}^3$). Since the weight of the beam ($71 \times 9.81 \times 10 = 6.97$ kN) is small compared with the applied load (80 kN), it is neglected.

The approximate or average maximum shear stress in beams with flanges may be obtained by dividing the shear force V by the web area:

$$\tau_{avg} = \frac{V}{A_{web}} = \frac{V}{ht} \tag{3.26}$$

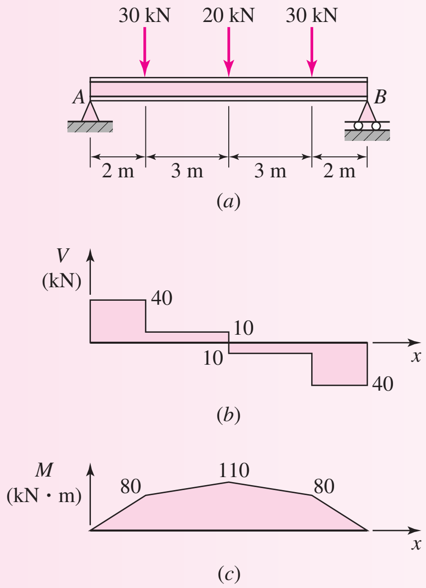


Figure 3.16 Example 3.7.

In this relationship, h and t represent the beam depth and web thickness, respectively. From Table A.6, the area of the web of a W 200 × 71 section is $216 \times 10.2 = 2.203(10^3)$ mm^2. We therefore have

$$\tau_{avg} = \frac{40 \times 10^3}{2.203(10^{-3})} = 18.16 \text{ MPa}$$

Comment: Inasmuch as this stress is well within the allowable limit of 90 MPa, the beam is acceptable.

Beams of Constant Strength

When a beam is stressed to a uniform allowable stress, σ_{all}, throughout, then it is clear that the beam material is used to its greatest capacity. For a prescribed material, such a design is of minimum weight. At any cross section, the required section modulus S is given by

$$S = \frac{M}{\sigma_{all}} \tag{3.27}$$

where M presents the bending moment on an arbitrary section. Tapered beams designed in this manner are called *beams of constant strength*. Note that shear stress at those beam locations where the moment is small controls the design.

Beams of uniform strength are exemplified by leaf springs and certain forged or cast machine components (see Section 14.10). For a structural member, fabrication and design constraints make it impractical to produce a beam of constant stress. So, welded cover plates are often used for parts of prismatic beams where the moment is large; for instance, in a bridge girder. If the angle between the sides of a tapered beam is small, the flexure formula allows little error. On the other hand, the results obtained by using the shear stress formula may not be sufficiently accurate for nonprismatic beams. Usually, a modified form of this formula is used for design purposes. The exact distribution in a rectangular wedge is obtained by the theory of elasticity [2].

Design of a Constant Strength Beam

EXAMPLE 3.8

A cantilever beam of uniform strength and rectangular cross section is to support a concentrated load P at the free end (Figure 3.17a). Determine the required cross-sectional area, for two cases: (a) the width b is constant; (b) the height h is constant.

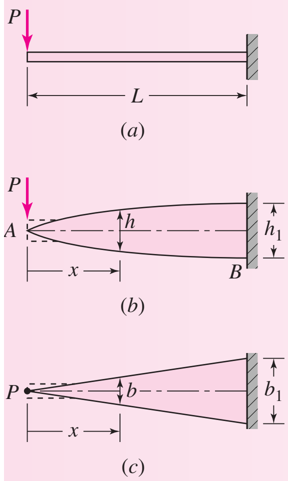


Figure 3.17 Example 3.8.
(a) Uniform strength cantilever;
(b) side view; (c) top view.

Solution:

(a) At a distance x from A, $M = Px$ and $S = bh^2/6$. Through the use of Eq. (3.27), we write

$$\frac{bh^2}{6} = \frac{Px}{\sigma_{\text{all}}} \tag{a}$$

Similarly, at a fixed end ($x = L$ and $h = h_1$),

$$\frac{bh_1^2}{6} = \frac{PL}{\sigma_{\text{all}}}$$

Dividing Eq. (a) by the preceeding relationship results in

$$h = h_1\sqrt{\frac{x}{L}} \tag{b}$$

Therefore, the depth of the beam varies parabolically from the free end (Figure 3.17b).

(b) Equation (a) now yields

$$b = \left(\frac{6P}{h^2\sigma_{\text{all}}}\right)x = \frac{b_1}{L}x \tag{c}$$

Comments: In Eq. (c), the expression in parentheses represents a constant and set equal to b_1/L so that when $x = L$ the width is b_1 (Figure 3.17c). In both cases, obviously the cross section of the beam near end A must be designed to resist the shear force, as shown by the dashed lines in the figure.

Composite Beams

Beams fabricated of two or more materials having different moduli of elasticity are called *composite beams*. The advantage of this type construction is that large quantities of low-modulus material can be used in regions of low stress, and small quantities of high-modulus materials can be used in regions of high stress. Two common examples are wooden beams whose bending strength is bolstered by metal strips, either along its sides or along its top or bottom, and reinforced concrete beams. The assumptions of the technical theory of homogenous beams, discussed in Section 3.7, are valid for a beam of more than one material. We use the common *transformed-section method* to ascertain the stresses in a composite beam. In this approach, the cross section of several materials is transformed into an equivalent cross section of one material in that the resisting forces are the same as on the original section. The flexure formula is then applied to the transformed section.

To demonstrate the method, a typical beam with symmetrical cross section built of two different materials is considered (Figure 3.18a). The moduli of elasticity of materials are denoted by E_1 and E_2. We define the modular ratio, n, as follows

$$n = \frac{E_2}{E_1} \tag{d}$$

Although $n > 1$ in Eq. (d), the choice is arbitrary; the technique applies well for $n < 1$. The transformed section is composed of only material 1 (Figure 3.18b). The moment of inertia of the entire transformed area about the neutral axis is then denoted by I_t. It can be

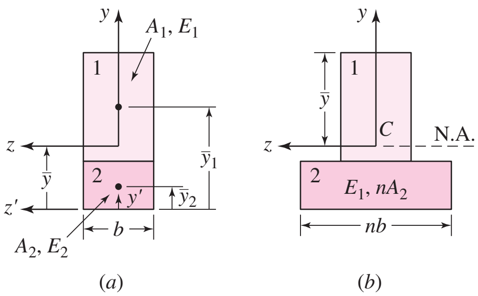


Figure 3.18 Beam of two materials: (a) Cross section; (b) equivalent section.

shown [1] that, the flexure formulas for a composite beam are in the forms

$$\sigma_{x1} = -\frac{My}{I_t}, \qquad \sigma_{x2} = -\frac{nMy}{I_t} \tag{3.28}$$

where σ_{x1} and σ_{x2} are the stresses in materials 1 and 2, respectively. Obviously, when $E_1 = E_2 = E$, this equation reduces to the flexure formula for a beam of homogeneous material, as expected. The following sample problems illustrate the use of Eqs. (3.28).

Determination of Stress in a Composite Beam

EXAMPLE 3.9

A composite beam is made of wood and steel having the cross-sectional dimensions shown in Figure 3.19a. The beam is subjected to a bending moment $M_z = 25$ kN · m. Calculate the maximum stresses in each material.

Given: The modulus of elasticity of wood and steel are $E_w = 10$ GPa and $E_s = 210$ GPa, respectively.

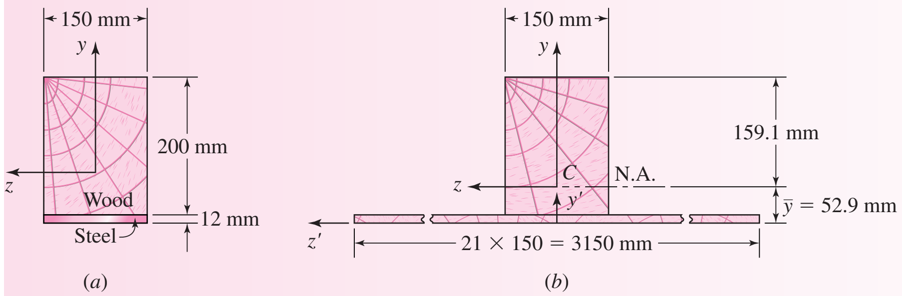


Figure 3.19 Example 3.9: (a) Composite beam and (b) equivalent section.

Solution: The modular ratio $n = E_s/E_w = 21$. We use a transformed section of wood (Figure 3.19b). The centroid and the moment of inertia about the neutral axis of this section are

$$\bar{y} = \frac{150(200)(112) + 3150(12)(6)}{150(200) + 3150(12)} = 52.9 \text{ mm}$$

$$I_t = \frac{1}{12}(150)(200)^3 + 150(200)(59.1)^2 + \frac{1}{12}(3150)(12)^3 + 3150(12)(46.9)^2$$

$$= 288 \times 10^6 \text{ mm}^4$$

The maximum stress in the wood and steel portions are therefore

$$\sigma_{w,\max} = \frac{Mc}{I_t} = \frac{25(10^3)(0.1591)}{288(10^{-6})} = 13.81 \text{ MPa}$$

$$\sigma_{s,\max} = \frac{nMc}{I_t} = \frac{21(25 \times 10^3)(0.0529)}{288(10^{-6})} = 96.43 \text{ MPa}$$

At the juncture of the two parts, we have

$$\sigma_{w,\min} = \frac{Mc}{I_t} = \frac{25(10^3)(0.0409)}{288(10^{-6})} = 3.55 \text{ MPa}$$

$$\sigma_{s,\min} = n(\sigma_{w,\min}) = 21(3.55) = 74.56 \text{ MPa}$$

Stress at any other location may be determined likewise.

EXAMPLE 3.10

Design of Steel Reinforced Concrete Beam

A concrete beam of width b and effective depth d is reinforced with three steel bars of diameter d_s (Figure 3.20a). Note that it is usual to use $a = 50$-mm allowance to protect the steel from corrosion or fire. Determine the maximum stresses in the materials produced by a positive bending moment of 50 kN · m.

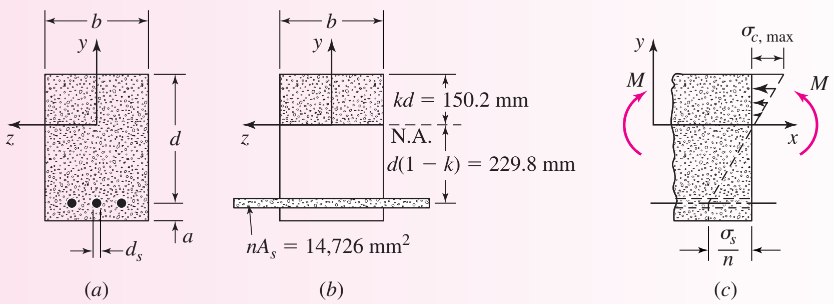


Figure 3.20 Example 3.10. Reinforced concrete beam.

Given: $b = 300$ mm, $d = 380$ mm, and $d_s = 25$ mm.

Assumptions: The modular ratio will be $n = E_s/E_c = 10$. The steel is uniformly stressed. Concrete resists only compression.

Solution: The portion of the cross section located a distance kd above the neutral axis is used in the transformed section (Figure 3.20b). The transformed area of the steel

$$nA_s = 10[3(\pi \times 25^2/4)] = 14{,}726 \text{ mm}^2$$

This is located by a single dimension from the neutral axis to its centroid. The compressive stress in the concrete is taken to vary linearly from the neutral axis. The first moment of the transformed section with respect to the neutral axis must be 0. Therefore,

$$b(kd)\frac{kd}{2} - nA_s(d - kd) = 0$$

from which

$$\boxed{(kd)^2 + (kd)\frac{2nA_s}{b} - \frac{2nA_s}{b}d = 0} \tag{3.29}$$

Introducing the required numerical values, Eq. (3.29) becomes

$$(kd)^2 + 98.17(kd) - 37.31 \times 10^3 = 0$$

Solving, $kd = 150.2$ mm, and hence $k = 0.395$. The moment of inertia of the transformed cross section about the neutral axis is

$$I_t = \frac{1}{12}(0.3)(0.1502)^3 + 0.3(0.1502)(0.0751)^2 + 0 + 14.73 \times 10^{-3}(0.2298)^2$$

$$= 1116.5 \times 10^{-6} \text{ m}^4$$

The peak compressive stress in the concrete and tensile stress in the steel are

$$\sigma_{c,\max} = \frac{Mc}{I_t} = \frac{50 \times 10^3(0.1502)}{1116.5 \times 10^{-6}} = 6.73 \text{ MPa}$$

$$\sigma_s = \frac{nMc}{I_t} = \frac{10(50 \times 10^3)(0.2298)}{1116.5 \times 10^{-6}} = 102.9 \text{ MPa}$$

These stresses act as shown in Figure 3.20c.

Comments: Often an alternative method of solution is used to estimate readily the stresses in reinforced concrete [6]. We note that, inasmuch as concrete is very weak in tension, the beam depicted in Figure 3.20 would become practically useless, should the bending moments act in the opposite direction. For balanced reinforcement, the beam must be designed so that stresses in concrete and steel are at their allowable levels simultaneously.

3.9 PLANE STRESS

The stresses and strains treated thus far have been found on sections perpendicular to the coordinates used to describe a member. This section deals with the states of stress at points located on inclined planes. In other words, we wish to obtain the stresses acting on the sides of a stress element oriented in any desired direction. This process is termed a *stress transformation*. The discussion that follows is limited to two-dimensional, or plane, stress. A two-dimensional state of stress exists when the stresses are independent of one of the coordinate axes, here taken as z. The plane stress is therefore specified by $\sigma_z = \tau_{yz} = \tau_{xz} = 0$, where σ_x, σ_y, and τ_{xy} have nonzero values. Examples include the stresses arising on inclined sections of an axially loaded bar, a shaft in torsion, a beam with transversely applied force, and a member subjected to more than one load simultaneously.

Consider the stress components σ_x, σ_y, τ_{xy} at a point in a body represented by a two-dimensional stress element (Figure 3.21a). To portray the stresses acting on an inclined section, an infinitesimal wedge is isolated from this element and depicted in Figure 3.21b. The angle θ, locating the x' axis or the unit normal n to the plane AB, is assumed positive when measured from the x axis in a counterclockwise direction. Note that, according to the sign convention (see Section 1.13), the stresses are indicated as positive values. It can be shown that equilibrium of the forces caused by stresses acting on the wedge-shaped element gives the following transformation equations for plane stress [1–3]:

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \tag{3.30a}$$

$$\tau_{x'y'} = \tau_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x)\sin\theta\cos\theta \tag{3.30b}$$

The stress $\sigma_{y'}$ may readily be obtained by replacing θ in Eq. (3.30a) by $\theta + \pi/2$ (Figure 3.21c). This gives

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \tag{3.30c}$$

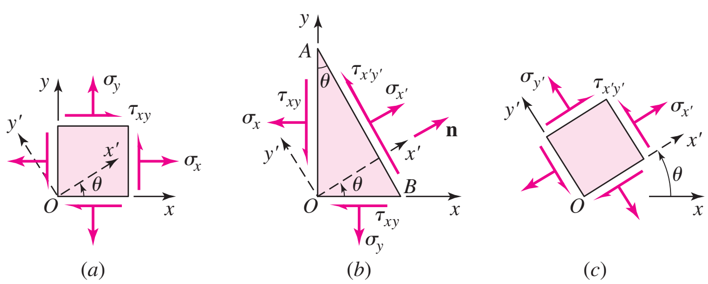


Figure 3.21 Elements in plane stress.

