95. ***Designer jeans.*** A pair of ordinary jeans at A-Mart costs \$50 less than a pair of designer jeans at Enrico's. In fact, you can buy four pairs of A-Mart jeans for less than one pair of Enrico's jeans. What is the price range for a pair of A-Mart jeans?

96. ***United Express.*** Al and Rita both drive parcel delivery trucks for United Express. Al averages 20 mph less than Rita. In fact, Al is so slow that in 5 hours he covered fewer miles than Rita did in 3 hours. What are the possible values for Al's rate of speed?

GETTING MORE INVOLVED

97. ***Discussion.*** If 3 is added to every number in $(4, \infty)$, the resulting set is $(7, \infty)$. In each of the following cases, write the resulting set of numbers in interval notation. Explain your results.

a) The number -6 is subtracted from every number in $[2, \infty)$.
b) Every number in $(-\infty, -3)$ is multiplied by 2.
c) Every number in $(8, \infty)$ is divided by 4.
d) Every number in $(6, \infty)$ is multiplied by -2.
e) Every number in $(-\infty, -10)$ is divided by -5.

98. ***Writing.*** Explain why saying that x is *at least* 9 is equivalent to saying that x is *greater than or equal to* 9. Explain why saying that x is *at most* 5 is equivalent to saying that x is *less than or equal to* 5.

3.2 COMPOUND INEQUALITIES

In this section

- Basics
- Graphing the Solution Set
- Applications

In this section we will use our knowledge of inequalities from Section 3.1 to work with compound inequalities.

Basics

The inequalities that we studied in Section 3.1 are referred to as **simple inequalities.** If we join two simple inequalities with the connective "and" or the connective "or," we get a **compound inequality.** A compound inequality using the connective "and" is true if and only if *both* simple inequalities are true.

EXAMPLE 1 **Compound inequalities using the connective "and"**

Determine whether each compound inequality is true.

a) $3 > 2$ and $3 < 5$ **b)** $6 > 2$ and $6 < 5$

Solution

a) The compound inequality is true because $3 > 2$ is true and $3 < 5$ is true.
b) The compound inequality is false because $6 < 5$ is false. ■

A compound inequality using the connective "or" is true if one or the other or both of the simple inequalities are true. It is false only if both simple inequalities are false.

EXAMPLE 2 **Compound inequalities using the connective "or"**

Determine whether each compound inequality is true.

a) $2 < 3$ or $2 > 7$ **b)** $4 < 3$ or $4 \geq 7$

Solution

a) The compound inequality is true because $2 < 3$ is true.
b) The compound inequality is false because both $4 < 3$ and $4 \geq 7$ are false. ■

If a compound inequality involves a variable, then we are interested in the solution set to the inequality. The solution set to an "and" inequality consists of all numbers that satisfy both simple inequalities, whereas the solution set to an "or" inequality consists of all numbers that satisfy at least one of the simple inequalities.

EXAMPLE 3 **Solutions of compound inequalities**

Determine whether 5 satisfies each compound inequality.

a) $x < 6$ and $x < 9$

b) $2x - 9 \le 5$ or $-4x \ge -12$

helpful hint

There is a big difference between "and" and "or." To get money from an automatic teller you must have a bank card *and* know a secret number (PIN). There would be a lot of problems if you could get money by having a bank card *or* knowing a PIN.

Solution

a) Because $5 < 6$ and $5 < 9$ are both true, 5 satisfies the compound inequality.

b) Because $2 \cdot 5 - 9 \le 5$ is true, it does not matter that $-4 \cdot 5 \ge -12$ is false. So 5 satisfies the compound inequality.

Graphing the Solution Set

If A and B are sets of numbers, then the **intersection** of A and B is the set of all numbers that are in both A and B. The intersection of A and B is denoted as $A \cap B$ (read "A intersect B"). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \cap B = \{2, 3\}$ because only 2 and 3 are in both A and B.

The solution set to a compound inequality using the connective "and" is the intersection of the solution sets to each of the simple inequalities. Using graphs, as shown in the next example, will help you understand compound inequalities.

EXAMPLE 4 **Graphing compound inequalities**

Graph the solution set to the compound inequality $x > 2$ and $x < 5$.

