2.1 THE ADDITION AND MULTIPLICATION PROPERTIES OF EQUALITY

In this section

- The Addition Property of Equality
- The Multiplication Property of Equality
- Variables on Both Sides
- Applications

In Section 1.6, you learned that an equation is a statement that two expressions are equal. You also learned how to determine whether a number is a solution to an equation. In this section you will learn systematic procedures for finding solutions to equations.

The Addition Property of Equality

The equations that we work with in this section and the next two are called linear equations.

Linear Equation

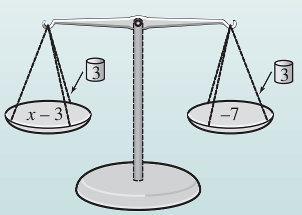
A **linear equation in one variable** x is an equation that can be written in the form

$$ax + b = 0,$$

where a and b are real numbers and $a \neq 0$.

helpful hint

Think of an equation like a balance scale. To keep the scale in balance, what you add to one side you must also add to the other side.



An equation such as $2x + 3 = 0$ is a linear equation. We also refer to equations such as

$$x + 8 = 0, \quad 3x = 7, \quad 2x + 5 = 9 - 5x, \quad \text{and} \quad 3 + 5(x - 1) = -7 + x$$

as linear equations, because these equations could be written in the form $ax + b = 0$ using the properties of equality, which we are about to discuss.

If two workers have equal salaries and each gets a \$1000 raise, then they still have equal salaries after the raise. This example illustrates the addition property of equality.

The Addition Property of Equality

Adding the same number to both sides of an equation does not change the solution to the equation. In symbols, if $a = b$, then

$$a + c = b + c.$$

To **solve** an equation means to find all of the solutions to the equation. The set of all solutions to an equation is the **solution set** to the equation. Equations that have the same solution set are **equivalent equations.** In our first example, we will use the addition property of equality to solve an equation.

EXAMPLE 1 **Adding the same number to both sides**

Solve $x - 3 = -7$.

Solution

We can remove the 3 from the left side of the equation by adding 3 to each side of the equation:

$$x - 3 = -7$$
$$x - 3 + 3 = -7 + 3 \quad \text{Add 3 to each side.}$$
$$x + 0 = -4 \quad \text{Simplify each side.}$$
$$x = -4 \quad \text{Zero is the additive identity.}$$

Since -4 satisfies the last equation, it should also satisfy the original equation because all of the previous equations are equivalent. Check that -4 satisfies the original equation by replacing x by -4:

$$x - 3 = -7 \quad \text{Original equation}$$
$$-4 - 3 = -7 \quad \text{Replace } x \text{ by } -4.$$
$$-7 = -7 \quad \text{Simplify.}$$

Since $-4 - 3 = -7$ is correct, $\{-4\}$ is the solution set to the equation.

In Example 1, we used addition to isolate the variable on the left-hand side of the equation. Once the variable is isolated, we can determine the solution to the equation. Because subtraction is defined in terms of addition, we can also use subtraction to isolate the variable.

EXAMPLE 2 **Subtracting the same number from both sides**

Solve $9 + x = -2$.

Solution

We can remove the 9 from the left side by adding -9 to each side or by subtracting 9 from each side of the equation:

$$9 + x = -2$$
$$9 + x - 9 = -2 - 9 \quad \text{Subtract 9 from each side.}$$
$$x = -11 \quad \text{Simplify each side.}$$

Check that -11 satisfies the original equation by replacing x by -11:

$$9 + x = -2 \quad \text{Original equation}$$
$$9 + (-11) = -2 \quad \text{Replace } x \text{ by } -11.$$

Since $9 + (-11) = -2$ is correct, $\{-11\}$ is the solution set to the equation.

study tip

Think! Thinking is the manipulation of facts and principles. Your thinking will be as clear as your understanding of the facts and principles.

Our goal in solving equations is to isolate the variable. In the first two examples, the variable was isolated on the left side of the equation. In the next example, we isolate the variable on the right side of the equation.

EXAMPLE 3 **Isolating the variable on the right side**

Solve $\frac{1}{2} = -\frac{1}{4} + y$.