Using the double-angle relationships, the foregoing equations can be expressed in the following useful alternative form:

$$\sigma_{x'} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \tag{3.31a}$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{3.31b}$$

$$\sigma_{y'} = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \tag{3.31c}$$

For design purposes, the largest stresses are usually needed. The two perpendicular directions (θ'_p and θ''_p) of planes on which the shear stress vanishes and the normal stress has extreme values can be found from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{3.32}$$

The angle θ_p defines the orientation of the principal planes (Figure 3.22). The in-plane principal stresses can be obtained by substituting each of the two values of θ_p from Eq. (3.32) into Eqs. (3.31a and c) as follows:

$$\sigma_{\max,\min} = \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{3.33}$$

The plus sign gives the algebraically larger maximum principal stress σ_1. The minus sign results in the minimum principal stress σ_2. It is necessary to substitute θ_p into Eq. (3.31a) to learn which of the two corresponds to σ_1.

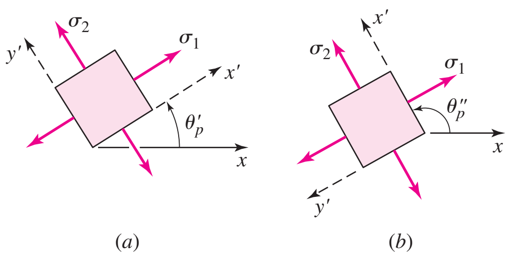


Figure 3.22 Planes of principal stresses.

EXAMPLE 3.11 Finding Stresses in a Cylindrical Pressure Vessel Welded along a Helical Seam

Figure 3.23a depicts a cylindrical pressure vessel constructed with a helical weld that makes an angle ψ with the longitudinal axis. Determine

(a) The maximum internal pressure p.

(b) The shear stress in the weld.

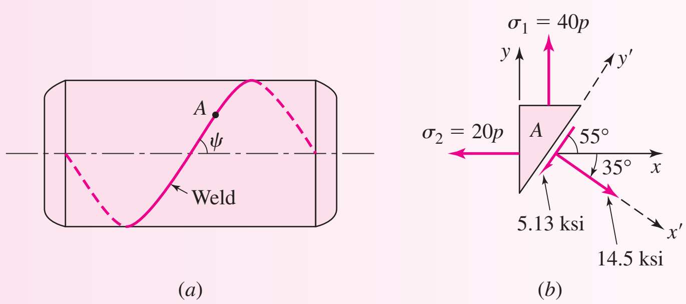


Figure 3.23 Example 3.11.

Given: $r = 10$ in., $t = \frac{1}{4}$ in., and $\psi = 55°$. Allowable tensile strength of the weld is 14.5 ksi.

Assumptions: Stresses are at a point A on the wall away from the ends. Vessel is a thin-walled cylinder.

Solution: The principal stresses in axial and tangential directions are, respectively,

$$\sigma_a = \frac{pr}{2t} = \frac{p(10)}{2\left(\frac{1}{4}\right)} = 20p = \sigma_2, \qquad \sigma_\theta = 2\sigma_a = 40p = \sigma_1$$

The state of stress is shown on the element of Figure 3.23b. We take the x' axis perpendicular to the plane of the weld. This axis is rotated $\theta = 35°$ clockwise with respect to the x axis.

(a) Through the use of Eq. (3.31a), the tensile stress in the weld:

$$\sigma_{x'} = \frac{\sigma_2 + \sigma_1}{2} + \frac{\sigma_2 - \sigma_1}{2}\cos 2(-35°)$$
$$= 30p - 10p\cos(-70°) \leq 14{,}500$$

from which $p_{\max} = 546$ psi.

(b) Applying Eq. (3.31b), the shear stress in the weld corresponding to the foregoing value of pressure is

$$\tau_{x'y'} = -\frac{\sigma_2 - \sigma_1}{2}\sin 2(-35°)$$
$$= 10p\sin(-70°) = -5.13 \text{ ksi}$$

The answer is presented in Figure 3.23b.

MOHR'S CIRCLE FOR STRESS

Transformation equations for plane stress, Eqs. (3.31), can be represented with σ and τ as coordinate axes in a graphical form known as *Mohr's circle* (Figure 3.24b). This representation is very useful in visualizing the relationships between normal and shear stresses acting on various inclined planes at a point in a stressed member. Also, with the aid of this graphical construction, a quicker solution of stress-transformation problem can be facilitated. The coordinates for point A on the circle correspond to the stresses on the x face or plane of the element shown in Figure 3.24a. Similarly, the coordinates of a point A' on Mohr's circle are to be interpreted representing the stress components $\sigma_{x'}$ and $\tau_{x'y'}$ that act on x' plane. The center is at $(\sigma', 0)$ and the circle radius r equals the length CA. In Mohr's circle representation the normal stresses obey the *sign convention* of Section 1.13. However, for the purposes of *only constructing and reading values* of stress from a Mohr's circle, the shear stresses on the y planes of the element are taken to be positive (as before) but those on the x faces are now negative, Figure 3.24c.

The magnitude of the maximum shear stress is equal to the radius r of the circle. From the geometry of Figure 3.24b, we obtain

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{3.34}$$

Mohr's circle shows the planes of maximum shear are always oriented at 45° from planes of principal stress (Figure 3.25). Note that a diagonal of a stress element along which the algebraically larger principal stress acts is called the *shear diagonal*. The maximum shear stress acts toward the shear diagonal. The normal stress occurring on planes of maximum shear stress is

$$\sigma' = \sigma_{\text{avg}} = \frac{1}{2}(\sigma_x + \sigma_y) \tag{3.35}$$

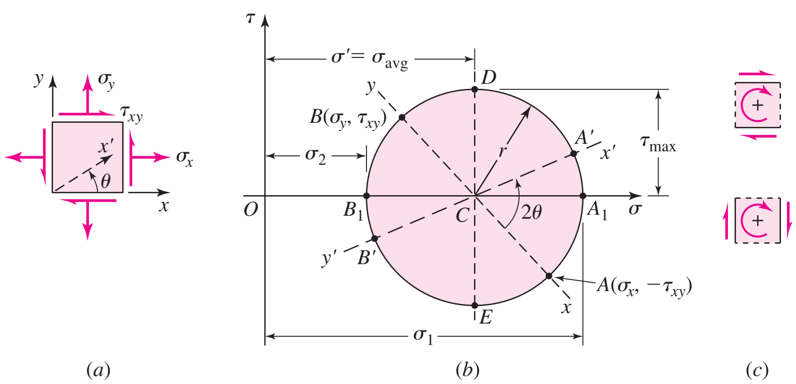


Figure 3.24 (a) Stress element; (b) Mohr's circle of stress; (c) interpretation of positive shear stress.

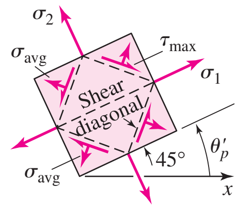


Figure 3.25 Planes of principal and maximum shear stresses.

It can readily be verified using Mohr's circle that, on any mutually perpendicular planes,

$$I_1 = \sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'} \qquad I_2 = \sigma_x\sigma_y - \tau_{xy}^2 = \sigma_{x'}\sigma_{y'} - \tau_{x'y'}^2 \tag{3.36}$$

The quantities I_1 and I_2 are known as two-dimensional *stress invariants,* because they do not change in value when the axes are rotated positions. Equations (3.36) are particularly useful in checking numerical results of stress transformation.

Note that, in the case of triaxial stresses σ_1, σ_2, and σ_3, a Mohr's circle is drawn corresponding to each projection of a three-dimensional element. The three-circle cluster represents Mohr's circle for triaxial stress (see Figure 3.28). The general state of stress at a point is discussed in some detail in the later sections of this chapter. Mohr's circle construction is of fundamental importance because it applies to all (second-rank) tensor quantities; that is, Mohr's circle may be used to determine strains, moments of inertia, and natural frequencies of vibration [7]. It is customary to draw for Mohr's circle only a rough sketch; distances and angles are determined with the help of trigonometry. Mohr's circle provides a convenient means of obtaining the results for the stresses under the following two common loadings.

Axial Loading

In this case, we have $\sigma_x = \sigma_1 = P/A$, $\sigma_y = 0$, and $\tau_{xy} = 0$, where A is the cross-sectional area of the bar. The corresponding points A and B define a circle of radius $r = P/2A$ that passes through the origin of coordinates (Figure 3.26b). Points D and E yield the orientation of the planes of the maximum shear stress (Figure 3.26a), as well as the values of τ_{max} and the corresponding normal stress σ':

$$\tau_{max} = \sigma' = r = \frac{P}{2A} \tag{a}$$

Observe that the normal stress is either maximum or minimum on planes for which shearing stress is 0.

Torsion

Now we have $\sigma_x = \sigma_y = 0$ and $\tau_{xy} = \tau_{max} = Tc/J$, where J is the polar moment of inertia of cross-sectional area of the bar. Points D and E are located on the τ axis, and Mohr's

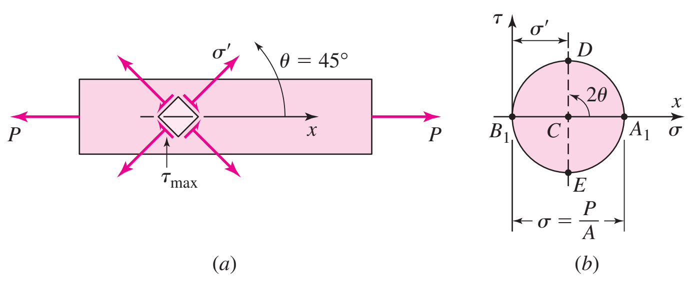


Figure 3.26 (a) Maximum shear stress acting on an element of an axially loaded bar; (b) Mohr's circle for uniaxial loading.

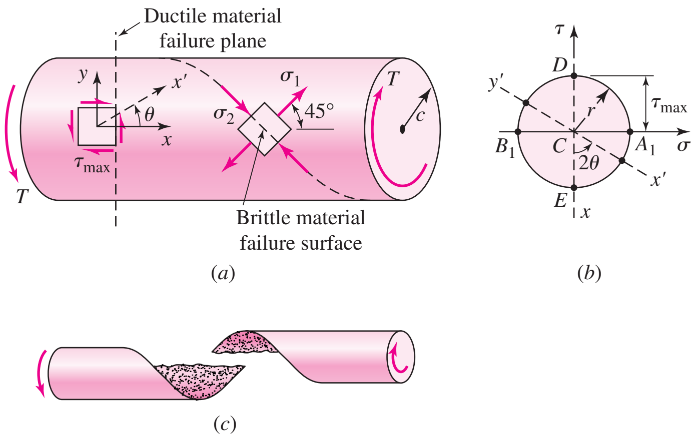


Figure 3.27 (a) Stress acting on a surface element of a twisted shaft; (b) Mohr's circle for torsional loading; (c) brittle material fractured in torsion.

circle is a circle of radius $r = Tc/J$ centered at the origin (Figure 3.27b). Points A_1 and B_1 define the principal stresses:

$$\sigma_{1,2} = \pm r = \pm \frac{Tc}{J} \qquad \text{(b)}$$

So, it becomes evident that, for a material such as cast iron that is weaker in tension than in shear, failure occurs in tension along a helix indicated by the dashed lines in Figure 3.27a. Fracture of a bar that behaves in a brittle manner in torsion is depicted in Figure 3.27c; ordinary chalk behaves this way. Shafts made of materials weak in shear strength (for example, structural steel) break along a line perpendicular to the axis. Experiments show that a very thin-walled hollow shaft buckles or wrinkles in the direction of maximum compression while, in the direction of maximum tension, tearing occurs.

Stress Analysis of Cylindrical Pressure Vessel Using Mohr's Circle

EXAMPLE 3.12

Redo Example 3.11 using Mohr's circle. Also determine maximum in-plane and absolute shear stresses at a point on the wall of the vessel.

Solution: Mohr's circle, Figure 3.28, constructed referring to Figure 3.23 and Example 3.11, describes the state of stress. The x' axis is rotated $2\theta = 70^\circ$ on the circle with respect to x axis.

(a) From the geometry of Figure 3.28, we have $\sigma_{x'} = 30p - 10p\cos 70^\circ \leq 14{,}500$. This results in $p_{max} = 546$ psi.

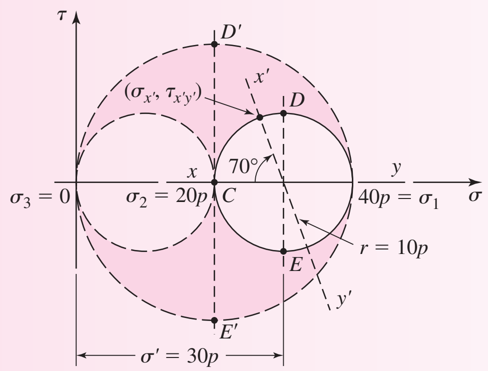


Figure 3.28 Example 3.12.

(b) For the preceding value of pressure the shear stress in the weld is

$$\tau_{x'y'} = \pm 10(546)\sin 70^\circ = \pm 5.13 \text{ ksi}$$

The largest in-plane shear stresses are given by points D and E on the circle. Hence,

$$\tau = \pm\frac{1}{2}(40p - 20p) = \pm 10(546) = \pm 5.46 \text{ ksi}$$

The third principal stress in the radial direction is 0, $\sigma_3 = 0$. The three principal stress circles are shown in the figure. The absolute maximum shear stresses are associated with points D' and E' on the major principal circle. Therefore,

$$\tau_{\max} = \pm\frac{1}{2}(40p - 0) = \pm 20(546) = \pm 10.92 \text{ ksi}$$

3.10 COMBINED STRESSES

Basic formulas of mechanics of materials for determining the state of stress in elastic members are developed in Sections 3.2 through 3.7. Often these formulas give either a normal stress or a shear stress caused by a single load component being axially, centric, or lying in one plane. Note that each formula leads to stress as directly proportional to the magnitude of the applied load. When a component is acted on simultaneously by two or more loads, causing various internal-force resultants on a section, it is assumed that each load produces the stress as if it were the only load acting on the member. The final or combined stress is then found by superposition of several states of stress. As we see throughout the text, under combined loading, the critical points may not be readily located. Therefore, it may be necessary to examine the stress distribution in some detail.

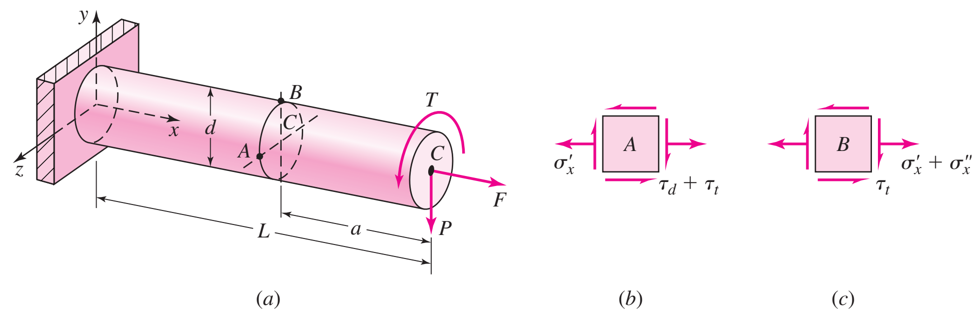


Figure 3.29 Combined stresses owing to torsion, tension, and direct shear.

