1.7 PROPERTIES OF THE REAL NUMBERS

In this section

- The Commutative Properties
- The Associative Properties
- The Distributive Property
- The Identity Properties
- The Inverse Properties
- Multiplication Property of Zero
- Applications

Everyone knows that the price of a hamburger plus the price of a Coke is the same as the price of a Coke plus the price of a hamburger. But do you know that this example illustrates the commutative property of addition? The properties of the real numbers are commonly used by anyone who performs the operations of arithmetic. In algebra we must have a thorough understanding of these properties.

The Commutative Properties

We get the same result whether we evaluate $3 + 5$ or $5 + 3$. This example illustrates the commutative property of addition. The fact that $4 \cdot 6$ and $6 \cdot 4$ are equal illustrates the commutative property of multiplication.

Commutative Properties

For any real numbers a and b,

$$a + b = b + a \qquad \text{and} \qquad ab = ba.$$

EXAMPLE 1

The commutative property of addition

Use the commutative property of addition to rewrite each expression.

a) $2 + (-10)$ **b)** $8 + x^2$ **c)** $2y - 4x$

Solution

a) $2 + (-10) = -10 + 2$

b) $8 + x^2 = x^2 + 8$

c) $2y - 4x = 2y + (-4x) = -4x + 2y$ ■

EXAMPLE 2

The commutative property of multiplication

Use the commutative property of multiplication to rewrite each expression.

a) $n \cdot 3$ **b)** $(x + 2) \cdot 3$ **c)** $5 - yx$

Solution

a) $n \cdot 3 = 3 \cdot n = 3n$ **b)** $(x + 2) \cdot 3 = 3(x + 2)$

c) $5 - yx = 5 - xy$ ■

Addition and multiplication are commutative operations, but what about subtraction and division? Since $5 - 3 = 2$ and $3 - 5 = -2$, subtraction is not commutative. To see that division is not commutative, try dividing \$8 among 4 people and \$4 among 8 people.

helpful hint

In arithmetic we would probably write $(2 + 3) + 7 = 12$ without thinking about the associative property. In algebra, we need the associative property to understand that

$$(x + 3) + 7 = x + (3 + 7) = x + 10.$$

The Associative Properties

Consider the computation of $2 + 3 + 6$. Using the order of operations, we add 2 and 3 to get 5 and then add 5 and 6 to get 11. If we add 3 and 6 first to get 9 and then add 2 and 9, we also get 11. So

$$(2 + 3) + 6 = 2 + (3 + 6).$$

We get the same result for either order of addition. This property is called the **associative property of addition.** The commutative and associative properties of addition are the reason that a hamburger, a Coke, and French fries cost the same as French fries, a hamburger, and a Coke.

We also have an **associative property of multiplication.** Consider the following two ways to find the product of 2, 3, and 4:

$$(2 \cdot 3)4 = 6 \cdot 4 = 24$$
$$2(3 \cdot 4) = 2 \cdot 12 = 24$$

We get the same result for either arrangement.

Associative Properties

For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (ab)c = a(bc).$$

EXAMPLE 3

Using the properties of multiplication

Use the commutative and associative properties of multiplication and exponential notation to rewrite each product.

a) $(3x)(x)$ **b)** $(xy)(5yx)$

Solution

a) $(3x)(x) = 3(x \cdot x) = 3x^2$

b) The commutative and associative properties of multiplication allow us to rearrange the multiplication in any order. We generally write numbers before variables, and we usually write variables in alphabetical order:

$$(xy)(5yx) = 5xxyy = 5x^2y^2$$

study tip

Find out what help is available at your school. Accompanying this text are video tapes, solution manuals, and a computer tutorial. Around most campuses you will find tutors available for hire, but most schools have a math lab where you can get help for free. Some schools even have free one-on-one tutoring available through special programs.

Consider the expression

$$3 - 9 + 7 - 5 - 8 + 4 - 13.$$

According to the accepted order of operations, we could evaluate this by computing from left to right. However, using the definition of subtraction, we can rewrite this expression as addition:

$$3 + (-9) + 7 + (-5) + (-8) + 4 + (-13)$$

The commutative and associative properties of addition allow us to add these numbers in any order we choose. It is usually faster to add the positive numbers, add the negative numbers, and then combine those two totals:

$$3 + 7 + 4 + (-9) + (-5) + (-8) + (-13) = 14 + (-35) = -21$$

Note that by performing the operations in this manner, we must subtract only once. There is no need to rewrite this expression as we have done here. We can sum the positive numbers and the negative numbers from the original expression and then combine their totals.

