8.8 LINEAR PROGRAMMING

In this section

- Graphing the Constraints
- Maximizing or Minimizing a Linear Function

In this section we graph the solution set to a system of several linear inequalities. We then use the solution set as the domain of a function for which we are seeking the maximum or minimum value. The method that we use is called **linear programming,** and it can be applied to problems such as finding maximum profit or minimum cost.

Graphing the Constraints

In linear programming we have two variables that must satisfy several linear inequalities. These inequalities are called the **constraints** because they restrict the variables to only certain values. A graph in the coordinate plane is used to indicate the points that satisfy all of the constraints.

EXAMPLE 1 **Graphing the constraints**

Graph the solution set to the system of inequalities and identify each vertex of the region:

$$x \geq 0, \quad y \geq 0$$
$$3x + 2y \leq 12$$
$$x + 2y \leq 8$$

Solution

Graph the line $3x + 2y = 12$ by using its intercepts (0, 6) and (4, 0). Graph $x + 2y = 8$ by using its intercepts (0, 4) and (8, 0) as shown in Fig. 8.21. The points that satisfy $x \geq 0$ are on or to the right of the y-axis. The points on or above the x-axis satisfy $y \geq 0$. The points on or below the line $3x + 2y = 12$ satisfy $3x + 2y \leq 12$. And the points on or below $x + 2y = 8$ satisfy $x + 2y \leq 8$. Points that satisfy all of the inequalities are in the region shaded in the figure. Three of the vertices of the region are easily identified as (0, 0), (4, 0), and (0, 4). To find the fourth vertex, we solve the system $3x + 2y = 12$ and $x + 2y = 8$. Substitute $x = 8 - 2y$ from the second equation into the first equation:

$$\begin{aligned} 3(8 - 2y) + 2y &= 12 \\ 24 - 6y + 2y &= 12 \\ -4y &= -12 \\ y &= 3 \\ x &= 8 - 2(3) = 2 \end{aligned}$$

So the fourth vertex is (2, 3).

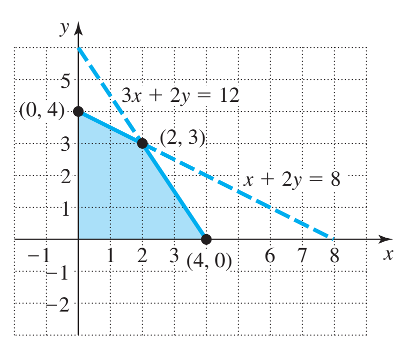

FIGURE 8.21

In linear programming the constraints usually come from physical limitations in some problem. In the next example we write the constraints and then graph the points in the coordinate plane that satisfy all of the constraints.

EXAMPLE 2 **Writing the constraints**

Jules is in the business of constructing dog houses. A small dog house requires 8 square feet (ft^2) of plywood and 6 ft^2 of insulation. A large dog house requires 16 ft^2 of plywood and 3 ft^2 of insulation. Jules has available only 48 ft^2 of plywood and 18 ft^2 of insulation. Write the constraints on the number of small and large dog houses that he can build with the available supplies and graph the solution set to the system of constraints.

Solution

Let x represent the number of small dog houses and y represent the number of large dog houses. We have two natural constraints, $x \geq 0$ and $y \geq 0$, since he cannot build a negative number of dog houses. A small dog house requires 8 ft^2 of plywood and a large dog house requires 16 ft^2 of plywood. Since only 48 ft^2 of plywood is available, we have $8x + 16y \leq 48$. A small dog house requires 6 ft^2 of insulation and a large dog house requires 3 ft^2 of insulation. Since the total insulation available is 18 ft^2, we have $6x + 3y \leq 18$. Simplify the inequalities to get the following constraints:

$$x \geq 0, \quad y \geq 0$$
$$x + 2y \leq 6$$
$$2x + y \leq 6$$

The line $x + 2y = 6$ has intercepts (0, 3) and (6, 0). The line $2x + y = 6$ has intercepts (0, 6) and (3, 0). These lines intersect at (2, 2). Points that satisfy all of the inequalities are below both lines and in the first quadrant as in Fig. 8.22