Solution

We first sketch the graph of $x > 2$ and then the graph of $x < 5$, as shown in the top two number lines in Fig. 3.10. The intersection of these two solution sets is the portion of the number line that is shaded on both graphs, just the part between 2 and 5, not including the endpoints. The graph of $\{x \mid x > 2 \text{ and } x < 5\}$ is shown at the bottom of Fig. 3.10. We write this set in interval notation as $(2, 5)$.

study tip

Never leave an exam early. Most papers turned in early contain careless errors that could be found and corrected. Every point counts. Reread the questions to be sure that you don't have a nice solution to a question that wasn't asked. Check all arithmetic. You can check many problems by using a different method. Some problems can be checked with a calculator. Make sure that you did not forget to answer a question.

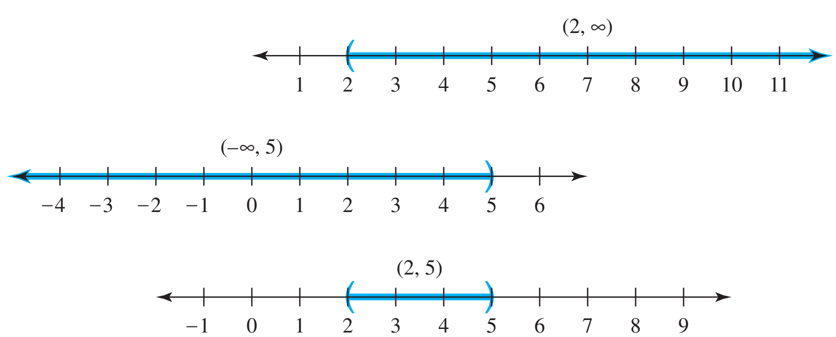


FIGURE 3.10

If A and B are sets of numbers, then the **union** of A and B is the set of all numbers that are in either A or B. The union of A and B is denoted as $A \cup B$ (read

"*A* union *B*"). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$ because all of these numbers are in *A* or *B*. Notice that the numbers in *A* and *B* are in $A \cup B$.

The solution set to a compound inequality using the connective "or" is the union of the solution sets to each of the simple inequalities.

EXAMPLE 5 **Graphing compound inequalities**

Graph the solution set to the compound inequality $x > 4$ or $x < -1$.

Solution

To find the union of the solution sets to the simple inequalities, we sketch their graphs as shown at the top of Fig. 3.11. We graph the union of these two sets by putting both shaded regions together on the same line as shown in the bottom graph in Fig. 3.11. This set is written in interval notation as $(-\infty, -1) \cup (4, \infty)$.

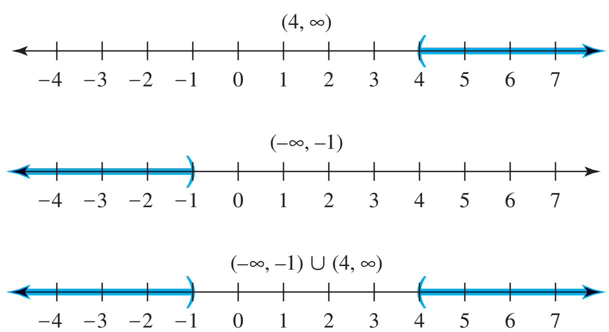


FIGURE 3.11

study tip

Make sure you know exactly how your grade in this course is determined. You should know how much weight is given to tests, quizzes, projects, homework, and the final exam. You should know the policy for making up work in case of illness. Record all scores and compute your final grade.

CAUTION When graphing the intersection of two simple inequalities, do not draw too much. For the intersection, graph only numbers that satisfy *both* inequalities. Omit numbers that satisfy one but not the other inequality. Graphing a union is usually easier because we can simply draw both solution sets on the same number line.

It is not always necessary to graph the solution set to each simple inequality before graphing the solution set to the compound inequality. We can save time and work if we learn to think of the two preliminary graphs but draw only the final one.

EXAMPLE 6 **Overlapping intervals**

Sketch the graph and write the solution set in interval notation to each compound inequality.

a) $x < 3$ and $x < 5$ **b)** $x > 4$ or $x > 0$

Solution

a) To graph $x < 3$ and $x < 5$, we shade only the numbers that are both less than 3 and less than 5. So numbers between 3 and 5 are not shaded in Fig. 3.12. The compound inequality $x < 3$ and $x < 5$ is equivalent to the simple inequality $x < 3$. The solution set can be written as $(-\infty, 3)$.