EXAMPLE 4

Using the properties of addition

Evaluate.

a) $3 - 7 + 9 - 5$ **b)** $4 - 5 - 9 + 6 - 2 + 4 - 8$

Solution

a) First add the positive numbers and the negative numbers:

$$\begin{aligned} 3 - 7 + 9 - 5 &= 12 + (-12) \\ &= 0 \end{aligned}$$

b) $$\begin{aligned} 4 - 5 - 9 + 6 - 2 + 4 - 8 &= 14 + (-24) \\ &= -10 \end{aligned}$$

It is certainly not essential that we evaluate the expressions of Example 4 as shown. We get the same answer by adding and subtracting from left to right. However, in algebra, just getting the answer is not always the most important point. Learning new methods often increases understanding.

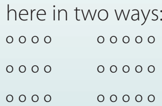
Even though addition is associative, subtraction is not an associative operation. For example, $(8 - 4) - 3 = 1$ and $8 - (4 - 3) = 7$. So

$$(8 - 4) - 3 \neq 8 - (4 - 3).$$

We can also use a numerical example to show that division is not associative. For instance, $(16 \div 4) \div 2 = 2$ and $16 \div (4 \div 2) = 8$. So

$$(16 \div 4) \div 2 \neq 16 \div (4 \div 2).$$

The Distributive Property

> **helpful hint**
>
> To visualize the distributive property, we can determine the number of circles shown here in two ways:
>
> 
>
> There are $3 \cdot 9$ or 27 circles, or there are $3 \cdot 4$ circles in the first group and $3 \cdot 5$ circles in the second group for a total of 27 circles.

If four men and five women pay \$3 each for a movie, there are two ways to find the total amount spent:

$$3(4 + 5) = 3 \cdot 9 = 27$$
$$3 \cdot 4 + 3 \cdot 5 = 12 + 15 = 27$$

Since we get \$27 either way, we can write

$$3(4 + 5) = 3 \cdot 4 + 3 \cdot 5.$$

We say that the multiplication by 3 is *distributed* over the addition. This example illustrates the **distributive property.**

Consider the following expressions involving multiplication and subtraction:

$$5(6 - 4) = 5 \cdot 2 = 10$$
$$5 \cdot 6 - 5 \cdot 4 = 30 - 20 = 10$$

Since both expressions have the same value, we can write

$$5(6 - 4) = 5 \cdot 6 - 5 \cdot 4.$$

Multiplication by 5 is distributed over each number in the parentheses. This example illustrates that multiplication distributes over subtraction.

> **Distributive Property**
>
> For any real numbers a, b, and c,
>
> $$a(b + c) = ab + ac \quad \text{and} \quad a(b - c) = ab - ac.$$

We can use the distributive property to remove parentheses. If we start with $4(x + 3)$ and write

$$4(x + 3) = 4x + 4 \cdot 3 = 4x + 12,$$

we are using it to multiply 4 and $x + 3$ or to remove the parentheses. We wrote the product $4(x + 3)$ as the sum $4x + 12$.

EXAMPLE 5 **Writing a product as a sum or difference**

Use the distributive property to remove the parentheses.

a) $a(3 - b)$ **b)** $-3(x - 2)$

Solution

a) $a(3 - b) = a3 - ab$ Distributive property

$= 3a - ab$ $a3 = 3a$

b) $-3(x - 2) = -3x - (-3)(2)$ Distributive property

$= -3x - (-6)$ $(-3)(2) = -6$

$= -3x + 6$ Simplify.

When we write a number or an expression as a product, we are **factoring.** If we start with $3x + 15$ and write

$$3x + 15 = 3x + 3 \cdot 5 = 3(x + 5),$$

we are using the distributive property to factor $3x + 15$. We factored out the common factor 3.

EXAMPLE 6 **Writing a sum or difference as a product**

Use the distributive property to factor each expression.

a) $7x - 21$ **b)** $5a + 5$

Solution

a) $7x - 21 = 7x - 7 \cdot 3$ Write 21 as $7 \cdot 3$.

$= 7(x - 3)$ Distributive property

b) $5a + 5 = 5a + 5 \cdot 1$ Write 5 as $5 \cdot 1$.