Consider, for example, a solid circular cantilevered bar subjected to a transverse force P, a torque T, and a centric load F at its free end (Figure 3.29a). Every section experiences an axial force F, torque T, a bending moment M, and a shear force $P = V$. The corresponding stresses may be obtained using the applicable relationships:

$$\sigma_x' = \frac{F}{A}, \qquad \tau_t = -\frac{Tc}{J}, \qquad \sigma_x'' = -\frac{Mc}{I}, \qquad \tau_d = -\frac{VQ}{Ib}$$

Here τ_t and τ_d are the torsional and direct shear stresses, respectively. In Figures 3.29b and 3.29c, the stresses shown are those acting on an element B at the top of the bar and on an element A on the side of the bar at the neutral axis. Clearly, B (when located at the support) and A represent the critical points at which most severe stresses occur. The principal stresses and maximum shearing stress at a critical point can now be ascertained as discussed in the preceding section.

The following examples illustrate the general approach to problems involving combined loadings. Any number of critical locations in the components can be analyzed. These either confirm the adequacy of the design or, if the stresses are too large (or too small), indicate the design changes required. This is used in a seemingly endless variety of practical situations, so it is often not worthwhile to develop specific formulas for most design use. We develop design formulas under combined loading of common mechanical components, such as shafts, shrink or press fits, flywheels, and pressure vessels in Chapters 9 and 16.

Determining the Allowable Combined Loading in a Cantilever Bar

EXAMPLE 3.13

A round cantilever bar is loaded as shown in Figure 3.29a. Determine the largest value of the load P.

Given: diameter $d = 60$ mm, $T = 0.1P$ N · m, and $F = 10P$ N.

Assumptions: Allowable stresses are 100 MPa in tension and 60 MPa in shear on a section at $a = 120$ mm from the free end.

Solution: The normal stress at all points of the bar is

$$\sigma'_x = \frac{F}{A} = \frac{10P}{\pi(0.03)^2} = 3536.8P \tag{a}$$

The torsional stress at the outer fibers of the bar is

$$\tau_t = -\frac{Tc}{J} = -\frac{0.1P(0.03)}{\pi(0.03)^4/2} = -2357.9P \tag{b}$$

The largest tensile bending stress occurs at point B of the section considered. Therefore, for $a = 120$ mm, we obtain

$$\sigma''_x = \frac{Mc}{I} = \frac{0.12P(0.03)}{\pi(0.03)^4/4} = 5658.8P$$

Since $Q = A\overline{y} = (\pi c^2/2)(4c/3\pi) = 2c^3/3$ and $b = 2c$, the largest direct shearing stress at point A is

$$\tau_d = -\frac{VQ}{Ib} = -\frac{4V}{3A} = -\frac{4P}{3\pi(0.03)^2} = -471.57P \tag{c}$$

The maximum principal stress and the maximum shearing stress at point A (Figure 3.29b), applying Eqs. (3.33) and (3.34) with $\sigma_y = 0$, Eqs. (a), (b), and (c) are

$$\begin{aligned}(\sigma_1)_A &= \frac{\sigma'_x}{2} + \left[\left(\frac{\sigma'_x}{2}\right)^2 + (\tau_d + \tau_t)^2\right]^{1/2}\\ &= \frac{3536.8P}{2} + \left[\left(\frac{3536.8P}{2}\right)^2 + (-2829.5P)^2\right]^{1/2}\\ &= 1768.4P + 3336.7P = 5105.1P\\ (\tau_{\max})_A &= 3336.7P\end{aligned}$$

Likewise, at point B (Figure 3.29c),

$$\begin{aligned}(\sigma_1)_B &= \frac{\sigma'_x + \sigma''_x}{2} + \left[\left(\frac{\sigma'_x + \sigma''_x}{2}\right)^2 + \tau_t^2\right]^{1/2}\\ &= \frac{9195.6P}{2} + \left[\left(\frac{9195.6P}{2}\right)^2 + (-2357.9P)^2\right]^{1/2}\\ &= 4597.8P + 5167.2P = 9765P\\ (\tau_{\max})_B &= 5167.2P\end{aligned}$$

It is observed that the stresses at B are more severe than those at A. Inserting the given data into the foregoing, we obtain

$$100(10^6) = 9765P \qquad \text{or} \qquad P = 10.24 \text{ kN}$$
$$60(10^6) = 5167.2P \qquad \text{or} \qquad P = 11.61 \text{ kN}$$

Comment: The magnitude of the largest allowable transverse, axial, and torsional loads that can be carried by the bar are $P = 10.24$ kN, $F = 102.4$ kN, and $T = 1.024$ kN · m, respectively.

Determination of Maximum Allowable Pressure in a Pipe under Combined Loading

EXAMPLE 3.14

A cylindrical pipe subjected to internal pressure p is simultaneously compressed by an axial load P through the rigid end plates, as shown in Figure 3.30a. Calculate the largest value of p that can be applied to the pipe.

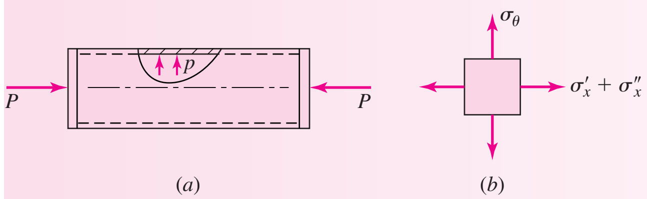


Figure 3.30 Example 3.14.

Given: The pipe diameter $d = 120$ mm, thickness $t = 5$ mm, and $P = 60$ kN. Allowable in-plane shear stress in the wall is 80 MPa.

Assumption: The critical stress is at a point on cylinder wall away from the ends.

Solution: The cross-sectional area of this thin-walled shell is $A = \pi dt$. Combined axial and tangential stresses act at a critical point on an element in the wall of the pipe (Figure 3.30b). We have

$$\sigma_x'' = -\frac{P}{\pi dt} = -\frac{60(10^3)}{\pi(0.12 \times 0.005)} = -31.83 \text{ MPa}$$

$$\sigma_x' = \frac{pr}{2t} = \frac{p(60)}{2(5)} = 6p$$

$$\sigma_\theta = \frac{pr}{t} = 12p$$

Applying Eq. (3.34),

$$\tau_{\max} = \frac{1}{2}[\sigma_\theta - (\sigma_x' + \sigma_x'')] = \frac{1}{2}[12p - (6p - 31.83)]$$

$$= 3p_{\max} + 15.915 \leq 80$$

from which

$$p_{\max} = 21.36 \text{ MPa}$$

Case Study 3-1 WINCH CRANE FRAME STRESS ANALYSIS

The frame of a winch crane is represented schematically in Figure 1.5. Determine the maximum stress and the factor of safety against yielding.

Given: The geometry and loading are known from Case Study 1-1. The frame is made of ASTM-A36 structural steel tubing. From Table B.1:

$$S_y = 250 \text{ MPa} \qquad E = 200 \text{ GPa}$$

Assumptions: The loading is static. The displacements of welded joint C are negligibly small, hence part CD of the frame is considered a cantilever beam.

Solution: See Figures 1.5 and 3.31 and Table B.1.

We observe from Figure 1.5 that the maximum bending moment occurs at points B and C and $M_B = M_C$. Since two vertical beams resist moment at B, the critical section is at C of cantilever CD carrying its own weight per unit length w and concentrated load P at the free end (Figure 3.31).

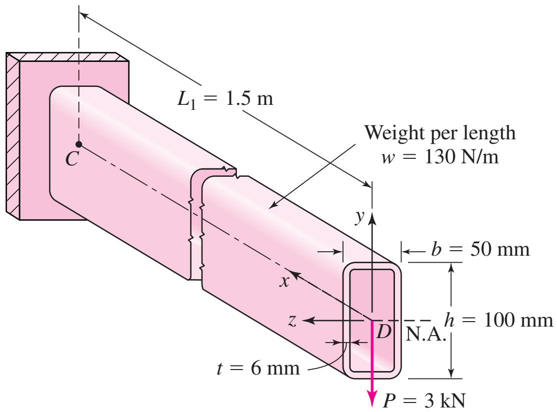


Figure 3.31 Part CD of the crane frame shown in Figure 1.4.

The bending moment M_C and shear force V_C at the cross section through the point C, from static equilibrium, have the following values:

$$M_C = PL_1 + \frac{1}{2}wL_1^2$$

$$= 3000(1.5) + \frac{1}{2}(130)(1.5)^2 = 4646 \text{ N} \cdot \text{m}$$

$$V_C = 3 \text{ kN}$$

The cross-sectional area properties of the tubular beam are

$$A = bh - (b - 2t)(h - 2t)$$

$$= 50 \times 100 - 38 \times 88 = 1.66(10^{-3}) \text{ m}^2$$

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b - 2t)(h - 2t)^3$$

$$= \frac{1}{12}[(50 \times 100^3) - (38)(88)^3] = 2.01(10^{-6}) \text{ m}^4$$

where I represents the moment of inertia about the neutral axis.

Therefore, the maximum bending stress at point C equals

$$\sigma_C = \frac{Mc}{I} = \frac{4646(0.05)}{2.01(10^{-6})} = 115.6 \text{ MPa}$$

The highest value of the shear stress occurs at the neutral axis. Referring to Figure 3.31, the first moment of the area about the N.A. is

$$Q_{\max} = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - 2t)\left(\frac{h}{2} - t\right)\left(\frac{h/2 - t}{2}\right)$$

$$= 50(50)(25) - (38)(44)(22) = 25.716(10^{-6}) \text{ m}^3$$

Hence,

$$\tau_C = \frac{V_C Q_{\max}}{Ib}$$

$$= \frac{3000(25.716)}{2.01(2 \times 0.006)} = 3.199 \text{ MPa}$$

Case Study (CONCLUDED)

We obtain the largest principal stress $\sigma_1 = \sigma_{max}$ from Eq. (3.33), which in this case reduces to

$$\sigma_{max} = \frac{\sigma_C}{2} + \sqrt{\left(\frac{\sigma_C}{2}\right)^2 + \tau_C^2}$$

$$= \frac{115.6}{2} + \left[\left(\frac{115.6}{2}\right)^2 + (3.199)^2\right]^{1/2}$$

$$= 115.7 \text{ MPa}$$

The factor of safety against yielding is then

$$n = \frac{S_y}{\sigma_{max}} = \frac{250}{115.7} = 2.16$$

This is satisfactory because the frame is made of average material operated in ordinary environment and subjected to known loads.

Comments: At joint C, as well as at B, a thin (about 6-mm) steel gusset should be added at each side (not shown in the figure). These enlarge the weld area of the joints and help reduce stress in the welds. Case Study 15-2 illustrates the design analysis of the welded joint at C.

Case Study 3-2 BOLT CUTTER STRESS ANALYSIS

A bolt cutting tool is shown in Figure 1.6. Determine the stresses in the members.

Given: The geometry and forces are known from Case Study 1-2. Material of all parts is AISI 1080 HR steel. Dimensions are in inches. We have

$$S_y = 60.9 \text{ ksi (Table B.3)}, \qquad S_{ys} = 0.5S_y = 30.45 \text{ ksi},$$

$$E = 30 \times 10^6 \text{ psi}$$

Assumptions:

1. The loading is taken to be static. The material is ductile, and stress concentration factors can be disregarded under steady loading.
2. The most likely failure points are in link 3, the hole where pins are inserted, the connecting pins in shear, and jaw 2 in bending.
3. Member 2 can be approximated as a simple beam with an overhang.

Solution: See Figures 1.6 and 3.32.

The largest force on any pin in the assembly is at joint A.

Member 3 is a pin-ended tensile link. The force on a pin is 128 lb, as shown in Figure 3.32a. The normal stress is therefore

$$\sigma = \frac{F_A}{(w_3 - d)t_3} = \frac{128}{\left(\frac{3}{8} - \frac{1}{8}\right)\left(\frac{1}{8}\right)} = 4.096 \text{ ksi}$$

For the bearing stress in the joint A, using Eq. (3.5), we have

$$\sigma_b = \frac{F_A}{dt_3} = \frac{128}{\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)} = 8.192 \text{ ksi}$$

The link and other members have ample material around holes to prevent tearout. The $\frac{1}{8}$-in. diameter pins are in single shear. The worst-case direct shear stress, from Eq. (3.4),

$$\tau = \frac{4F_A}{\pi d^2} = \frac{4(128)}{\pi\left(\frac{1}{8}\right)^2} = 10.43 \text{ ksi}$$

(continued)

Case Study (CONCLUDED)

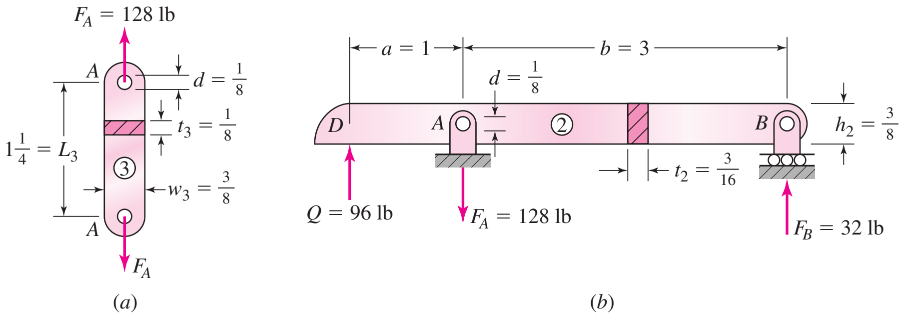


Figure 3.32 Some free-body diagrams of bolt cutter shown in Figure 1.6: (a) link 3; (b) jaw 2.

Member 2, the jaw, is supported and loaded as shown in Figure 3.32b. The moment of inertia of the cross-sectional area is

$$I = \frac{t_2}{12}\left(h_2^3 - d^3\right)$$

$$= \frac{3/16}{12}\left[\left(\frac{3}{8}\right)^3 - \left(\frac{1}{8}\right)^3\right] = 0.793(10^{-3}) \text{ in.}^4$$

The maximum moment, that occurs at point A of the jaw, equals $M = F_B b = 32(3) = 96$ lb · in. The bending stress is then

$$\sigma_C = \frac{Mc}{I} = \frac{96\left(\frac{3}{16}\right)}{0.793 \times 10^{-3}} = 22.7 \text{ ksi}$$

It can readily be shown that, the shear stress is negligibly small in the jaw.

Member 1, the handle, has an irregular geometry and is relatively massive compared to the other components of the assembly. Accurate values of stresses as well as deflections in the handle may be obtained by the finite element analysis.

Comment: The results show that the maximum stresses in members are well under the yield strength of the material.

3.11 PLANE STRAIN

In the case of two-dimensional, or plane, strain, all points in the body before and after the application of the load remain in the same plane. Therefore, in the *xy* plane the strain components ε_x, ε_y, and γ_{xy} may have nonzero values. The normal and shear strains at a point in a member vary with direction in a way analogous to that for stress. We briefly discuss expressions that give the strains in the inclined directions. These in-plane strain transformation equations are particularly significant in experimental investigations, where strains are measured by means of strain gages. The site at www.measurementsgroup.com includes general information on strain gages as well as instrumentation.