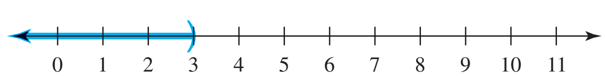


FIGURE 3.12

b) To graph $x > 4$ or $x > 0$, we shade both regions on the same number line as shown in Fig. 3.13. The compound inequality $x > 4$ or $x > 0$ is equivalent to the simple inequality $x > 0$. The solution set is $(0, \infty)$.

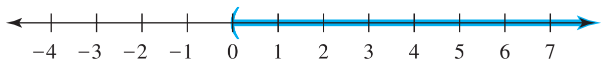


FIGURE 3.13

The next example shows a compound inequality that has no solution and one that is satisfied by every real number.

EXAMPLE 7

All or nothing

Sketch the graph and write the solution set in interval notation to each compound inequality.

a) $x < 2$ and $x > 6$ **b)** $x < 3$ or $x > 1$

Solution

a) A number satisfies $x < 2$ and $x > 6$ if it is both less than 2 *and* greater than 6. There are no such numbers. The solution set is the empty set, $\emptyset$.

b) To graph $x < 3$ or $x > 1$, we shade both regions on the same number line as shown in Fig. 3.14. Since the two regions cover the entire line, the solution set is the set of all real numbers $(-\infty, \infty)$.

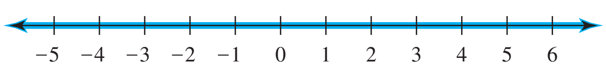


FIGURE 3.14

If we start with a more complicated compound inequality, we first simplify each part of the compound inequality and then find the union or intersection.

EXAMPLE 8

Intersection

Solve $x + 2 > 3$ and $x - 6 < 7$. Graph the solution set.

Solution

First simplify each simple inequality:

$$x + 2 - 2 > 3 - 2 \quad \text{and} \quad x - 6 + 6 < 7 + 6$$

$$x > 1 \quad \text{and} \quad x < 13$$

The intersection of these two solution sets is the set of numbers between (but not including) 1 and 13. Its graph is shown in Fig. 3.15. The solution set is written in interval notation as $(1, 13)$.

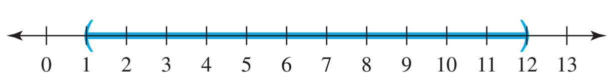


FIGURE 3.15

calculator close-up

To check Example 8, press Y= and let $y_1 = x + 2$ and $y_2 = x - 6$. Now scroll through a table of values for y_1 and y_2. From the table you can see that y_1 is greater than 3 and y_2 is less than 7 precisely when x is between 1 and 13.

X	Y1	Y2
1	3	-5
3	5	-3
5	7	-1
7	9	1
9	11	3
11	13	5
13	15	7

$Y_1 = X+2$

EXAMPLE 9

Union

Graph the solution set to the inequality

$$5 - 7x \geq 12 \qquad \text{or} \qquad 3x - 2 < 7.$$

Solution

First solve each of the simple inequalities:

$$5 - 7x - 5 \geq 12 - 5 \qquad \text{or} \qquad 3x - 2 + 2 < 7 + 2$$

$$-7x \geq 7 \qquad \text{or} \qquad 3x < 9$$

$$x \leq -1 \qquad \text{or} \qquad x < 3$$

The union of the two solution intervals is $(-\infty, 3)$. The graph is shown in Fig. 3.16.

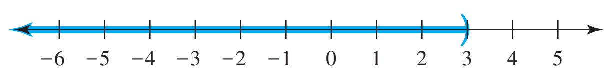


FIGURE 3.16

calculator close-up

To check Example 9, press Y= and let $y_1 = 5 - 7x$ and $y_2 = 3x - 2$. Now scroll through a table of values for y_1 and y_2. From the table you can see that either $y_1 \geq 12$ or $y_2 < 7$ is true for $x < 3$. Note also that for $x \geq 3$ both $y_1 \geq 12$ and $y_2 < 7$ are incorrect. The table supports the conclusion of Example 9.