$= 5(a + 1)$ Factor out the common factor 5.

study tip

Don't cram for a test. Some students try to cram weeks of work into one "all-nighter." These same students are seen frantically paging through the text up until the moment that the test papers are handed out. These practices create a lot of test anxiety and will only make you sick. Start studying for a test several days in advance, and get a good night's sleep before a test. If you keep up with homework, then there will be no need to cram.

The Identity Properties

The numbers 0 and 1 have special properties. Multiplication of a number by 1 does not change the number, and addition of 0 to a number does not change the number. That is why 1 is called the **multiplicative identity** and 0 is called the **additive identity.**

Identity Properties

For any real number a,

$$a \cdot 1 = 1 \cdot a = a \quad \text{and} \quad a + 0 = 0 + a = a.$$

The Inverse Properties

The idea of additive inverses was introduced in Section 1.3. Every real number a has an **additive inverse** or **opposite,** $-a$, such that $a + (-a) = 0$. Every nonzero real number a also has a **multiplicative inverse** or **reciprocal,** written $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = 1$. Note that the sum of additive inverses is the additive identity and that the product of multiplicative inverses is the multiplicative identity.

Inverse Properties

For any real number a there is a number $-a$, such that

$$a + (-a) = 0.$$

For any nonzero real number a there is a number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = 1.$$

We are already familiar with multiplicative inverses for rational numbers. For example, the multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$ because

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1.$$

EXAMPLE 7

Multiplicative inverses

Find the multiplicative inverse of each number.

a) 5 **b)** 0.3 **c)** $-\frac{3}{4}$ **d)** 1.7

calculator close-up

You can find multiplicative inverses with a calculator as shown here.

```
1/.3▸Frac
            10/3
1/(-3/4)▸Frac
            -4/3
1/1.7▸Frac
           10/17
```

When the divisor is a fraction, it must be in parentheses.

Solution

a) The multiplicative inverse of 5 is $\frac{1}{5}$ because

$$5 \cdot \frac{1}{5} = 1.$$

b) To find the reciprocal of 0.3, we first write 0.3 as a ratio of integers:

$$0.3 = \frac{3}{10}$$

The multiplicative inverse of 0.3 is $\frac{10}{3}$ because

$$\frac{3}{10} \cdot \frac{10}{3} = 1.$$

c) The reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$ because

$$\left(-\frac{3}{4}\right)\left(-\frac{4}{3}\right) = 1.$$

d) First convert 1.7 to a ratio of integers:

$$1.7 = 1\frac{7}{10} = \frac{17}{10}$$

The multiplicative inverse is $\frac{10}{17}$. ■

Multiplication Property of Zero

Zero has a property that no other number has. Multiplication involving zero always results in zero.

Multiplication Property of Zero

For any real number a,

$$0 \cdot a = 0 \quad \text{and} \quad a \cdot 0 = 0.$$

EXAMPLE 8

Identifying the properties

Name the property that justifies each equation.

a) $5 \cdot 7 = 7 \cdot 5$

b) $4 \cdot \frac{1}{4} = 1$

c) $1 \cdot 864 = 864$

d) $6 + (5 + x) = (6 + 5) + x$

e) $3x + 5x = (3 + 5)x$

f) $6 + (x + 5) = 6 + (5 + x)$

g) $\pi x^2 + \pi y^2 = \pi(x^2 + y^2)$

h) $325 + 0 = 325$

i) $-3 + 3 = 0$

j) $455 \cdot 0 = 0$

Solution

a) Commutative

b) Multiplicative inverse

c) Multiplicative identity

d) Associative

e) Distributive

f) Commutative

g) Distributive

h) Additive identity

i) Additive inverse

j) Multiplication property of 0

Applications

Reciprocals are important in problems involving work. For example, if you wax one car in 3 hours, then your rate is $\frac{1}{3}$ of a car per hour. If you can wash one car in 12 minutes $\left(\frac{1}{5} \text{ of an hour}\right)$, then you are washing cars at the rate of 5 cars per hour. In general, if you can complete a task in x hours, then your rate is $\frac{1}{x}$ tasks per hour.

EXAMPLE 9

helpful hint

When machines or people are working together, we can add their rates provided they do not interfere with each other's work. If operating both car wash machines causes a traffic jam, then the rate together might not be 22.5 cars per hour.