Mathematically, in every respect, the transformation of strain is the same as the stress transformation. It can be shown that [2] *transformation* expressions of stress are converted into strain relationships by substitution:

$$\sigma \rightarrow \varepsilon \quad \text{and} \quad \tau \rightarrow \gamma/2 \tag{a}$$

These replacements can be made in all the analogous two- and three-dimensional transformation relations. Therefore, the principal strain directions are obtained from Eq. (3.32) in the form, for example,

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \tag{3.37}$$

Using Eq. (3.33), the magnitudes of the in-plane principal strains are

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{3.38}$$

In a like manner, the in-plane transformation of strain in an arbitrary direction proceeds from Eqs. (3.31):

$$\varepsilon_{x'} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \tag{3.39a}$$

$$\gamma_{x'y'} = -(\varepsilon_x - \varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta \tag{3.39b}$$

$$\varepsilon_{y'} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) - \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \tag{3.39c}$$

An expression for the maximum shear strain may also be found from Eq. (3.34). Similarly, the transformation equations of three-dimensional strain may be deduced from the corresponding stress relations, given in Section 3.18.

In *Mohr's circle for strain,* the normal strain ε is plotted on the horizontal axis, positive to the right. The vertical axis is measured in terms of $\gamma/2$. The center of the circle is at $(\varepsilon_x + \varepsilon_y)/2$. When the shear strain is *positive,* the point representing the x axis strain is plotted a distance $\gamma/2$ *below* the axis and vice versa when shear strain is negative. Note that this convention for shearing strain, used *only* in constructing and reading values from Mohr's circle, agrees with the convention used for stress in Section 3.9.

Determination of Principal Strains Using Mohr's Circle

EXAMPLE 3.15

It is observed that an element of a structural component elongates 450μ along the x axis, contracts 120μ in the y direction, and distorts through an angle of -360μ (see Section 1.14). Calculate

(a) The principal strains.

(b) The maximum shear strains.

Given: $\varepsilon_x = 450\mu,\ \varepsilon_y = -120\mu,\ \gamma_{xy} = -360\mu$

Assumption: Element is in a state of plane strain.

Solution: A sketch of Mohr's circle is shown in Figure 3.33, constructed by finding the position of point C at $\varepsilon' = (\varepsilon_x + \varepsilon_y)/2 = 165\mu$ on the horizontal axis and of point A at $(\varepsilon_x, -\gamma_{xy}/2) = (450\mu, 180\mu)$ from the origin O.

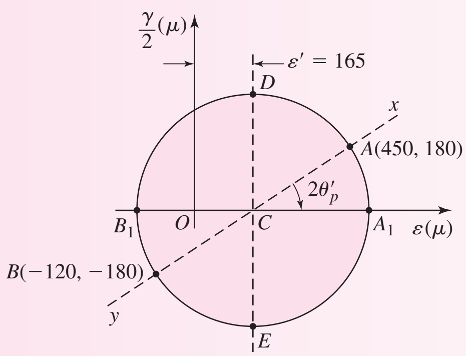


Figure 3.33 Example 3.15.

(a) The in-plane principal strains are represented by points A and B. Hence,

$$\varepsilon_{1,2} = 165\mu \pm \left[\left(\frac{450+120}{2}\right)^2 + (-180)^2\right]^{1/2}$$

$$\varepsilon_1 = 502\mu \qquad \varepsilon_2 = -172\mu$$

Note, as a check, that $\varepsilon_x + \varepsilon_y = \varepsilon_1 + \varepsilon_2 = 330\mu$. From geometry,

$$\theta'_p = \frac{1}{2}\tan^{-1}\frac{180}{285} = 16.14°$$

It is seen from the circle that θ'_p locates the ε_1 direction.

(b) The maximum shear strains are given by points D and E. Hence,

$$\gamma_{\max} = \pm(\varepsilon_1 - \varepsilon_2) = \pm 674\mu$$

Comments: Mohr's circle depicts that the axes of maximum shear strain make an angle of 45° with respect to principal axes. In the directions of maximum shear strain, the normal strains are equal to $\varepsilon' = 165\mu$.

3.12 STRESS CONCENTRATION FACTORS

The condition where high localized stresses are produced as a result of an abrupt change in geometry is called the stress concentration. The abrupt change in form or discontinuity occurs in such frequently encountered stress raisers as holes, notches, keyways, threads, grooves, and fillets. Note that the stress concentration is a primary cause of fatigue failure and static failure in brittle materials, discussed in the next section. The formulas of mechanics of materials apply as long as the material remains linearly elastic and shape variations are gradual. In some cases, the stress and accompanying deformation near a discontinuity can be analyzed by applying the theory of elasticity. In those instances that do not yield to analytical methods, it is more usual to rely on experimental techniques or the finite element method (see Case Study 17-4). In fact, much research centers on determining stress concentration effects for combined stress.

A geometric or theoretical *stress concentration factor* K_t is used to relate the maximum stress at the discontinuity to the nominal stress. The factor is defined by

$$K_t = \frac{\sigma_{\max}}{\sigma_{\text{nom}}} \quad \text{or} \quad K_t = \frac{\tau_{\max}}{\tau_{\text{nom}}} \tag{3.40}$$

Here the nominal stresses are stresses that would occur if the abrupt change in the cross section did not exist or had no influence on stress distribution. It is important to note that a stress concentration factor is applied to the stress computed for the net or reduced cross section. Stress concentration factors for several types of configuration and loading are available in technical literature [8–13].

The stress concentration factors for a variety of geometries, provided in Appendix C, are useful in the design of machine parts. Curves in the Appendix C figures are plotted on the basis of dimensionless ratios: the shape, but not the size, of the member is involved. Observe that all these graphs indicate the advisability of streamlining junctures and transitions of portions that make up a member; that is, stress concentration can be reduced in intensity by properly proportioning the parts. Large fillet radii help at reentrant corners.

The values shown in Figures C.1, C.2, and C.7 through C.9 are for fillets of radius r that join a part of depth (or diameter) d to the one of larger depth (or diameter) D at a step or shoulder in a member (see Figure 3.34). A full fillet is a 90° arc with radius $r = (D - d_f)/2$. The stress concentration factor decreases with increases in r/d or d/D. Also, results for the axial tension pertain equally to cases of axial compression. However, the stresses obtained are valid only if the loading is not significant relative to that which would cause failure by buckling.

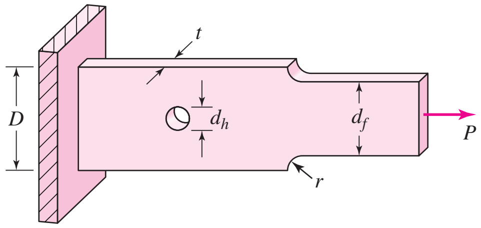


Figure 3.34 A flat bar with fillets and a centric hole under axial loading.

EXAMPLE 3.16

Design of Axially Loaded Thick Plate with a Hole and Fillets

A filleted plate of thickness t supports an axial load P (Figure 3.34). Determine the radius r of the fillets so that the same stress occurs at the hole and the fillets.

Given: $P = 50$ kN, $D = 100$ mm, $d_f = 66$ mm, $d_h = 20$ mm, $t = 10$ mm

Design Decisions: The plate will be made of a relatively brittle metallic alloy; we must consider stress concentration.

Solution: For the *circular hole,*

$$\frac{d_h}{D} = \frac{20}{100} = 0.2, \qquad A = (D - d_h)t = (100 - 20)10 = 800 \text{ mm}^2$$

Using the lower curve in Figure C.5, we find that $K_t = 2.44$ corresponding to $d_h/D = 0.2$. Hence,

$$\sigma_{\max} = K_t \frac{P}{A} = 2.44 \frac{50 \times 10^3}{800(10^{-6})} = 152.5 \text{ MPa}$$

For *fillets,*

$$\sigma_{\max} = K_t \frac{P}{A} = K_t \frac{50 \times 10^3}{660(10^{-6})} = 75.8 K_t \text{ MPa}$$

The requirement that the maximum stress for the hole and fillets be identical is satisfied by

$$152.5 = 75.8 K_t \quad \text{or} \quad K_t = 2.01$$

From the curve in Figure C.1, for $D/d_f = 100/66 = 1.52$, we find that $r/d_f = 0.12$ corresponding to $K_t = 2.01$. The necessary fillet radius is therefore

$$r = 0.12 \times 66 = 7.9 \text{ mm}$$

3.13 IMPORTANCE OF STRESS CONCENTRATION FACTORS IN DESIGN

Under certain conditions, a normally ductile material behaves in a brittle manner and vice versa. So, for a specific application, the distinction between ductile and brittle materials must be inferred from the discussion of Section 2.9. Also remember that the determination of stress concentration factors is based on the use of Hooke's law.

Fatigue Loading

Most engineering materials may fail as a result of propagation of cracks originating at the point of high dynamic stress. The presence of stress concentration in the case of fluctuating (and impact) loading, as found in some machine elements, must be considered, regardless of whether the material response is brittle or ductile. In machine design, then, fatigue stress concentrations are of paramount importance. However, its effect on the nominal stress is not as large, as indicated by the theoretical factors (see Section 8.7).

STATIC LOADING

For static loading, stress concentration is important only for *brittle* material. However, for some brittle materials having internal irregularities, such as cast iron, stress raisers usually have little effect, regardless of the nature of loading. Hence, the use of a stress concentration factor appears to be unnecessary for cast iron. Customarily, stress concentration is ignored in static loading of *ductile* materials. The explanation for this restriction is quite simple. For ductile materials slowly and steadily loaded beyond the yield point, the stress concentration factors decrease to a value approaching unity because of the redistribution of stress around a discontinuity.

To illustrate the foregoing inelastic action, consider the behavior of a mild-steel flat bar that contains a hole and is subjected to a gradually increasing load P (Figure 3.35). When $\sigma_{\max}$ reaches the yield strength S_y, stress distribution in the material is of the form of curve mn, and yielding impends at A. Some fibers are stressed in the plastic range but enough others remain elastic, and the member can carry additional load. We observe that the area under stress distribution curve is equal to the load P. This area increases as overload P increases, and a contained plastic flow occurs in the material [14]. Therefore, with the increase in the value of P, the stress-distribution curve assumes forms such as those shown by line mp and finally mq. That is, the effect of an abrupt change in geometry is nullified and $\sigma_{\max} = \sigma_{\text{nom}}$, or $K_t = 1$; prior to necking, a nearly *uniform* stress distribution across the net section occurs. Hence, for most practical purposes, the bar containing a hole carries the same static load as the bar with no hole.

The effect of ductility on the strength of the shafts and beams with stress raisers is similar to that of axially loaded bars. That is, localized inelastic deformations enable these members to support high stress concentrations. Interestingly, material ductility introduces a certain element of forgiveness in analysis while producing acceptable design results; for example, rivets can carry equal loads in a riveted connection (see Section 15.13).

When a member is yielded nonuniformly throughout a cross section, *residual stresses* remain in this cross section after the load is removed. An overload produces residual stresses favorable to future loads in the same direction and unfavorable to future loads in the opposite direction. Based on the idealized stress-strain curve, the increase in load

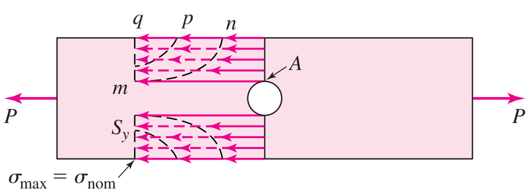


Figure 3.35 Redistribution of stress in a flat bar of mild steel.

capacity in one direction is the same as the decrease in load capacity in the opposite direction. Note that coil springs in compression are good candidates for favorable residual stresses caused by yielding.

3.14 CONTACT STRESS DISTRIBUTIONS

The application of a load over a small area of contact results in unusually high stresses. Situations of this nature are found on a microscopic scale whenever force is transmitted through bodies in contact. The original analysis of elastic contact stresses, by H. Hertz, was published in 1881. In his honor, the stresses at the mating surfaces of curved bodies in compression are called *Hertz contact stresses*. The Hertz problem relates to the stresses owing to the contact surface of a sphere on a plane, a sphere on a sphere, a cylinder on a cylinder, and the like. In addition to rolling bearings, the problem is of importance to cams, push rod mechanisms, locomotive wheels, valve tappets, gear teeth, and pin joints in linkages.

Consider the contact without deflection of two bodies having curved surfaces of different radii (r_1 and r_2), in the vicinity of contact. If a collinear pair of forces (F) presses the bodies together, deflection occurs and the point of contact is replaced by a small area of contact. The first steps taken toward the solution of this problem are the determination of the size and shape of the contact area as well as the distribution of normal pressure acting on the area. The deflections and subsurface stresses resulting from the contact pressure are then evaluated. The following *basic assumptions* are generally made in the solution of the Hertz problem:

1. The contacting bodies are isotropic, homogeneous, and elastic.
2. The contact areas are essentially flat and small relative to the radii of curvature of the undeflected bodies in the vicinity of the interface.
3. The contacting bodies are perfectly smooth, therefore friction forces need not be taken into account.

The foregoing set of presuppositions enables elastic analysis by theory of elasticity. Without going into the rather complex derivations, in this section, we introduce some of the results for both cylinders and spheres. The next section concerns the contact of two bodies of any general curvature. Contact problems of rolling bearings and gear teeth are discussed in the later chapters.*

SPHERICAL AND CYLINDRICAL SURFACES IN CONTACT

Figure 3.36 illustrates the contact area and corresponding stress distribution between two spheres, loaded with force F. Similarly, two parallel cylindrical rollers compressed by forces F is shown in Figure 3.37. We observe from the figures that, in each case, the maximum contact pressure exist on the load axis. The area of contact is defined by dimension a for the spheres and a and L for the cylinders. The relationships between the force of contact F,

*A summary and complete list of references dealing with contact stress problems are given by References [2, 4, 15–17].

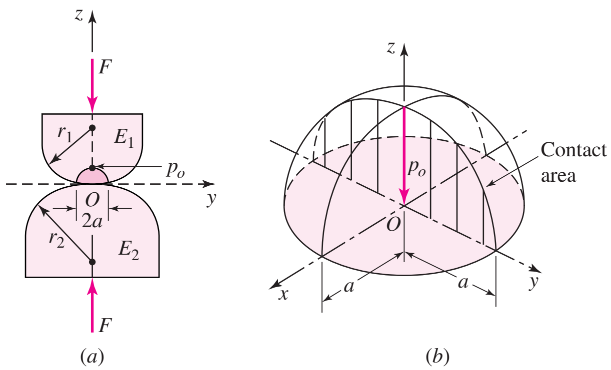


Figure 3.36 (a) Spherical surfaces of two members held in contact by force *F*. (b) Contact stress distribution. Note: The contact area is a circle of radius *a*.

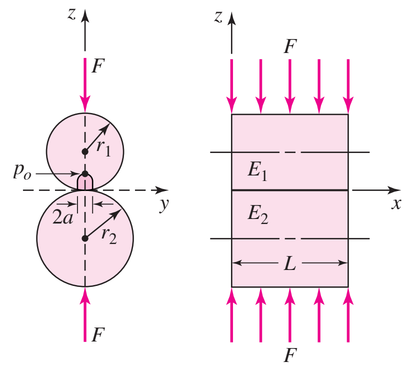


Figure 3.37 Two cylinders held in contact by force *F* uniformly distributed along cylinder length *L*. Note: The contact area is a narrow rectangle of $2a \times L$.

maximum pressure p_o, and the *deflection* δ in the point of contact are given in Table 3.2. Obviously, the δ represents the relative displacement of the centers of the two bodies, owing to local deformation. The contact pressure within each sphere or cylinder has a semi-elliptical distribution; it varies from 0 at the side of the contact area to a maximum value p_o at its center, as shown in the figures. For spheres, a is the radius of the circular contact area (πa^2). But, for cylinders, a represents the half-width of the rectangular contact area ($2aL$), where L is the length of the cylinder. Poisson's ratio ν in the formulas is taken as 0.3.