X	Y1	Y2
1	-2	1
2	-9	4
3	-16	7
4	-23	10
5	-30	13
6	-37	16
7	-44	19

$Y_1 = 5-7X$

An inequality may be read from left to right or from right to left. Consider the inequality $1 < x$. If we read it in the usual way, we say, "1 is less than x." The meaning is clearer if we read the variable first. Reading from right to left, we say, "x is greater than 1."

Another notation is commonly used for the compound inequality

$$x > 1 \qquad \text{and} \qquad x < 13.$$

This compound inequality can also be written as

$$1 < x < 13.$$

Reading from left to right, we read $1 < x < 13$ as "1 is less than x is less than 13." The meaning of this inequality is clearer if we read the variable first and read the first inequality symbol from right to left. Reading the variable first, $1 < x < 13$ is read as "x is greater than 1 and less than 13." So x is between 1 and 13, and reading x first makes it clear.

CAUTION We write $a < x < b$ only if $a < b$, and we write $a > x > b$ only if $a > b$. Similar rules hold for $\leq$ and $\geq$. So $4 < x < 9$ and $-6 \geq x \geq -8$ are correct uses of this notation, but $5 < x < 2$ is not correct. Also, the inequalities should *not* point in opposite directions as in $5 < x > 7$.

EXAMPLE 10 **Another notation**

Solve the inequality and graph the solution set:

$$-2 \le 2x - 3 < 7$$

Solution

This inequality could be written as the compound inequality

$$2x - 3 \ge -2 \qquad \text{and} \qquad 2x - 3 < 7.$$

However, there is no need to rewrite the inequality because we can solve it in its original form.

$$-2 + 3 \le 2x - 3 + 3 < 7 + 3 \qquad \text{Add 3 to each part.}$$

$$1 \le 2x \le 10$$

$$\frac{1}{2} \le \frac{2x}{2} < \frac{10}{2} \qquad \text{Divide each part by 2.}$$

$$\frac{1}{2} \le x < 5$$

The solution set is $\left[\frac{1}{2}, 5\right)$, and its graph is shown in Fig. 3.17.

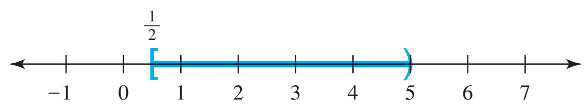


FIGURE 3.17

EXAMPLE 11 **Solving a compound inequality**

Solve the inequality $-1 < 3 - 2x < 9$ and graph the solution set.

Solution

$$-1 - 3 < 3 - 2x - 3 < 9 - 3 \qquad \text{Subtract 3 from each part of the inequality.}$$

$$-4 < -2x < 6$$

$$2 > x > -3 \qquad \text{Divide each part by } -2 \text{ and reverse both inequality symbols.}$$

$$-3 < x < 2 \qquad \text{Rewrite the inequality with the smallest number on the left.}$$

The solution set is $(-3, 2)$, and its graph is shown in Fig. 3.18.

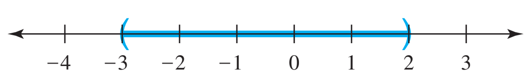
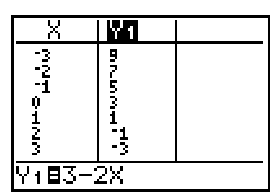


FIGURE 3.18

calculator close-up

Let $y_1 = 3 - 2x$ and make a table. Scroll through the table to see that y_1 is between -1 and 9 when x is between -3 and 2. The table supports the conclusion of Example 11.



Applications

When final exams are approaching, students are often interested in finding the final exam score that would give them a certain grade for a course.

EXAMPLE 12 **Final exam scores**

Fiana made a score of 76 on her midterm exam. For her to get a B in the course, the average of her midterm exam and final exam must be between 80 and 89 inclusive. What possible scores on the final exam would give Fiana a B in the course?

Solution

Let x represent her final exam score. Between 80 and 89 inclusive means that an average between 80 and 89 as well as an average of exactly 80 or 89 will get a B. So the average of the two scores must be greater than or equal to 80 and less than or equal to 89.

$$80 \le \frac{x + 76}{2} \le 89$$

$$160 \le x + 76 \le 178 \quad \text{Multiply by 2.}$$

$$160 - 76 \le x \le 178 - 76 \quad \text{Subtract 76.}$$

$$84 \le x \le 102$$

If Fiana scores between 84 and 102 inclusive, she will get a B in the course.

helpful hint

When you use two inequality symbols as in Example 12, they must both point in the same direction. In fact, we usually have them both point to the left so that the numbers increase in size from left to right.