Solution

We can remove $-\frac{1}{4}$ from the right side by adding $\frac{1}{4}$ to both sides of the equation:

$$\frac{1}{2} = -\frac{1}{4} + y$$
$$\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} + y + \frac{1}{4} \quad \text{Add } \tfrac{1}{4} \text{ to each side.}$$
$$\frac{3}{4} = y \quad \text{Simplify each side.}$$

study tip

Don't simply work exercises to get answers. Keep reminding yourself of what it is that you are doing. Keep trying to get the big picture. How does this section relate to what we did in the last section? Where are we going next? When is the picture complete?

Check that $\frac{3}{4}$ satisfies the original equation by replacing y by $\frac{3}{4}$:

$$\frac{1}{2} = -\frac{1}{4} + y \quad \text{Original equation}$$

$$\frac{1}{2} = -\frac{1}{4} + \frac{3}{4} \quad \text{Replace } y \text{ by } \tfrac{3}{4}.$$

$$\frac{1}{2} = \frac{2}{4} \quad \text{Simplify.}$$

Since $\frac{1}{2} = \frac{2}{4}$ is correct, $\left\{\frac{3}{4}\right\}$ is the solution set to the equation. ■

The Multiplication Property of Equality

To isolate a variable that is involved in a product or a quotient, we need the multiplication property of equality.

The Multiplication Property of Equality

Multiplying both sides of an equation by the same nonzero number does not change the solution to the equation. In symbols, if $a = b$ and $c \neq 0$, then

$$ac = bc.$$

If the variable in an equation is divided by a number, we can isolate the variable by multiplying each side of the equation by the divisor as in the next example.

EXAMPLE 4 **Multiplying both sides by the same number**

Solve $\frac{z}{2} = 6$.

Solution

We isolate the variable z by multiplying each side of the equation by 2.

$$\frac{z}{2} = 6 \quad \text{Original equation}$$

$$2 \cdot \frac{z}{2} = 2 \cdot 6 \quad \text{Multiply each side by 2.}$$

$$1z = 12 \quad \text{Because } 2 \cdot \tfrac{z}{2} = 2 \cdot \tfrac{1}{2}z = 1z$$

$$z = 12 \quad \text{Multiplicative identity}$$

Because $\frac{12}{2} = 6$, $\{12\}$ is the solution set to the equation. ■

Because dividing by a number is the same as multiplying by its reciprocal, the multiplication property of equality allows us to divide each side of the equation by any nonzero number.

EXAMPLE 5

Dividing both sides by the same number

Solve $-5w = 30$.

Solution

Since w is multiplied by -5, we can isolate w by multiplying by $-\frac{1}{5}$ or by dividing each side by -5:

$$-5w = 30 \quad \text{Original equation}$$

$$\frac{-5w}{-5} = \frac{30}{-5} \quad \text{Divide each side by } -5.$$

$$1 \cdot w = -6 \quad \text{Because } \frac{-5}{-5} = 1$$

$$w = -6 \quad \text{Multiplicative identity}$$

Because $-5(-6) = 30$, $\{-6\}$ is the solution set to the equation.

In the next example, the coefficient of the variable is a fraction. We could divide each side by the coefficient as we did in Example 5, but it is easier to multiply each side by the reciprocal of the coefficient.

EXAMPLE 6

Multiplying by the reciprocal

Solve $\frac{2}{3}p = 40$.

Solution

Multiply each side by $\frac{3}{2}$, the reciprocal of $\frac{2}{3}$, to isolate p on the left side.

$$\frac{2}{3}p = 40$$

$$\frac{3}{2} \cdot \frac{2}{3}p = \frac{3}{2} \cdot 40 \quad \text{Multiply each side by } \frac{3}{2}.$$

$$1 \cdot p = 60 \quad \text{Multiplicative inverses}$$

$$p = 60 \quad \text{Multiplicative identity}$$

Because $\frac{2}{3} \cdot 60 = 40$, we can be sure that the solution set is $\{60\}$.

helpful hint

You could solve this equation by multiplying each side by 3 to get $2p = 120$, and then dividing each side by 2 to get $p = 60$.

If the coefficient of the variable is an integer, we usually divide each side by that integer, as in Example 5. If the coefficient of the variable is a fraction, we usually multiply each side by the reciprocal of the fraction as in Example 6.

If $-x$ appears in an equation, we can multiply by -1 to get x, because $-1(-x) = -(-x) = x$.