The material along the axis compressed in the z direction tends to expand in the x and y directions. However, the surrounding material does not permit this expansion; hence, the compressive stresses are produced in the x and y directions. The maximum stresses occur along the load axis z, and they are principal stresses (Figure 3.38). These and the resulting maximum shear stresses are given in terms of the maximum contact pressure p_o by the equations to follow [3, 16].

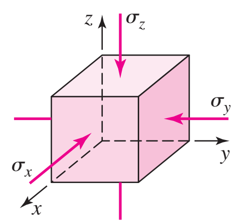


Figure 3.38 Principal stress below the surface along the load axis *z*.

Two Spheres in Contact (*Figure 3.36*)

$$\sigma_x = \sigma_y = -p_o\left\{\left(1 - \frac{z}{a}\tan^{-1}\frac{1}{z/a}\right)(1+\nu) - \frac{1}{2[1+(z/a)^2]}\right\} \tag{3.41a}$$

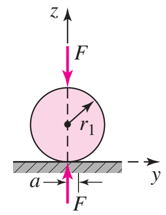
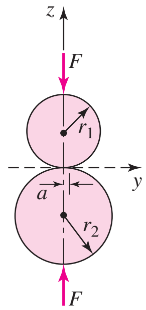
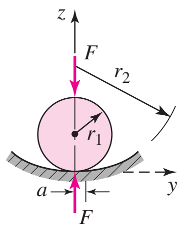
$$\sigma_z = -\frac{p_o}{1+(z/a)^2} \tag{3.41b}$$

Therefore, we have $\tau_{xy} = 0$ and

$$\tau_{yz} = \tau_{xz} = \frac{1}{2}(\sigma_x - \sigma_z) \tag{3.41c}$$

A plot of these equations is shown in Figure 3.39a.

Table 3.2 Maximum pressure p_o and deflection δ of two bodies in point of contact

Configuration	**Spheres:** $p_o = 1.5\dfrac{F}{\pi a^2}$	**Cylinders:** $p_o = \dfrac{2}{\pi}\dfrac{F}{aL}$
A. 	Sphere on a Flat Surface $a = 0.880\sqrt[3]{Fr_1\Delta}$ $\delta = 0.775\sqrt[3]{F^2\dfrac{\Delta^2}{r_1}}$	Cylinder on a Flat Surface $a = 1.076\sqrt{\dfrac{F}{L}r_1\Delta}$ For $E_1 = E_2 = E$: $\delta = \dfrac{0.579F}{EL}\left(\dfrac{1}{3} + \ln\dfrac{2r_1}{a}\right)$
B. 	Two Spherical Balls $a = 0.880\sqrt[3]{F\dfrac{\Delta}{m}}$ $\delta = 0.775\sqrt[3]{F^2\Delta^2 m}$	Two Cylindrical Rollers $a = 1.076\sqrt{\dfrac{F\Delta}{Lm}}$
C. 	Sphere on a Spherical Seat $a = 0.880\sqrt[3]{F\dfrac{\Delta}{n}}$ $\delta = 0.775\sqrt[3]{F^2\Delta^2 n}$	Cylinder on a Cylindrical Seat $a = 1.076\sqrt{\dfrac{F\Delta}{Ln}}$

Note: $\Delta = \dfrac{1}{E_1} + \dfrac{1}{E_2}, \qquad m = \dfrac{1}{r_1} + \dfrac{1}{r_2}, \qquad n = \dfrac{1}{r_1} - \dfrac{1}{r_2}$

where the modulus of elasticity (E) and radius (r) are for the contacting members, 1 and 2. The L represents the length of the cylinder (Figure 3.37). The total force pressing two spheres or cylinders is F.

Two Cylinders in Contact (Figure 3.37)

$$\sigma_x = -2\nu p_o\left[\sqrt{1 + \left(\frac{z}{a}\right)^2} - \frac{z}{a}\right] \tag{3.42a}$$

$$\sigma_y = -p_o\left\{\left[2 - \frac{1}{1 + (z/a)^2}\right]\sqrt{1 + \left(\frac{z}{a}\right)^2} - 2\frac{z}{a}\right\} \tag{3.42b}$$

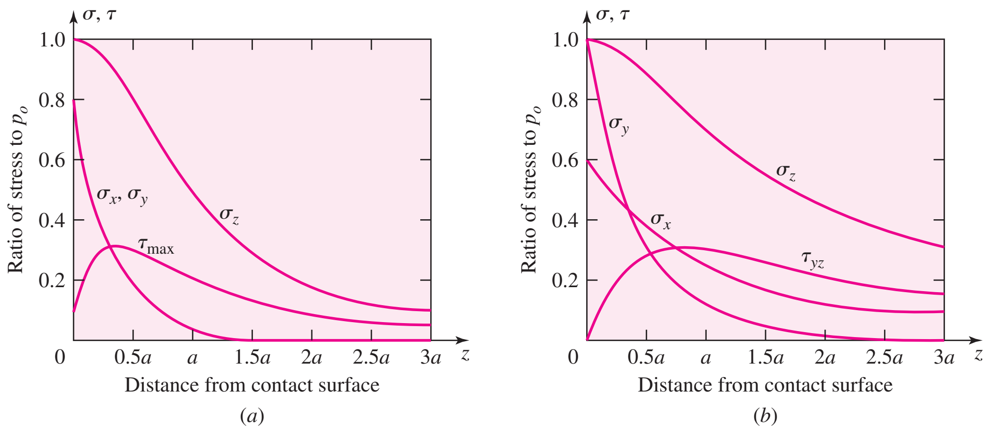


Figure 3.39 Stresses below the surface along the load axis (for $\nu = 0.3$): (a) two spheres; (b) two parallel cylinders. Note: All normal stresses are compressive stresses.

$$\sigma_z = -\frac{p_o}{\sqrt{1 + (z/a)^2}} \tag{3.42c}$$

$$\tau_{xy} = \frac{1}{2}(\sigma_x - \sigma_y), \qquad \tau_{yz} = \frac{1}{2}(\sigma_y - \sigma_z), \qquad \tau_{xz} = \frac{1}{2}(\sigma_x - \sigma_z) \tag{3.42d}$$

Equations (3.42a–3.42c) and the second of Eqs. (3.42d) are plotted in Figure 3.39b. For each case, Figure 3.39 illustrates how principal stresses diminish below the surface. It also shows how the shear stress reaches a maximum value slightly below the surface and diminishes. The maximum shear stresses act on the planes bisecting the planes of maximum and minimum principal stresses.

The subsurface shear stresses is believed to be responsible for the surface fatigue failure of contacting bodies (see Section 8.15). The explanation is that minute cracks originate at the point of maximum shear stress below the surface and propagate to the surface to permit small bits of material to separate from the surface. As already noted, all stresses considered in this section exist along the load axis z. The states of stress off the z axis are not required for design purposes, because the maxima occur on the z axis.

Determining Maximum Contact Pressure between a Cylindrical Rod and a Beam — EXAMPLE 3.17

A concentrated load F at the center of a narrow, deep beam is applied through a rod of diameter d laid across the beam width of width b. Determine

(a) The contact area between rod and beam surface.

(b) The maximum contact stress.

Given: $F = 4$ kN, $d = 12$ mm, $b = 125$ mm

Assumptions: Both the beam and the rod are made of steel having $E = 200$ GPa and $\nu = 0.3$.

Solution: We use the equations on the second column of case A in Table 3.2.

(a) Since $E_1 = E_2 = E$ or $\Delta = 2/E$, the half-width of contact area is

$$a = 1.076\sqrt{\frac{F}{L} r_1 \Delta}$$

$$= 1.076\sqrt{\frac{4(10^3)}{0.125}\,\frac{(0.006)2}{200(10^9)}} = 0.0471 \text{ mm}$$

The rectangular contact area equals

$$2aL = 2(0.0471)(125) = 11.775 \text{ mm}^2$$

(b) The maximum contact pressure is therefore

$$p_o = \frac{2}{\pi}\frac{F}{aL} = \frac{2}{\pi}\frac{4(10^3)}{5.888(10^{-6})} = 432.5 \text{ MPa}$$

Case Study 3-3 CAM AND FOLLOWER STRESS ANALYSIS OF AN INTERMITTENT-MOTION MECHANISM

Figure 3.40 shows a camshaft and follower of an intermittent-motion mechanism. For the position indicated, the cam exerts a force P_{max} on the follower. What are the maximum stress at the contact line between the cam and follower and the deflection?

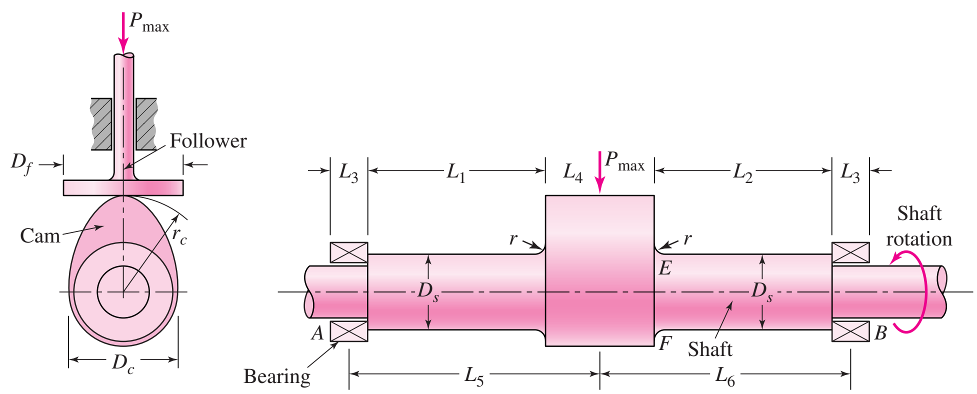


Figure 3.40 Layout of camshaft and follower of an intermittent-motion mechanism.

Case Study (CONCLUDED)

Given: The shapes of the contacting surfaces are known. The material of all parts is AISI 1095, carburized on the surfaces, oil quenched and tempered (Q&T) at 650°C.

Data:

$$P_{max} = 1.6 \text{ kips}, \quad r_c = 1.5 \text{ in.}, \quad D_f = L_4 = 1.5 \text{ in.},$$
$$E = 29 \times 10^6 \text{ psi}, \quad S_y = 80 \text{ ksi},$$

Assumptions: Frictional forces can be neglected. The rotational speed is slow so that the loading is considered static.

Solution: See Figure 3.40, Tables 3.2, B.1, and B.4.

Equations on the second column of case A of Table 3.2 apply. We first determine the half-width a of the contact patch. Since $E_1 = E_2 = E$ and $\Delta = 2/E$, we have

$$a = 1.076\sqrt{\frac{P_{max}}{L_4} r_c \Delta}$$

Substitution of the given data yield

$$a = 1.076\left[\frac{1600}{1.5}(1.5)\left(\frac{2}{30 \times 10^6}\right)\right]^{1/2}$$
$$= 11.113(10^{-3}) \text{ in.}$$

The rectangular patch area:

$$2aL = 2(11.113 \times 10^{-3})(1.5) = 33.34(10^{-3}) \text{ in.}^2$$

Maximum contact pressure is then

$$p_o = \frac{2}{\pi}\frac{P_{max}}{aL_4}$$
$$= \frac{2}{\pi}\frac{1600}{(11.113 \times 10^{-3})(1.5)} = 61.11 \text{ ksi}$$

The deflection δ of the cam and follower at the line of contact is obtained as follows

$$\delta = \frac{0.579 P_{max}}{EL_4}\left(\frac{1}{3} + \ln\frac{2r_c}{a}\right)$$

Introducing the numerical values,

$$\delta = \frac{0.579(1600)}{30 \times 10^6(1.5)}\left(\frac{1}{3} + \ln\frac{2 \times 1.5}{11.113 \times 10^{-3}}\right)$$
$$= 0.122(10^{-3}) \text{ in.}$$

Comments: The contact stress is determined to be less than the yield strength and the design is satisfactory. The calculated deflection between the cam and the follower is very small and does not effect the system performance.

*3.15 MAXIMUM STRESS IN GENERAL CONTACT

In this section, we introduce some formulas for the determination of the maximum contact stress or pressure p_o between the two contacting bodies that have any general curvature [2,15]. Since the radius of curvature of each member in contact is different in every direction, the equations for the stress given here are more complex than those presented in the preceding section. A brief discussion on factors affecting the contact pressure is given in Section 8.15.

Consider two rigid bodies of equal elastic modulus E, compressed by F, as shown in Figure 3.41. The load lies along the axis passing through the centers of the bodies and through the point of contact and is perpendicular to the plane tangent to both bodies at the point of contact. The minimum and maximum radii of curvature of the surface of the upper body are r_1 and r_1'; those of the lower body are r_2 and r_2' at the point of contact. Therefore,

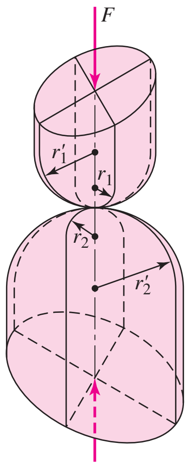


Figure 3.41 Curved surfaces of different radii of two bodies compressed by forces *F*.

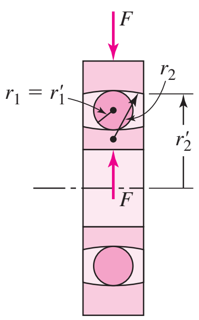


Figure 3.42 Contact load in a single-row ball bearing.

$1/r_1$, $1/r_1'$, $1/r_2$, and $1/r_2'$ are the principal curvatures. The *sign convention* of the *curvature* is such that it is positive if the corresponding center of curvature is inside the body; if the center of the curvature is outside the body, the curvature is negative. (For instance, in Figure 3.42, r_1, r_1' are positive, while r_2, r_2' are negative.)

Let θ be the angle between the normal planes in which radii r_1 and r_2 lie (Figure 3.41). Subsequent to the loading, the area of contact will be an ellipse with semiaxes a and b. The *maximum contact pressure* is

$$p_o = 1.5\frac{F}{\pi ab} \tag{3.43}$$

where

$$a = c_a\sqrt[3]{\frac{Fm}{n}} \qquad b = c_b\sqrt[3]{\frac{Fm}{n}} \tag{3.44}$$

In these formulas, we have

$$m = \frac{4}{\frac{1}{r_1}+\frac{1}{r_1'}+\frac{1}{r_2}+\frac{1}{r_2'}} \qquad n = \frac{4E}{3(1-\nu^2)} \tag{3.45}$$

The constants c_a and c_b are given in Table 3.3 corresponding to the value of α calculated from the formula

$$\cos\alpha = \frac{B}{A} \tag{3.46}$$

Here

$$A = \frac{2}{m}, \qquad B = \pm\frac{1}{2}\left[\left(\frac{1}{r_1}-\frac{1}{r_1'}\right)^2 + \left(\frac{1}{r_2}-\frac{1}{r_2'}\right)^2 + 2\left(\frac{1}{r_1}-\frac{1}{r_1'}\right)\left(\frac{1}{r_2}-\frac{1}{r_2'}\right)\cos 2\theta\right]^{1/2} \tag{3.47}$$

The proper sign in B must be chosen so that its values are positive.