Washing rates

A car wash has two machines. The old machine washes one car in 0.1 hour, while the new machine washes one car in 0.08 hour. If both machines are operating, then at what rate (in cars per hour) are the cars being washed?

Solution

The old machine is working at the rate of $\frac{1}{0.1}$ cars per hour, and the new machine is working at the rate of $\frac{1}{0.08}$ cars per hour. Their rate working together is the sum of their individual rates:

$$\frac{1}{0.1} + \frac{1}{0.08} = 10 + 12.5 = 22.5$$

So working together, the machines are washing 22.5 cars per hour.

WARM-UPS

True or false? Explain your answer.

1. $24 \div (4 \div 2) = (24 \div 4) \div 2$
2. $1 \div 2 = 2 \div 1$
3. $6 - 5 = -5 + 6$
4. $9 - (4 - 3) = (9 - 4) - 3$
5. Multiplication is a commutative operation.
6. $5x + 5 = 5(x + 1)$ for any value of x.
7. The multiplicative inverse of 0.02 is 50.
8. $-3(x - 2) = -3x + 6$ for any value of x.
9. $3x + 2x = (3 + 2)x$ for any value of x.
10. The additive inverse of 0 is 0.

1.7 EXERCISES

Reading and Writing *After reading this section write out the answers to these questions. Use complete sentences.*

1. What is the difference between the commutative property of addition and the associative property of addition?

2. Which property involves two different operations?

3. What is factoring?

4. Which two numbers play a prominent role in the properties studied here?

5. What is the purpose of studying the properties of real numbers?

6. What is the relationship between rate and time?

Use the commutative property of addition to rewrite each expression. See Example 1.

7. $9 + r$ **8.** $t + 6$ **9.** $3(2 + x)$

10. $P(1 + rt)$ **11.** $4 - 5x$ **12.** $b - 2a$

Use the commutative property of multiplication to rewrite each expression. See Example 2.

13. $x \cdot 6$ **14.** $y \cdot (-9)$ **15.** $(x - 4)(-2)$

16. $a(b + c)$ **17.** $4 - y \cdot 8$ **18.** $z \cdot 9 - 2$

Use the commutative and associative properties of multiplication and exponential notation to rewrite each product. See Example 3.

19. $(4w)(w)$ **20.** $(y)(2y)$ **21.** $3a(ba)$

22. $(x \cdot x)(7x)$ **23.** $(x)(9x)(xz)$ **24.** $y(y \cdot 5)(wy)$

Evaluate by finding first the sum of the positive numbers and then the sum of the negative numbers. See Example 4.

25. $8 - 4 + 3 - 10$

26. $-3 + 5 - 12 + 10$

27. $8 - 10 + 7 - 8 - 7$

28. $6 - 11 + 7 - 9 + 13 - 2$

29. $-4 - 11 + 7 - 8 + 15 - 20$

30. $-8 + 13 - 9 - 15 + 7 - 22 + 5$

31. $-3.2 + 2.4 - 2.8 + 5.8 - 1.6$

32. $5.4 - 5.1 + 6.6 - 2.3 + 9.1$

33. $3.26 - 13.41 + 5.1 - 12.35 - 5$

34. $5.89 - 6.1 + 8.58 - 6.06 - 2.34$

Use the distributive property to remove the parentheses. See Example 5.

35. $3(x - 5)$ **36.** $4(b - 1)$

37. $a(2 + t)$ **38.** $b(a + w)$

39. $-3(w - 6)$ **40.** $-3(m - 5)$

41. $-4(5 - y)$ **42.** $-3(6 - p)$

43. $-1(a - 7)$ **44.** $-1(c - 8)$

45. $-1(t + 4)$ **46.** $-1(x + 7)$

Use the distributive property to factor each expression. See Example 6.

47. $2m + 12$ **48.** $3y + 6$

49. $4x - 4$ **50.** $6y + 6$

51. $4y - 16$ **52.** $5x + 15$

53. $4a + 8$ **54.** $7a - 35$

Find the multiplicative inverse (reciprocal) of each number. See Example 7.

55. $\frac{1}{2}$ **56.** $\frac{1}{3}$ **57.** -5

58. -6 **59.** 7 **60.** 8

61. 1 **62.** -1 **63.** -0.25

64. 0.75 **65.** 2.5 **66.** 3.5

Name the property that justifies each equation. See Example 8.