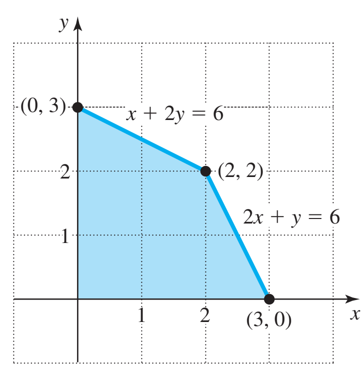

FIGURE 8.22

Maximizing or Minimizing a Linear Function

In Example 2 any ordered pair within the region is a possible solution to the number of dog houses of each type that could be built. If a small dog house sells for \$15 and a large dog house sells for \$20, then the total revenue in dollars from x small and y large dog houses is $R = 15x + 20y$. Since the revenue is a function of x and y, we write $R(x, y) = 15x + 20y$. The function R is a linear function of x and y. The domain of R is the region graphed in Fig. 8.22.

Linear Function of Two Variables

A function of the form $f(x, y) = Ax + By + C$, where A, B, and C are real numbers, is called a **linear function of two variables.**

Naturally, we are interested in the maximum revenue subject to the constraints on x and y. To investigate some possible revenues, replace R in $R = 15x + 20y$ with, say 35, 50, and 60. The graphs of the parallel lines $15x + 20y = 35$, $15x + 20y = 50$, and $15x + 20y = 60$ are shown in Fig. 8.23. The revenue at any point on the line $15x + 20y = 35$ is \$35. We get a larger revenue on a higher revenue line (and lower revenue on a lower line). The maximum revenue possible will be on the highest revenue line that still intersects the region. Because the sides of the region are straight-line segments, the intersection of the highest (or lowest) revenue line with the region must include a vertex of the region. This is the fundamental principle behind linear programming.

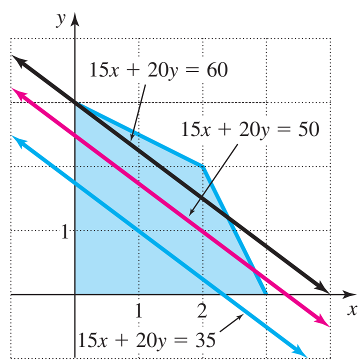

FIGURE 8.23

The Principle of Linear Programming

The maximum or minimum value of a linear function subject to linear constraints occurs at a vertex of the region determined by the constraints.

EXAMPLE 3 **Maximizing a linear function with linear constraints**

A small dog house requires 8 ft^2 of plywood and 6 ft^2 of insulation. A large dog house requires 16 ft^2 of plywood and 3 ft^2 of insulation. Only 48 ft^2 of plywood and 18 ft^2 of insulation are available. If a small dog house sells for \$15 and a large dog house sells for \$20, then how many dog houses of each type should be built to maximize the revenue and to satisfy the constraints?

Solution

Let x be the number of small dog houses and y be the number of large dog houses. We wrote and graphed the constraints for this problem in Example 2, so we will not repeat that here. The graph in Fig. 8.22 has four vertices: (0, 0), (0, 3), (3, 0), and (2, 2). The revenue function is $R(x, y) = 15x + 20y$. Since the maximum value of this function must occur at a vertex, we evaluate the function at each vertex:

$$R(0, 0) = 15(0) + 20(0) = \$0$$
$$R(0, 3) = 15(0) + 20(3) = \$60$$
$$R(3, 0) = 15(3) + 20(0) = \$45$$
$$R(2, 2) = 15(2) + 20(2) = \$70$$

From this list we can see that the maximum revenue is \$70 when two small and two large dog houses are built. We also see that the minimum revenue is \$0 when no dog houses of either type are built.

We can summarize the procedure for solving linear programming problems with the following strategy.

Strategy for Linear Programming

Use the following steps to find the maximum or minimum value of a linear function subject to linear constraints.

1. Graph the region that satisfies all of the constraints.
2. Determine the coordinates of each vertex of the region.
3. Evaluate the function at each vertex of the region.
4. Identify which vertex gives the maximum or minimum value of the function.

In the next example we solve another linear programming problem.

EXAMPLE 4 **Minimizing a linear function with linear constraints**

One serving of food A contains 2 grams of protein and 6 grams of carbohydrates. One serving of food B contains 4 grams of protein and 3 grams of carbohydrates. A dietitian wants a meal that contains at least 12 grams of protein and at least 18 grams of carbohydrates. If the cost of food A is 9 cents per serving and the cost of food B is 20 cents per serving, then how many servings of each food would minimize the cost and satisfy the constraints?