Table 3.3 Factors for use in Equation (3.44)

α (degrees)	c_a	c_b	α (degrees)	c_a	c_b
20	3.778	0.408	60	1.486	0.717
30	2.731	0.493	65	1.378	0.759
35	2.397	0.530	70	1.284	0.802
40	2.136	0.567	75	1.202	0.846
45	1.926	0.604	80	1.128	0.893
50	1.754	0.641	85	1.061	0.944
55	1.611	0.678	90	1.000	1.000

Using Eq. (3.43), many problems of practical importance may be solved. These include contact stresses in rolling bearings (Figure 3.42), contact stresses in cam and push-rod mechanisms (see Problem P3.42), and contact stresses between a cylindrical wheel and rail (see Problem P3.44).

Ball Bearing Capacity Analysis

EXAMPLE 3.18

A single-row ball bearing supports a radial load F as shown in Figure 3.42. Calculate

(a) The maximum pressure at the contact point between the outer race and a ball.

(b) The factor of safety, if the ultimate strength is the maximum usable stress.

Given: $F = 1.2$ kN, $E = 200$ GPa, $\nu = 0.3$, and $S_u = 1900$ MPa. Ball diameter is 12 mm; the radius of the groove, 6.2 mm; and the diameter of the outer race is 80 mm.

Assumptions: The basic assumptions listed in Section 3.14 apply. The loading is static.

Solution: See Figure 3.42 and Table 3.3.

For the situation described $r_1 = r_1' = 0.006$ m, $r_2 = -0.0062$ m, and $r_2' = -0.04$ m.

(a) Substituting the given data into Eqs. (3.45) and (3.47), we have

$$m = \frac{4}{\frac{2}{0.006} - \frac{1}{0.0062} - \frac{1}{0.04}} = 0.0272, \qquad n = \frac{4(200 \times 10^9)}{3(0.91)} = 293.0403 \times 10^9$$

$$A = \frac{2}{0.0272} = 73.5294, \quad B = \frac{1}{2}[(0)^2 + (-136.2903)^2 + 2(0)^2]^{1/2} = 68.1452$$

Using Eq. (3.46),

$$\cos\alpha = \pm\frac{68.1452}{73.5294} = 0.9268, \qquad \alpha = 22.06^\circ$$

Corresponding to this value of α, interpolating in Table 3.3, we obtain $c_a = 3.5623$ and $c_b = 0.4255$. The semiaxes of the ellipsoidal contact area are found by using Eq. (3.44):

$$a = 3.5623\left[\frac{1200 \times 0.0272}{293.0403 \times 10^9}\right]^{1/3} = 1.7140 \text{ mm}$$

$$b = 0.4255\left[\frac{1200 \times 0.0272}{293.0403 \times 10^9}\right]^{1/3} = 0.2047 \text{ mm}$$

The maximum contact pressure is then

$$p_o = 1.5\frac{1200}{\pi(1.7140 \times 0.2047)} = 1633 \text{ MPa}$$

(b) Since contact stresses are not linearly related to load F, the safety factor is defined by Eq. (1.1):

$$n = \frac{F_u}{F} \tag{a}$$

in which F_u is the ultimate loading. The maximum principal stress theory of failure gives

$$S_u = \frac{1.5F_u}{\pi ab} = \frac{1.5F_u}{\pi c_a c_b \sqrt[3]{(F_u m/n)^2}}$$

This may be written as

$$S_u = \frac{1.5\sqrt[3]{F_u}}{\pi c_a c_b (m/n)^{2/3}} \tag{3.48}$$

Introducing the numerical values into the preceding expression, we have

$$1900(10^6) = \frac{1.5\sqrt[3]{F_u}}{\pi(3.5623 \times 0.4255)\left(\frac{0.0272}{293.0403 \times 10^9}\right)^{2/3}}$$

Solving, $F_u = 1891$ N. Equation (a) gives then

$$n = \frac{1891}{1200} = 1.58$$

Comments: In this example, the magnitude of the contact stress obtained is quite large in comparison with the values of the stress usually found in direct tension, bending, and torsion. In all contact problems, three-dimensional compressive stresses occur at the point, and hence a material is capable of resisting higher stress levels.

3.16 THREE-DIMENSIONAL STRESS

In the most general case of three-dimensional stress, an element is subjected to stresses on the orthogonal x, y, and z planes, as shown in Figure 1.10. Consider a tetrahedron, isolated from this element and represented in Figure 3.43. Components of stress on the perpendicular planes (intersecting at the origin O) can be related to the normal and shear stresses on the oblique plane ABC, by using an approach identical to that employed for the two-dimensional state of stress.

Orientation of plane ABC may be defined in terms of the *direction cosines,* associated with the angles between a unit normal $\mathbf{n}$ to the plane and the x, y, z coordinate axes:

$$\cos(\mathbf{n}, x) = l, \qquad \cos(\mathbf{n}, y) = m, \qquad \cos(\mathbf{n}, z) = n \tag{3.49}$$

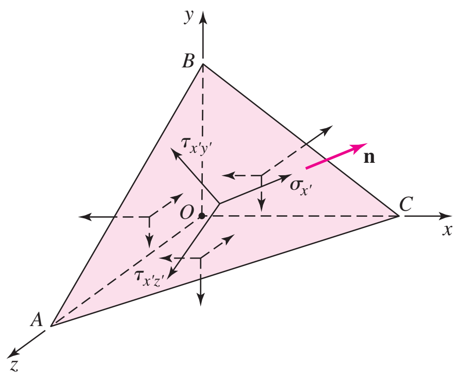


Figure 3.43 Components of stress on a tetrahedron.

The sum of the squares of these quantities is unity:

$$l^2 + m^2 + n^2 = 1 \tag{3.50}$$

Consider now a new coordinate system x', y', z', where x' coincides with **n** and y', z' lie on an oblique plane. It can readily be shown that [2] the normal stress acting on the oblique x' plane shown in Figure 3.43 is expressed in the form

$$\sigma_{x'} = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2(\tau_{xy} lm + \tau_{yz} mn + \tau_{xz} ln) \tag{3.51}$$

where l, m, and n are direction cosines of angles between x' and the x, y, z axes, respectively. The shear stresses $\tau_{x'y'}$ and $\tau_{x'z'}$ may be written similarly. The stresses on the three mutually perpendicular planes are required to specify the stress at a point. One of these planes is the oblique (x') plane in question. The other stress components $\sigma_{y'}$, $\sigma_{z'}$, and $\tau_{y'z'}$ are obtained by considering those (y' and z') planes perpendicular to the oblique plane. In so doing, the resulting six expressions represent *transformation equations* for three-dimensional stress.

PRINCIPAL STRESSES IN THREE DIMENSIONS

For the three-dimensional case, three mutually perpendicular planes of zero shear exist; and on these planes, the normal stresses have maximum or minimum values. The foregoing normal stresses are called *principal stresses* σ_1, σ_2, and σ_3. The algebraically largest stress is represented by σ_1 and the smallest by σ_3. Of particular importance are the direction cosines of the plane on which $\sigma_{x'}$ has a maximum value, determined from the equations:

$$\begin{bmatrix} \sigma_x - \sigma_i & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_i \end{bmatrix} \begin{Bmatrix} l_i \\ m_i \\ n_i \end{Bmatrix} = 0, \qquad (i = 1, 2, 3) \tag{3.52}$$

A nontrivial solution for the direction cosines requires that the characteristic determinant vanishes. Thus

$$\begin{vmatrix} \sigma_x - \sigma_i & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_i & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_i \end{vmatrix} = 0 \tag{3.53}$$

Expanding Eq. (3.53), we obtain the following stress cubic equation:

$$\sigma_i^3 - I_1\sigma_i^2 + I_2\sigma_i - I_3 = 0 \tag{3.54}$$

where

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 \end{aligned} \tag{3.55}$$

The quantities I_1, I_2, and I_3 represent *invariants* of the three-dimensional stress. For a given state of stress, Eq. (3.54) may be solved for its three roots, σ_1, σ_2, and σ_3. Introducing each of these principal stresses into Eq. (3.52) and using $l_i^2 + m_i^2 + n_i^2 = 1$, we can obtain three sets of direction cosines for three principal planes. Note that the direction cosines of the principal stresses are occasionally required to predict the behavior of members. A convenient way of determining the roots of the stress cubic equation and solving for the direction cosines is given in Appendix D.

After obtaining the three-dimensional principal stresses, we can readily determine the maximum shear stresses. Since no shear stress acts on the principal planes, it follows that an element oriented parallel to the principal directions is in a state of triaxial stress (Figure 3.44). Therefore,

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{3.56}$$

The maximum shear stress acts on the planes that bisect the planes of the maximum and minimum principal stresses as shown in the figure.

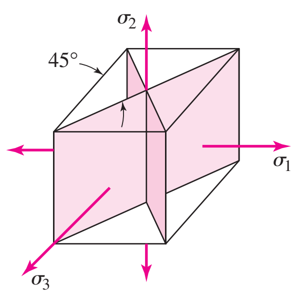


Figure 3.44 Planes of maximum three-dimensional shear stress.

EXAMPLE 3.19

Three-Dimensional State of Stress in a Member

At a critical point in a loaded machine component, the stresses relative to x, y, z coordinate system are given by

$$\begin{bmatrix} 60 & 20 & 20 \\ 20 & 0 & 40 \\ 20 & 40 & 0 \end{bmatrix} \text{MPa} \tag{a}$$

Determine the principal stresses σ_1, σ_2, σ_3, and the orientation of σ_1 with respect to the original coordinate axes.

Solution: Substitution of Eq. (a) into Eq. (3.54) gives

$$\sigma_i^3 - 60\sigma_i^2 - 2400\sigma_i + 64{,}000 = 0, \qquad (i = 1, 2, 3)$$

The three principal stresses representing the roots of this equation are

$$\sigma_1 = 80 \text{ MPa}, \qquad \sigma_2 = 20 \text{ MPa}, \qquad \sigma_3 = -40 \text{ MPa}$$

Introducing σ_1 into Eq. (3.52), we have

$$\begin{bmatrix} 60-80 & 20 & 20 \\ 20 & 0-80 & 40 \\ 20 & 40 & 0-80 \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix} = 0 \tag{b}$$

Here l_1, m_1, and n_1 represent the direction cosines for the orientation of the plane on which σ_1 acts.

It can be shown that only two of Eqs. (b) are independent. From these expressions, together with $l_1^2 + m_1^2 + n_1^2 = 1$, we obtain

$$l_1 = \frac{2}{\sqrt{6}} = 0.8165, \qquad m_1 = \frac{1}{\sqrt{6}} = 0.4082, \qquad n_1 = \frac{1}{\sqrt{6}} = 0.4082$$

The direction cosines for σ_2 and σ_3 are ascertained in a like manner. The foregoing computations may readily be performed by using the formulas given in Appendix D.

Simplified Transformation for Three-Dimensional Stress

Often we need the normal and shear stresses acting on an arbitrary oblique plane of a tetrahedron in terms of the principal stresses acting on perpendicular planes (Figure 3.45). In this case, the x, y, and z coordinate axes are parallel to the principal axes: $\sigma_{x'} = \sigma$, $\sigma_x = \sigma_1$, $\tau_{xy} = \tau_{xz} = 0$, and so on, as depicted in the figure. Let l, m, and n denote the direction cosines of oblique plane ABC. The normal stress σ on the oblique plane, from Eq. (3.51), is

$$\boxed{\sigma = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2} \tag{3.57a}$$

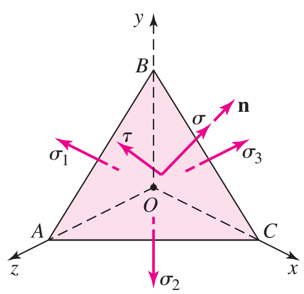


Figure 3.45 Triaxial stress on a tetrahedron.

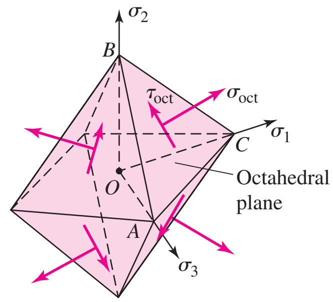


Figure 3.46 Stresses on a octahedron.

It can be verified that, the shear stress τ on this plane may be expressed in the convenient form:

$$\tau = [(\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2 + (\sigma_3 - \sigma_1)^2 n^2 l^2]^{1/2} \tag{3.57b}$$

The preceding expressions are the simplified transformation equations for three-dimensional state of stress.

Octahedral Stresses

Let us consider an oblique plane that forms equal angles with each of the principal stresses, represented by face ABC in Figure 3.45 with $OA = OB = OC$. Thus, the normal **n** to this plane has equal direction cosines relative to the principal axes. Inasmuch as $l^2 + m^2 + n^2 = 1$, we have

$$l = m = n = \frac{1}{\sqrt{3}}$$

There are eight such plane or octahedral planes, all of which have the same intensity of normal and shear stresses at a point O (Figure 3.46). Substitution of the preceding equation into Eqs. (3.57) results in, the magnitudes of the *octahedral normal stress* and *octahedral shear stress,* in the following forms:

$$\sigma_{\text{oct}} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{3.58a}$$

$$\tau_{\text{oct}} = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} \tag{3.58b}$$

Equation (3.58a) indicates that the normal stress acting on an octahedral plane is the mean of the principal stresses. The octahedral stresses play an important role in certain failure criteria, discussed in Sections 5.3 and 7.8.

Determining Principal Stresses Using Mohr's Circle

EXAMPLE 3.20

Figure 3.47a depicts a point in a loaded machine base subjected to the three-dimensional stresses. Determine at the point

(a) The principal planes and principal stresses.

(b) The maximum shear stress.

(c) The octahedral stresses.

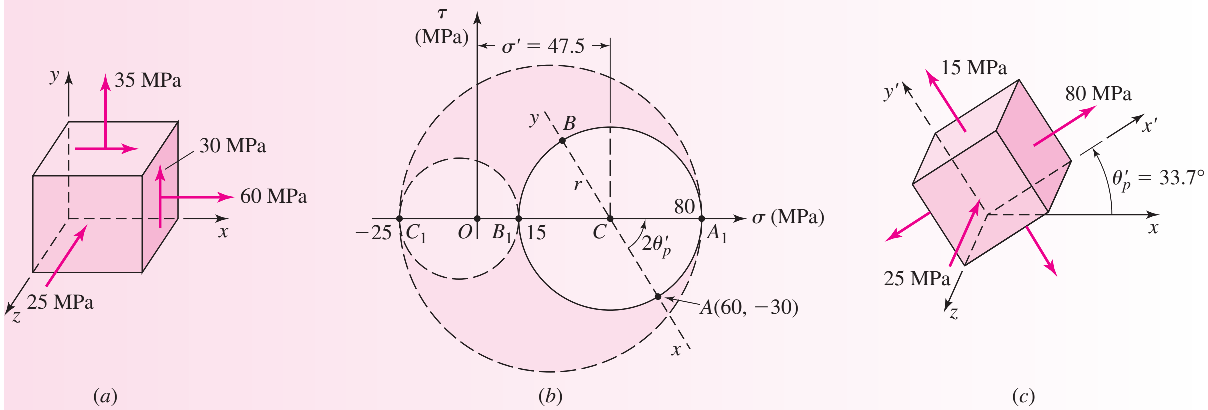


Figure 3.47 Example 3.20.

Solution: We construct Mohr's circle for the transformation of stress in the xy plane as indicated by the solid lines in Figure 3.47b. The radius of the circle is $r = (12.5^2 + 30^2)^{1/2} = 32.5$ MPa.