67. $3 \cdot x = x \cdot 3$

68. $x + 5 = 5 + x$

69. $2(x - 3) = 2x - 6$

70. $a(bc) = (ab)c$

71. $-3(xy) = (-3x)y$

72. $3(x + 1) = 3x + 3$

73. $4 + (-4) = 0$

74. $1.3 + 9 = 9 + 1.3$

75. $x^2 \cdot 5 = 5x^2$

76. $0 \cdot \pi = 0$

77. $1 \cdot 3y = 3y$

78. $(0.1)(10) = 1$

79. $2a + 5a = (2 + 5)a$

80. $3 + 0 = 3$

81. $-7 + 7 = 0$

82. $1 \cdot b = b$

83. $(2346)0 = 0$

84. $4x + 4 = 4(x + 1)$

85. $ay + y = y(a + 1)$

86. $ab + bc = b(a + c)$

Complete each equation, using the property named.

87. $a + y =$ ____, commutative

88. $6x + 6 =$ ____, distributive

89. $5(aw) =$ ____, associative

90. $x + 3 =$ ____, commutative

91. $\frac{1}{2}x + \frac{1}{2} =$ ____, distributive

92. $-3(x - 7) =$ ____, distributive

93. $6x + 15 =$ ____, distributive

94. $(x + 6) + 1 =$ ____, associative

95. $4(0.25) =$ ____, inverse property

96. $-1(5 - y) =$ ____, distributive

97. $0 = 96($____$)$, multiplication property of zero

98. $3 \cdot ($____$) = 3$, identity property

99. $0.33($____$) = 1$, inverse property

100. $-8(1) =$ ____, identity property

Solve each problem. See Example 9.

101. ***Laying bricks.*** A bricklayer lays one brick in 0.04 hour, while his apprentice lays one brick in 0.05 hour.

a) If both are working, then at what combined rate (in bricks per hour) are they laying bricks?

b) Which person is working faster?

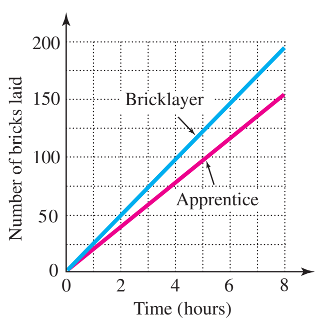


FIGURE FOR EXERCISE 101

102. ***Recovering golf balls.*** Susan and Joan are diving for golf balls in a large water trap. Susan recovers a golf ball every 0.016 hour while Joan recovers a ball every 0.025 hour. If both are working, then at what rate (in golf balls per hour) are they recovering golf balls?

103. ***Population explosion.*** In 1998, the population of the earth was increasing by one person every 0.3801 second (*World Population Data Sheet* 1998, www.prb.org).

a) At what rate in people per second is the population of the earth increasing?

b) At what rate in people per week is the population of the earth increasing?

104. ***Farmland conversion.*** The amount of farmland in the United States is decreasing by one acre every 0.00876 hours as farmland is being converted to nonfarm use (American Farmland Trust, www.farmland.org). At what rate in acres per day is the farmland decreasing?

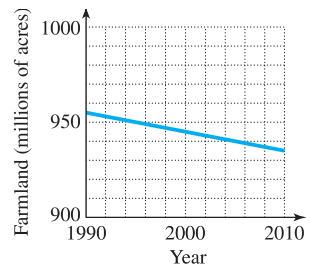


FIGURE FOR EXERCISE 104

GETTING MORE INVOLVED

105. ***Writing.*** The perimeter of a rectangle is the sum of twice the length and twice the width. Write in words another way to find the perimeter that illustrates the distributive property.

106. ***Discussion.*** Eldrid bought a loaf of bread for $1.69 and a gallon of milk for $2.29. Using a tax rate of 5%, he correctly figured that the tax on the bread would be 8 cents and the tax on the milk would be 11 cents, for a total of $4.17. However, at the cash register he was correctly charged $4.18. How could this happen? Which property of the real numbers is in question in this case?

107. ***Exploration.*** Determine whether each of the following pairs of tasks are "commutative." That is, does the order in which they are performed produce the same result?

a) Put on your coat; put on your hat.

b) Put on your shirt; put on your coat.

Find another pair of "commutative" tasks and another pair of "noncommutative" tasks.