Solution

Let x represent the number of servings of food A and y represent the number of servings of food B. Each serving of A contains 2 grams of protein and each serving of B contains 4 grams of protein. If the meal is to contain at least 12 grams of protein, then $2x + 4y \geq 12$. Each serving of A contains 6 grams of carbohydrates and each serving of B contains 3 grams of carbohydrates. If the meal is to contain at least 18 grams of carbohydrates, then $6x + 3y \geq 18$. Simplify each inequality and use the two natural constraints to get the following system:

$$\begin{aligned} x \geq 0, \quad y &\geq 0 \\ x + 2y &\geq 6 \\ 2x + y &\geq 6 \end{aligned}$$

The graph of the constraints is shown in Fig. 8.24. The vertices are (0, 6), (6, 0), and (2, 2). The cost in cents for x servings of A and y servings of B is $C(x, y) = 9x + 20y$. Evaluate the cost at each vertex:

$$\begin{aligned} C(0, 6) &= 9(0) + 20(6) = 120 \text{ cents} \\ C(6, 0) &= 9(6) + 20(0) = 54 \text{ cents} \\ C(2, 2) &= 9(2) + 20(2) = 58 \text{ cents} \end{aligned}$$

The minimum cost of 54 cents is attained by using six servings of food A and no servings of food B.

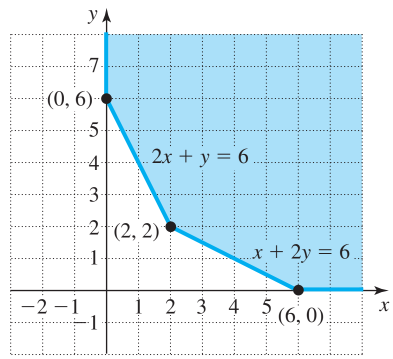

FIGURE 8.24

WARM-UPS

True or false? Explain your answer.

1. The graph of $x \geq 0$ in the coordinate plane consists of the points on or above the x-axis.
2. The graph of $y \geq 0$ in the coordinate plane consists of the points on or to the right of the y-axis.
3. The graph of $x + y \leq 6$ consists of the points below the line $x + y = 6$.
4. The graph of $2x + 3y = 30$ has x-intercept (0, 10) and y-intercept (15, 0).
5. The graph of a system of inequalities is a union of their individual solution sets.
6. In linear programming, constraints are inequalities that restrict the possible values that the variables can assume.
7. The function $F(x, y) = Ax^2 + By^2 + C$ is a linear function of x and y.
8. The value of $R(x, y) = 3x + 5y$ at the point (2, 4) is 26.
9. If $C(x, y) = 12x + 10y$, then $C(0, 5) = 62$.
10. In solving a linear programming problem, we must determine the vertices of the region defined by the constraints.

8.8 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a constraint?
2. What is linear programming?
3. Where do the constraints come from in a linear programming problem?
4. What is a linear function of two variables?

5. Where does the maximum or minimum value of a linear function subject to linear constraints occur?

6. What is the strategy for solving a linear programming problem?

Graph the solution set to each system of inequalities and identify each vertex of the region. See Example 1.

7. $x \geq 0, y \geq 0$
$x + y \leq 5$

8. $x \geq 0, y \geq 0$
$y \leq 5, y \geq x$

9. $x \geq 0, y \geq 0$
$2x + y \leq 4$
$x + y \leq 3$

10. $x \geq 0, y \geq 0$
$x + y \leq 4$
$x + 2y \leq 6$

11. $x \geq 0, y \geq 0$
$2x + y \geq 3$
$x + y \geq 2$

12. $x \geq 0, y \geq 0$
$3x + 2y \geq 12$
$2x + y \geq 7$

13. $x \geq 0, y \geq 0$
$x + 3y \leq 15$
$2x + y \leq 10$

14. $x \geq 0, y \geq 0$
$2x + 3y \leq 15$
$x + y \leq 7$

15. $x \geq 0, y \geq 0$
$x + y \geq 4$
$3x + y \geq 6$

16. $x \geq 0, y \geq 0$
$x + 3y \geq 6$
$2x + y \geq 7$

Solve each problem. See Examples 2–4.