(a) The principal stresses in the plane are represented by points A and B:

$$\sigma_1 = 47.5 + 32.5 = 80 \text{ MPa}$$

$$\sigma_2 = 47.5 - 32.5 = 15 \text{ MPa}$$

The z faces of the element define one of the principal stresses: $\sigma_3 = -25$ MPa. The planes of the maximum principal stress are defined by θ'_p, the angle through which the element should rotate about the z axis:

$$\theta'_p = \frac{1}{2}\tan^{-1}\frac{30}{12.5} = 33.7^\circ$$

The result is shown on a sketch of the rotated element (Figure 3.47c).

(b) We now draw circles of diameters C_1B_1 and C_1A_1, which correspond, respectively, to the projections in the $y'z'$ and $x'z'$ planes of the element (Figure 3.47b). The maximum shearing stress, the radius of the circle of diameter C_1A_1, is therefore

$$\tau_{max} = \frac{1}{2}(75 + 25) = 50 \text{ MPa}$$

Planes of the maximum shear stress are inclined at 45° with respect to the x' and z faces of the element of Figure 3.47c.

(c) Through the use of Eqs. (3.58), we have

$$\sigma_{\text{oct}} = \frac{1}{3}(80 + 15 - 25) = 23.3 \text{ MPa}$$

$$\tau_{\text{oct}} = \frac{1}{3}[(80 - 15)^2 + (15 + 25)^2 + (-25 - 80)^2]^{1/2} = 43.3 \text{ MPa}$$

*3.17 VARIATION OF STRESS THROUGHOUT A MEMBER

As noted earlier, the components of stress generally vary from point to point in a loaded member. Such variations of stress, accounted for by the theory of elasticity, are governed by the equations of statics. Satisfying these conditions, the differential equations of equilibrium are obtained. To be physically possible, a stress field must satisfy these equations at every point in a load carrying component.

For the two-dimensional case, the stresses acting on an element of sides dx, dy, and of unit thickness are depicted in Figure 3.48. The body forces per unit volume acting on the element, F_x and F_y, are independent of z, and the component of the body force $F_z = 0$. In general, stresses are functions of the coordinates (x, y). For example, from the lower-left corner to the upper-right corner of the element, one stress component, say, σ_x, changes in value: $\sigma_x + (\partial\sigma_x/\partial x)\,\mathrm{d}x$. The components σ_y and τ_{xy} change in a like manner. The stress

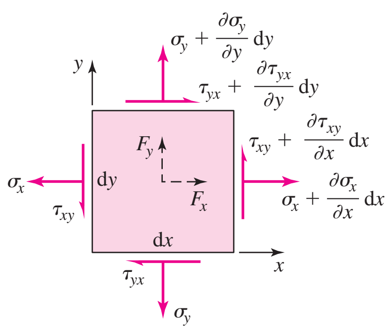


Figure 3.48 Stresses and body forces on an element.

element must satisfy the equilibrium condition $\sum M_z = 0$. Hence,

$$\left(\frac{\partial \sigma_y}{\partial_y} \mathrm{d}x\, \mathrm{d}y\right)\frac{\mathrm{d}x}{2} - \left(\frac{\partial \sigma_x}{\partial x} \mathrm{d}x\, \mathrm{d}y\right)\frac{\mathrm{d}y}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \mathrm{d}x\right) \mathrm{d}x\, \mathrm{d}y$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial_y} \mathrm{d}y\right) \mathrm{d}x\, \mathrm{d}y + F_y\, \mathrm{d}x\, \mathrm{d}y \frac{\mathrm{d}x}{2} - F_x\, \mathrm{d}x\, \mathrm{d}y\, \frac{\mathrm{d}y}{2} = 0$$

After neglecting the triple products involving dx and dy, this equation results in $\tau_{xy} = \tau_{yx}$. Similarly, for a general state of stress, it can be shown that $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$. Hence, the shear stresses in mutually perpendicular planes of the element are equal.

The equilibrium condition that x-directed forces must sum to 0, $\sum F_x = 0$. Therefore, referring to Figure 3.48,

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \mathrm{d}x\right) \mathrm{d}y - \sigma_x\, \mathrm{d}y + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \mathrm{d}y\right) \mathrm{d}x - \tau_{xy}\, \mathrm{d}x + F_x\, \mathrm{d}x\, \mathrm{d}y = 0$$

Summation of the forces in the y direction yields an analogous result. After reduction, we obtain the differential equations of equilibrium for a *two-dimensional stress* in the form [2]

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y &= 0 \end{aligned} \tag{3.59a}$$

In the general case of an element under *three-dimensional stresses,* it can be shown that the differential equations of equilibrium are given by

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z &= 0 \end{aligned} \tag{3.59b}$$

Note that, in many practical applications, the weight of the member is only body force. If we take the y axis as upward and designate by ρ the mass density per unit volume of the member and by g the gravitational acceleration, then $F_x = F_z = 0$ and $F_y = -\rho g$ in the foregoing equations.

We observe that two relations of Eqs. (3.59a) involve the three unknowns $(\sigma_x, \sigma_y, \tau_{xy})$ and the three relations of Eqs. (3.59b) contain the six unknown stress components. Therefore, problems in stress analysis are *internally* statically indeterminate. In the mechanics of materials method, this indeterminacy is eliminated by introducing simplifying assumptions regarding the stresses and considering the equilibrium of the finite segments of a load-carrying component.

3.18 THREE-DIMENSIONAL STRAIN

If deformation is distributed uniformly over the original length, the normal strain may be written $\varepsilon_x = \delta/L$, where L and δ are the original length and the change in length of the member, respectively (see Figure 1.12a). However, the strains generally vary from point to point in a member. Hence, the expression for strain must relate to a line of length dx which elongates by an amount du under the axial load. The definition of normal strain is therefore

$$\varepsilon_x = \frac{du}{dx} \tag{3.60}$$

This represents the strain at a point.

As noted earlier, in the case of two-dimensional or *plane strain,* all points in the body, before and after application of load, remain in the same plane. Therefore, the deformation of an element of dimensions dx, dy, and of unit thickness can contain normal strain (Figure 3.49a) and a shear strain (Figure 3.49b). Note that the partial derivative notation is used, since the displacement u or v is function of x and y. Recalling the basis of Eqs. (3.60) and (1.22), an examination of Figure 3.49 yields

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{3.61a}$$

Obviously, γ_{xy} is the shear strain between the x and y axes (or y and x axes), hence, $\gamma_{xy} = \gamma_{yx}$. A long prismatic member subjected to a lateral load (e.g., a cylinder under pressure) exemplifies the state of plane strain.

In an analogous manner, the strains at a point in a rectangular prismatic element of sides dx, dy, and dz are found in terms of the displacements u, v, and w. It can be shown that these *three-dimensional strain* components are ε_x, ε_y, γ_{xy}, and

$$\varepsilon_z = \frac{\partial w}{\partial z}, \qquad \gamma_{yz} = \frac{\partial w}{\partial z} + \frac{\partial v}{\partial z}, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{3.61b}$$

where $\gamma_{yz} = \gamma_{zy}$ and $\gamma_{xz} = \gamma_{zx}$. Equations (3.61) represent the components of strain tensor, which is similar to the stress tensor discussed in Section 1.13.

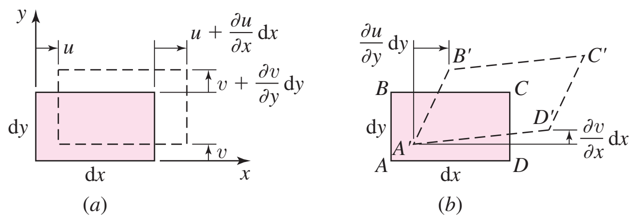


Figure 3.49 Deformations of a two-dimensional element: (a) normal strain; (b) shear strain.

PROBLEMS IN ELASTICITY

In many problems of practical importance, the stress or strain condition is one of plane stress or plane strain. These two-dimensional problems in elasticity are simpler than those involving three-dimensions. A finite element solution of two-dimensional problems is taken up in Chapter 17. In examining Eqs. (3.61), we see that the six strain components depend linearly on the derivatives of the three displacement components. Therefore, the strains cannot be independent of one another. Six equations, referred to as the *conditions of compatibility,* can be developed showing the relationships among ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, and γ_{xz} [2]. The number of such equations reduce to one for a two-dimensional problem. The conditions of compatibility assert that the displacements are continuous. Physically, this means that the body must be pieced together.

To conclude, the theory of elasticity is based on the following requirements: strain compatibility, stress equilibrium (Eqs. 3.59), general relationships between the stresses and strains (Eqs. 2.8), and boundary conditions for a given problem. In Chapter 16, we discuss various axisymmetrical problems using the elasticity approaches. In the method of mechanics of materials, simplifying assumptions are made with regard to the distribution of strains in the body as a whole or the finite portion of the member. Thus, the difficult task of solving the conditions of compatibility and the differential equations of equilibrium are avoided.

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PROBLEMS

Sections 3.1 through 3.8

3.1 Two plates are fastened by a bolt and nut as shown in Figure P3.1. Calculate

(*a*) The normal stress in the bolt shank.

(*b*) The average shear stress in the head of the bolt.

(*c*) The shear stress in the threads.

(*d*) The bearing stress between the head of the bolt and the plate.

Assumption: The nut is tightened to produce a tensile load in the shank of the bolt of 10 kips.

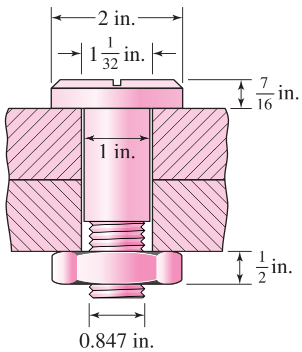

Figure P3.1

3.2 A short steel pipe of yield strength S_y is to support an axial compressive load P with factor of safety of n against yielding. Determine the minimum required inside radius a.

Given: $S_y = 280$ MPa, $P = 1.2$ MN, and $n = 2.2$.
Assumption: The thickness t of the pipe is to be one-fourth of its inside radius a.

3.3 The landing gear of an aircraft is depicted in Figure P3.3. What are the required pin diameters at A and B.

Given: Maximum usable stress of 28 ksi in shear.
Assumption: Pins act in double shear.

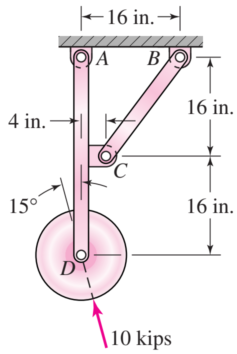

Figure P3.3

3.4 The frame of Figure P3.4 supports a concentrated load P. Calculate

(a) The normal stress in the member BD if it has a cross-sectional area A_{BD}.

(b) The shearing stress in the pin at A if it has a diameter of 25 mm and is in double shear.

Given: $P = 5$ kN, $A_{BD} = 8 \times 10^3$ mm^2.

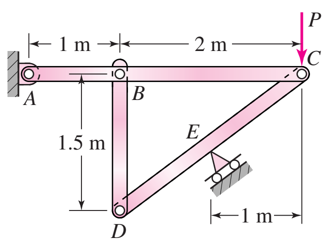

Figure P3.4

3.5 Two bars AC and BC are connected by pins to form a structure for supporting a vertical load P at C (Figure P3.5). Determine the angle α if the structure is to be of minimum weight.

Assumption: The normal stresses in both bars are to be the same.

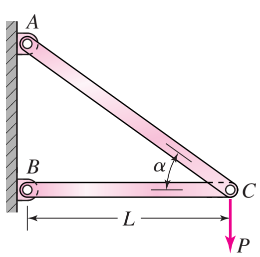

Figure P3.5

3.6 Two beams AC and BD are supported as shown in Figure P3.6. A roller fits snugly between the two beams at point B. Draw the shear and moment diagrams of the lower beam AC.

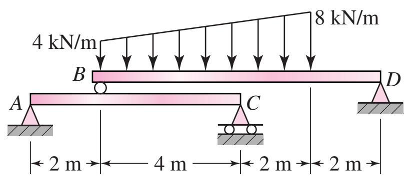

Figure P3.6

3.7 Design the cross section (determine h) of the simply supported beam loaded at two locations as shown in Figure P3.7.

Assumption: The beam will be made of timber of $\sigma_{\text{all}} = 1.8$ ksi and $\tau_{\text{all}} = 100$ psi.

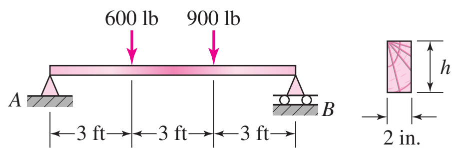

Figure P3.7

3.8 A rectangular beam is to be cut from a circular bar of diameter d (Figure P3.8). Determine the dimensions b and h so that the beam will resist the largest bending moment.

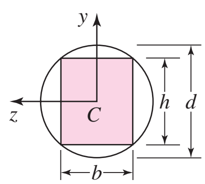

Figure P3.8

3.9 The T-beam, whose cross section is shown in Figure P3.9, is subjected to a shear force V. Calculate the maximum shear stress in the web of the beam.

Given: $b = 200$ mm, $t = 15$ mm, $h_1 = 175$ mm, $h_2 = 150$ mm, $V = 22$ kN.

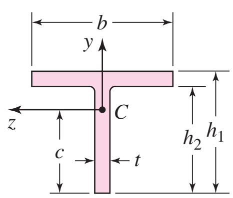

Figure P3.9

3.10 A box beam is made of four 50-mm × 200-mm planks, nailed together as shown in Figure P3.10. Determine the maximum allowable shear force V.

Given: The longitudinal spacing of the nails, $s = 100$ mm; the allowable load per nail, $F = 15$ kN.

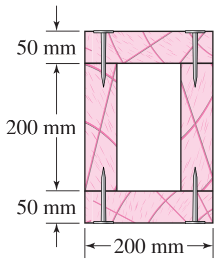

Figure P3.10

3.11 For the beam and loading shown in Figure P3.11, design the cross section of the beam for $\sigma_{\text{all}} = 12$ MPa and $\tau_{\text{all}} = 810$ kPa.

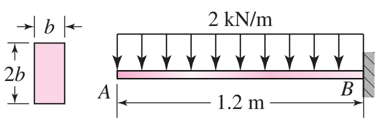

Figure P3.11

3.12 Select the S shape of a simply supported 6-m long beam subjected a uniform load of intensity 50 kN/m, for $\sigma_{\text{all}} = 170$ MPa and $\tau_{\text{all}} = 100$ MPa.

3.13 and **3.14** The beam AB has the rectangular cross section of constant width b and variable depth h (Figures P3.13 and P3.14). Derive an expression for h in terms of x, L, and h_1, as required.

Assumption: The beam is to be of constant strength.

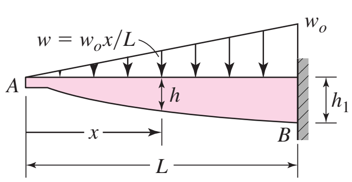

Figure P3.13

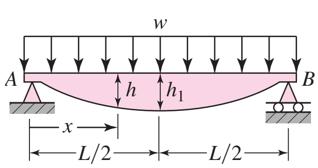

Figure P3.14

3.15 A wooden beam 8 in. wide × 12 in. deep is reinforced on both top and bottom by steel plates 0.5 in. thick (Figure P3.15). Calculate the maximum bending moment M about the z axis.

Design Assumptions: The allowable bending stresses in the wood and steel are 1.05 ksi and 18 ksi, respectively. Use $n = E_s/E_w = 20$.

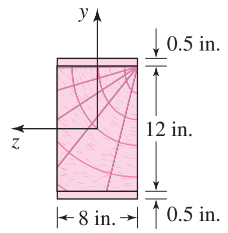

Figure P3.15

3.16 A simply supported beam of span length 8 ft carries a uniformly distributed load of 2.5 kip/ft. Determine the required thickness t of the steel plates.