EXAMPLE 7

Multiplying by −1

Solve $-h = 12$.

Solution

Multiply each side by -1 to get h on the left side.

$$-h = 12$$

$$-1(-h) = -1 \cdot 12$$

$$h = -12$$

Since $-(-12) = 12$, the solution set is $\{-12\}$.

Variables on Both Sides

In the next example, the variable occurs on both sides of the equation. Because the variable represents a real number, we can still isolate the variable by using the addition property of equality.

EXAMPLE 8

Subtracting an algebraic expression from both sides

Solve $-9 + 6y = 7y$.

Solution

The expression $6y$ can be removed from the left side of the equation by subtracting $6y$ from both sides.

$$-9 + 6y = 7y$$

$$-9 + 6y - 6y = 7y - 6y \quad \text{Subtract } 6y \text{ from each side.}$$

$$-9 = y \quad \text{Simplify each side.}$$

Check by replacing y by -9 in the original equation:

$$-9 + 6(-9) = 7(-9)$$

$$-63 = -63$$

The solution set to the equation is $\{-9\}$.

helpful hint

It does not matter whether the variable ends up on the left or right side of the equation. Whether we get $y = -9$ or $-9 = y$ we can still conclude that the solution is -9.

Applications

In the next example, we use the multiplication property of equality in an applied situation.

EXAMPLE 9

Population density

In 1990, San Francisco had $\frac{2}{3}$ as many people per hectare as New York (U.S. Bureau of Census, www.census.gov). The population density of San Francisco was 60 people per hectare. What was the population density of New York?

Solution

If p represents the population density of New York, then $\frac{2}{3}p = 60$. To find p, solve the equation:

$$\frac{2}{3}p = 60$$

$$\frac{3}{2} \cdot \frac{2}{3}p = \frac{3}{2} \cdot 60 \quad \text{Multiply each side by } \tfrac{3}{2}.$$

$$p = 90 \quad \text{Simplify.}$$

So the population density of New York was 90 people per hectare.

WARM-UPS

True or false? Explain your answer.

1. The solution to $x - 5 = 5$ is 10.
2. The equation $\frac{x}{2} = 4$ is equivalent to the equation $x = 8$.
3. To solve $\frac{3}{4}y = 12$, we should multiply each side by $\frac{3}{4}$.
4. The equation $\frac{x}{7} = 4$ is equivalent to $\frac{1}{7}x = 4$.
5. Multiplying each side of an equation by any real number will result in an equation that is equivalent to the original equation.
6. To isolate t in $2t = 7 + t$, subtract t from each side.
7. To solve $\frac{2r}{3} = 30$, we should multiply each side by $\frac{3}{2}$.
8. Adding any real number to both sides of an equation will result in an equation that is equivalent to the original equation.
9. The equation $5x = 0$ is equivalent to $x = 0$.
10. The solution to $2x - 3 = x + 1$ is 4.

2.1 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What does the addition property of equality say?
2. What are equivalent equations?
3. What is the multiplication property of equality?
4. What is a linear equation in one variable?
5. How can you tell if your solution to an equation is correct?
6. To obtain an equivalent equation, what are you not allowed to do to both sides of the equation?

Solve each equation. Show your work and check your answer. See Example 1.

7. $x - 6 = -5$ **8.** $x - 7 = -2$

9. $-13 + x = -4$ **10.** $-8 + x = -12$

11. $y - \frac{1}{2} = \frac{1}{2}$ **12.** $y - \frac{1}{4} = \frac{1}{2}$

13. $w - \frac{1}{3} = \frac{1}{3}$ **14.** $w - \frac{1}{3} = \frac{1}{2}$

Solve each equation. Show your work and check your answer. See Example 2.

15. $x + 3 = -6$ **16.** $x + 4 = -3$

17. $12 + x = -7$ **18.** $19 + x = -11$

19. $t + \frac{1}{2} = \frac{3}{4}$ **20.** $t + \frac{1}{3} = 1$

21. $\frac{1}{19} + m = \frac{1}{19}$ **22.** $\frac{1}{3} + n = \frac{1}{2}$

Solve each equation. Show your work and check your answer. See Example 3.