17. ***Phase I advertising.*** The publicity director for Mercy Hospital is planning to bolster the hospital's image by running a TV ad and a radio ad. Due to budgetary and other constraints, the number of times that she can run the TV ad, x, and the number of times that she can run the radio ad, y, must be in the region shown in the figure on page 457. The function

$$A = 9000x + 4000y$$

gives the total number of people reached by the ads.

a) Find the total number of people reached by the ads at each vertex of the region.

b) What mix of TV and radio ads maximizes the number of people reached?

18. ***Phase II advertising.*** Suppose the radio station in Exercise 17 starts playing country music and the function for the total number of people changes to

$$A = 9000x + 2000y.$$

a) Find A at each vertex of the region using this function.

b) What mix of TV and radio ads maximizes the number of people reached?

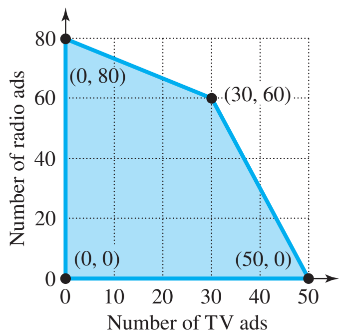

FIGURE FOR EXERCISES 17 AND 18

19. At Burger Heaven a double contains 2 meat patties and 6 pickles, whereas a triple contains 3 meat patties and 3 pickles. Near closing time one day, only 24 meat patties and 48 pickles are available. If a double burger sells for \$1.20 and a triple burger sells for \$1.50, then how many of each should be made to maximize the total revenue?

20. Sam and Doris manufacture rocking chairs and porch swings in the Ozarks. Each rocker requires 3 hours of work from Sam and 2 hours from Doris. Each swing requires 2 hours of work from Sam and 2 hours from Doris. Sam cannot work more than 48 hours per week, and Doris cannot work more than 40 hours per week. If a rocker sells for \$160 and a swing sells for \$100, then how many of each should be made per week to maximize the revenue?

21. If a double burger sells for \$1.00 and a triple burger sells for \$2.00, then how many of each should be made to maximize the total revenue subject to the constraints of Exercise 19?

22. If a rocker sells for \$120 and a swing sells for \$100, then how many of each should be made to maximize the total revenue subject to the constraints of Exercise 20?

23. One cup of Doggie Dinner contains 20 grams of protein and 40 grams of carbohydrates. One cup of Puppy Power contains 30 grams of protein and 20 grams of carbohydrates. Susan wants her dog to get at least 200 grams of protein and 180 grams of carbohydrates per day. If Doggie Dinner costs 16 cents per cup and Puppy Power costs 20 cents per cup, then how many cups of each would satisfy the constraints and minimize the total cost?

24. Mammoth Muffler employs supervisors and helpers. According to the union contract, a supervisor does 2 brake jobs and 3 mufflers per day, whereas a helper does 6 brake jobs and 3 mufflers per day. The home office requires enough staff for at least 24 brake jobs and for at least 18 mufflers per day. If a supervisor makes \$90 per day and a helper makes \$100 per day, then how many of each should be employed to satisfy the constraints and to minimize the daily labor cost?

25. Suppose in Exercise 23 Doggie Dinner costs 4 cents per cup and Puppy Power costs 10 cents per cup. How many cups of each would satisfy the constraints and minimize the total cost?

26. Suppose in Exercise 24 the supervisor makes \$110 per day and the helper makes \$100 per day. How many of each should be employed to satisfy the constraints and to minimize the daily labor cost?

27. Anita has at most \$24,000 to invest in her brother-in-law's laundromat and her nephew's car wash. Her brother-in-law has high blood pressure and heart disease but he will pay 18%, whereas her nephew is healthier but will pay only 12%. So the amount she will invest in the car wash will be at least twice the amount that she will invest in the laundromat but not more than three times as much. How much should she invest in each to maximize her total income from the two investments?

28. Herbert assembles computers in his shop. The parts for each economy model are shipped to him in a carton with a volume of 2 cubic feet (ft^3) and the parts for each deluxe model are shipped to him in a carton with a volume of 3 ft^3. After assembly, each economy model is shipped out in a carton with a volume of 4 ft^3, and each deluxe model is shipped out in a carton with a volume of 4 ft^3. The truck that delivers the parts has a maximum capacity of 180 ft^3, and the truck that takes out the completed computers has a maximum capacity of 280 ft^3. He can receive only one shipment of parts and send out one shipment of computers per week. If his profit on an economy model is \$60 and his profit on a deluxe model is \$100, then how many of each should he order per week to maximize his profit?