Given: The cross section of the beam is a hollow box with wood flanges ($E_w = 1.5 \times 10^6$ psi) and steel ($E_s = 30 \times 10^6$ psi), as shown in Figure P3.16.
Assumptions: The allowable stresses are 19 ksi for the steel and 1.1 ksi for the wood.

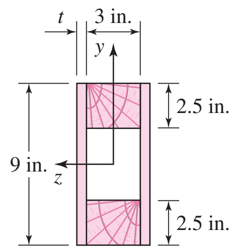

Figure P3.16

3.17 and **3.18** For the composite beam with cross section as shown (Figures P3.17 and P3.18), determine the maximum permissible value of the bending moment M about the z axis.

Given: $(\sigma_b)_{\text{all}} = 120$ MPa $\quad (\sigma_s)_{\text{all}} = 140$ MPa

$E_b = 100$ GPa $\quad E_s = 200$ GPa

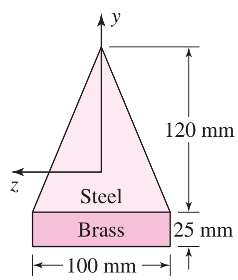

Figure P3.17

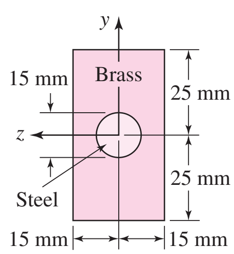

Figure P3.18

3.19 A round brass tube of outside diameter d and an aluminum core of diameter $d/2$ are bonded together to form a composite beam (Figure P3.19). Determine the maximum bending moment M that can be carried by the beam, in terms of E_b, E_s, σ_b, and d, as required. What is the value of M for $E_b = 15 \times 10^6$ psi, $E_a = 10 \times 10^6$ psi, $\sigma_b = 50$ ksi, and $d = 2$ in.?

Design Requirement: The allowable stress in the brass is σ_b.

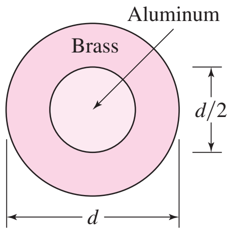

Figure P3.19

Sections 3.9 through 3.13

3.20 The state of stress at a point in a loaded machine component is represented in Figure P3.20. Determine

(*a*) The normal and shear stresses acting on the indicated inclined plane *a*-*a*.

(*b*) The principal stresses.

Sketch results on properly oriented elements.

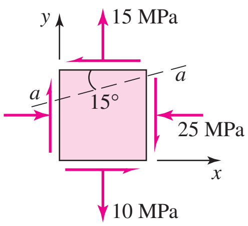

Figure P3.20

3.21 At a point A on the upstream face of a dam (Figure P3.21), the water pressure is -70 kPa and a measured tensile stress parallel to this surface is 30 kPa. Calculate

(*a*) The stress components σ_x, σ_y, and τ_{xy}.

(*b*) The maximum shear stress.

Sketch the results on a properly oriented element.

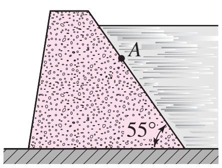

Figure P3.21

3.22 The stress acting uniformly over the sides of a skewed plate is shown in Figure P3.22. Determine

(*a*) The stress components on a plane parallel to *a-a*.

(*b*) The magnitude and orientation of principal stresses.

Sketch the results on properly oriented elements.

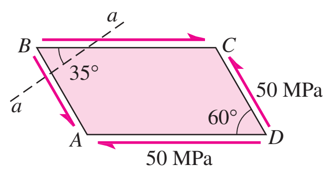

Figure P3.22

3.23 A thin skewed plate is depicted in Figure P3.22. Calculate the change in length of

(*a*) The edge *AB*.

(*b*) The diagonal *AC*.

Given: $E = 200$ GPa, $\nu = 0.3$, $AB = 40$ mm, and $BC = 60$ mm.

3.24 The stresses acting uniformly at the edges of a thin skewed plate are shown in Figure P3.24. Determine

(*a*) The stress components σ_x, σ_y, and τ_{xy}.

(*b*) The maximum principal stresses and their orientations.

Sketch the results on properly oriented elements.

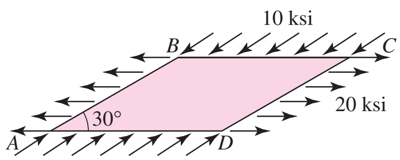

Figure P3.24

3.25 For the thin skewed plate shown in Figure P3.24, determine the change in length of the diagonal *BD*.

Given: $E = 30 \times 10^6$ psi, $\nu = \frac{1}{4}$, $AB = 2$ in., and $BC = 3$ in.

3.26 The stresses acting uniformly at the edges of a wall panel of a flight structure are depicted in Figure P3.26. Calculate the stress components on planes parallel and perpendicular to *a-a*. Sketch the results on a properly oriented element.

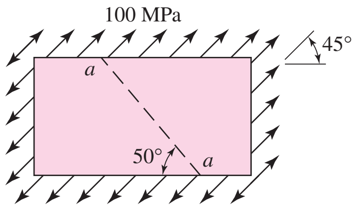

Figure P3.26

3.27 A rectangular plate is subjected to uniformly distributed stresses acting along its edges (Figure P3.27). Determine

(*a*) The normal and shear stresses on planes parallel and perpendicular to *a-a*.

(*b*) The maximum shear stress.

Sketch the results on properly oriented elements.

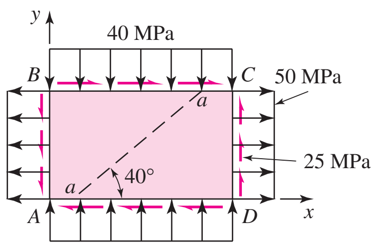

Figure P3.27

3.28 For the plate shown in Figure P3.27, calculate the change in the diagonals *AC* and *BD*.

Given: $E = 210$ GPa, $\nu = 0.3$, $AB = 50$ mm, and $BC = 75$ mm.

3.29 A cylindrical pressure vessel of diameter $d = 3$ ft and wall thickness $t = \frac{1}{8}$ in. is simply supported by two cradles as depicted in Figure P3.29. Calculate, at points *A* and *C* on the surface of the vessel,

(*a*) The principal stresses.

(*b*) The maximum shear stress.

Given: The vessel and its contents weigh 84 lb per ft of length, and the contents exert a uniform internal pressure of $p = 6$ psi on the vessel.

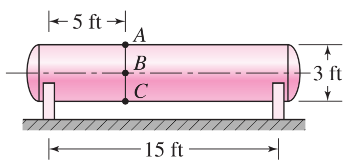

Figure P3.29

3.30 Redo Problem 3.29, considering point B on the surface of the vessel.

3.31 Calculate and sketch the normal stress acting perpendicular and shear stress acting parallel to the helical weld of the hollow cylinder loaded as depicted in Figure P3.31.

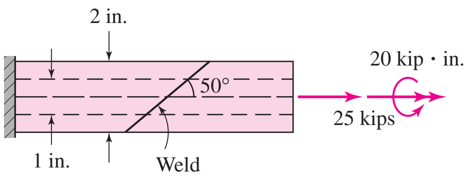

Figure P3.31

3.32 A 40-mm wide × 120-mm deep bracket supports a load of $P = 30$ kN (Figure P3.32). Determine the principal stresses and maximum shear stress at point A. Show the results on a properly oriented element.

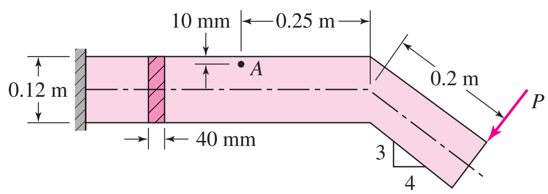

Figure P3.32

3.33 A pipe of 120-mm outside diameter and 10-mm thickness is constructed with a helical weld making an angle of 45° with the longitudinal axis, as shown in Figure P3.33. What is the largest torque T that may be applied to the pipe?

Given: Allowable tensile stress in the weld, $\sigma_{all} = 80$ MPa.

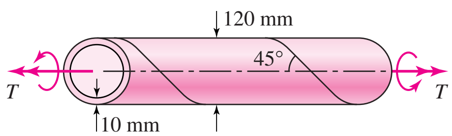

Figure P3.33

3.34 The strains at a point on a loaded shell has components $\varepsilon_x = 500\mu$, $\varepsilon_y = 800\mu$, $\varepsilon_z = 0$, and $\gamma_{xy} = 350\mu$. Determine

(a) The principal strains.

(b) The maximum shear stress at the point.

Given: $E = 70$ GPa and $\nu = 0.3$.

3.35 A thin rectangular steel plate shown in Figure P3.35 is acted on by a stress distribution, resulting in the uniform strains $\varepsilon_x = 200\mu$, $\varepsilon_y = 600\mu$, and $\gamma_{xy} = 400\mu$. Calculate

(*a*) The maximum shear strain.

(*b*) The change in length of diagonal *AC*.

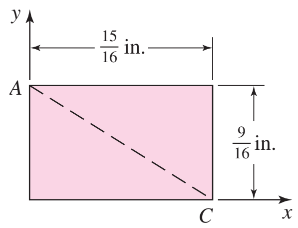

Figure P3.35

3.36 The strains at a point in a loaded bracket has components $\varepsilon_x = 50\mu$, $\varepsilon_y = 250\mu$, and $\gamma_{xy} = -150\mu$. Determine the principal stresses.

Assumptions: The bracket is made of a steel of $E = 210$ GPa and $\nu = 0.3$.

3.W Review the website at www.measurementsgroup.com. Search and identify

(*a*) Websites of three strain gage manufacturers.

(*b*) Three grid configurations of typical foil electrical resistance strain gages.

3.37 A thin-walled cylindrical tank of 500-mm radius and 10-mm wall thickness has a welded seam making an angle of 40° with respect to the axial axis (Figure P3.37). What is the allowable value of *p*?

Given: The tank carries an internal pressure of *p* and an axial compressive load of $P = 20\pi$ kN.
Assumption: The normal and shear stresses acting simultaneously in the plane of welding are not to exceed 50 and 20 MPa, respectively.

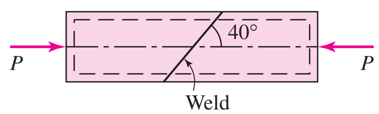

Figure P3.37

3.38 The 15-mm thick metal bar is to support an axial tensile load of 25 kN as shown in Figure P3.38 with a factor of safety of $n = 1.9$ (see Appendix C). Design the bar for minimum allowable width *h*.

Assumption: The bar is made of a relatively brittle metal having $S_y = 150$ MPa.

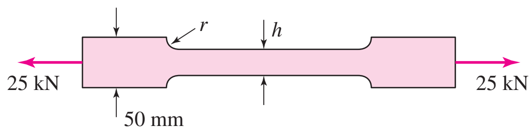

Figure P3.38

3.39 Calculate the largest load P that may be carried by a relatively brittle flat bar consisting of two portions, both 12-mm thick, and respectively 30-mm and 45-mm wide, connected by fillets of radius $r = 6$ mm (see Appendix C).

Given: $S_y = 210$ MPa and a factor of safety of $n = 1.5$.

Sections 3.14 through 3.18

3.40 Two identical 300-mm diameter balls of a rolling mill are pressed together with a force of 500 N. Determine

(*a*) The width of contact.
(*b*) The maximum contact pressure.
(*c*) The maximum principal stresses and shear stress in the center of the contact area.

Assumption: Both balls are made of steel of $E = 210$ GPa and $\nu = 0.3$.

3.41 A 14-mm diameter cylindrical roller runs on the inside of a ring of inner diameter 90 mm (see Figure 10.21a). Calculate

(*a*) The half-width a of the contact area.
(*b*) The value of the maximum contact pressure p_o.

Given: The roller load is $F = 200$ kN per meter of axial length.
Assumption: Both roller and ring are made of steel having $E = 210$ GPa and $\nu = 0.3$.

3.42 A spherical-faced (mushroom) follower or valve tappet is operated by a cylindrical cam (Figure P3.42). Determine the maximum contact pressure p_o.

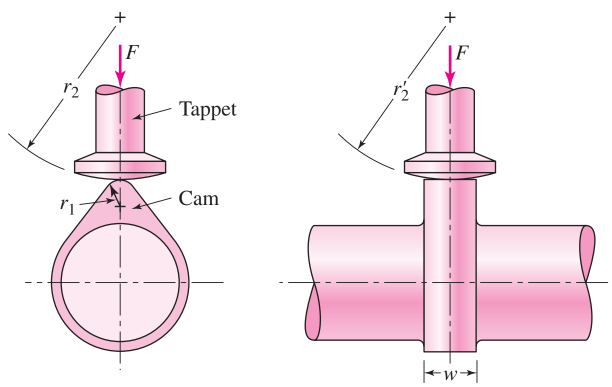

Figure P3.42

Given: $r_2 = r_2' = 10$ in., $r_1 = \frac{3}{8}$ in., and contact force $F = 500$ lb.
Assumptions: Both members are made of steel of $E = 30 \times 10^6$ psi and $\nu = 0.3$.

3.43 Resolve Problem 3.42, for the case in which the follower is flat faced.

Given: $w = \frac{1}{4}$ in.

3.44 Determine the maximum contact pressure p_o between a wheel of radius $r_1 = 500$ mm and a rail of crown radius of the head $r_2 = 350$ mm (Figure P3.44).

Given: Contact force $F = 5$ kN.
Assumptions: Both wheel and rail are made of steel of $E = 206$ GPa and $\nu = 0.3$.

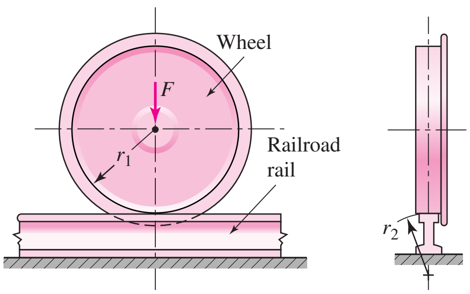

Figure P3.44

3.45 Redo Example 3.18 for a double-row ball bearing having $r_1 = r_1' = 5$ mm, $r_2 = -5.2$ mm, $r_2' = -30$ mm, $F = 600$ N, and $S_y = 1500$ MPa.

Assumptions: The remaining data are unchanged. The factor of safety is based on the yield strength.

3.46 At a point in a structural member, stresses with respect to an x, y, z coordinate system are

$$\begin{bmatrix} -10 & 0 & -8 \\ 0 & 2 & 0 \\ -8 & 0 & 2 \end{bmatrix} \text{ ksi}$$

Calculate

(*a*) The magnitude and direction of the maximum principal stress.

(*b*) The maximum shear stress.

(*c*) The octahedral stresses.

3.47 The state of stress at a point in a member relative to an x, y, z coordinate system is

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -18 \end{bmatrix} \text{ ksi}$$

Determine

(*a*) The maximum shear stress.

(*b*) The octahedral stresses.

3.48 At a critical point in a loaded component, the stresses with respect to an x, y, z coordinate system are

$$\begin{bmatrix} 42.5 & 0 & 0 \\ 0 & 5.26 & 0 \\ 0 & 0 & -7.82 \end{bmatrix} \text{MPa}$$

Determine the normal stress σ and the shear stress τ on a plane whose outer normal is oriented at angles of 40°, 60°, and 66.2° relative to the x, y, and z axes, respectively.