23. $2 = x + 7$ **24.** $3 = x + 5$

25. $-13 = y - 9$ **26.** $-14 = z - 12$

27. $0.5 = -2.5 + x$ **28.** $0.6 = -1.2 + x$

29. $\frac{1}{8} = -\frac{1}{8} + r$ **30.** $\frac{1}{6} = -\frac{1}{6} + h$

Solve each equation. Show your work and check your answer. See Example 4.

31. $\frac{x}{2} = -4$ **32.** $\frac{x}{3} = -6$

33. $0.03 = \frac{y}{60}$ **34.** $0.05 = \frac{y}{80}$

35. $\frac{a}{2} = \frac{1}{3}$

36. $\frac{b}{2} = \frac{1}{5}$

37. $\frac{1}{6} = \frac{c}{3}$

38. $\frac{1}{12} = \frac{d}{3}$

Solve each equation. Show your work and check your answer. See Example 5.

39. $-3x = 15$

40. $-5x = -20$

41. $20 = 4y$

42. $18 = -3a$

43. $2w = 2.5$

44. $-2x = -5.6$

45. $5 = 20x$

46. $-3 = 27d$

Solve each equation. Show your work and check your answer. See Example 6.

47. $\frac{3}{2}x = -3$

48. $\frac{2}{3}x = -8$

49. $90 = \frac{3y}{4}$

50. $14 = \frac{7y}{8}$

51. $-\frac{3}{5}w = -\frac{1}{3}$

52. $-\frac{5}{2}t = -\frac{3}{5}$

53. $\frac{2}{3} = -\frac{4x}{3}$

54. $\frac{1}{14} = -\frac{6p}{7}$

Solve each equation. Show your work and check your answer. See Example 7.

55. $-x = 8$

56. $-x = 4$

57. $-y = -\frac{1}{3}$

58. $-y = -\frac{7}{8}$

59. $3.4 = -z$

60. $4.9 = -t$

61. $-k = -99$

62. $-m = -17$

Solve each equation. Show your work and check your answer. See Example 8.

63. $4x = 3x - 7$

64. $3x = 2x + 9$

65. $9 - 6y = -5y$

66. $12 - 18w = -17w$

67. $-6x = 8 - 7x$

68. $-3x = -6 - 4x$

69. $\frac{1}{2}c = 5 - \frac{1}{2}c$

70. $-\frac{1}{2}h = 13 - \frac{3}{2}h$

Use the appropriate property of equality to solve each equation.

71. $12 = x + 17$

72. $-3 = x + 6$

73. $\frac{3}{4}y = -6$

74. $\frac{5}{9}z = -10$

75. $-3.2 + x = -1.2$

76. $t - 3.8 = -2.9$

77. $2a = \frac{1}{3}$

78. $-3w = \frac{1}{2}$

79. $-9m = 3$

80. $-4h = -2$

81. $-b = -44$

82. $-r = 55$

83. $\frac{2}{3}x = \frac{1}{2}$

84. $\frac{3}{4}x = \frac{1}{3}$

85. $-5x = 7 - 6x$

86. $-\frac{1}{2} + 3y = 4y$

87. $\frac{5a}{7} = -10$

88. $\frac{7r}{12} = -14$

89. $\frac{1}{2}v = -\frac{1}{2}v + \frac{3}{8}$

90. $\frac{1}{3}s + \frac{7}{9} = \frac{4}{3}s$

Solve each problem by writing and solving an equation. See Example 9.

91. ***Cigarette consumption.*** In 1999, cigarette consumption in the U.S. was 125 packs per capita. This rate of consumption was $\frac{5}{8}$ of what it was in 1980. Find the rate of consumption in 1980.

92. ***World grain demand.*** Freeport McMoRan projects that in 2010 world grain supply will be 1.8 trillion metric tons and the supply will be only $\frac{3}{4}$ of world grain demand. What will world grain demand be in 2010?

FIGURE FOR EXERCISE 92

93. ***Advancers and decliners.*** On Thursday, $\frac{13}{25}$ of the stocks traded on the New York Stock Exchange advanced in price. If 1495 stocks advanced, then how many stocks were traded on that day?

94. ***Accidental deaths.*** In 1996, $\frac{23}{50}$ of all accidental deaths in the U.S. were the result of automobile accidents (National Center for Health Statistics, www.nchs.gov). If there were 43,194 deaths due to automobile accidents, then how many accidental deaths were there in 1996?