WARM-UPS

True or false? Explain your answer.

1. $3 < 5$ and $3 \le 10$
2. $3 < 5$ or $3 < 10$
3. $3 > 5$ and $3 < 10$
4. $3 \ge 5$ or $3 \le 10$
5. $4 < 8$ and $4 > 2$
6. $4 < 8$ or $4 > 2$
7. $-3 < 0 < -2$
8. $(3, \infty) \cap (8, \infty) = (8, \infty)$
9. $(3, \infty) \cup [8, \infty) = [8, \infty)$
10. $(-2, \infty) \cap (-\infty, 9) = (-2, 9)$

3.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a compound inequality?
2. When is a compound inequality using "and" true?
3. When is a compound inequality using "or" true?
4. How do we solve compound inequalities?
5. What is the meaning of $a < b < c$?
6. What is the meaning of $5 < x > 7$?

Determine whether each compound inequality is true. See Examples 1 and 2.

7. $-6 < 5$ and $-6 > -3$

8. $3 < 5$ or $0 < -3$

9. $4 \leq 4$ and $-4 \leq 0$

10. $1 < 5$ and $1 > -3$

11. $6 < 5$ or $-4 > -3$

12. $4 \leq -4$ or $0 \leq 0$

Determine whether −4 satisfies each compound inequality. See Example 3.

13. $x < 5$ and $x > -3$

14. $x < 5$ or $x > -3$

15. $x - 3 \geq -7$ or $x + 1 > 1$

16. $2x \leq -8$ and $5x \leq 0$

17. $2x - 1 < -7$ or $-2x > 18$

18. $-3x > 0$ and $3x - 4 < 11$

Graph the solution set to each compound inequality. See Examples 4–7.

19. $x > -1$ and $x < 4$

20. $x \leq 3$ and $x \leq 0$

21. $x \geq 2$ or $x \geq 5$

22. $x < -1$ or $x < 3$

23. $x \leq 6$ or $x > -2$

24. $x > -2$ and $x \leq 4$

25. $x \leq 6$ and $x > 9$

26. $x < 7$ or $x > 0$

27. $x \leq 6$ or $x > 9$

28. $x \geq 4$ and $x \leq -4$

29. $x \geq 6$ and $x \leq 1$

30. $x > 3$ or $x < -3$

Solve each compound inequality. Write the solution set using interval notation and graph it. See Examples 8 and 9.

31. $x - 3 > 7$ or $3 - x > 2$

32. $x - 5 > 6$ or $2 - x > 4$

33. $3 < x$ and $1 + x > 10$

34. $-0.3x < 9$ and $0.2x > 2$

35. $\frac{1}{2}x > 5$ or $-\frac{1}{3}x < 2$

36. $5 < x$ or $3 - \frac{1}{2}x < 7$

37. $2x - 3 \leq 5$ and $x - 1 > 0$

38. $\frac{3}{4}x < 9$ and $-\frac{1}{3}x \leq -15$

39. $\frac{1}{2}x - \frac{1}{3} \geq -\frac{1}{6}$ or $\frac{2}{7}x \leq \frac{1}{10}$

40. $\frac{1}{4}x - \frac{1}{3} > -\frac{1}{5}$ and $\frac{1}{2}x < 2$

41. $0.5x < 2$ and $-0.6x < -3$

42. $0.3x < 0.6$ or $0.05x > -4$

Solve each compound inequality. Write the solution set in interval notation and graph it. See Examples 10 and 11.

43. $5 < 2x - 3 < 11$

44. $-2 < 3x + 1 < 10$

45. $-1 < 5 - 3x \leq 14$

46. $-1 \leq 3 - 2x < 11$

47. $-3 < \frac{3m + 1}{2} \le 5$

48. $0 \le \frac{3 - 2x}{2} < 5$

49. $-2 < \frac{1 - 3x}{-2} < 7$

50. $-3 < \frac{2x - 1}{3} < 7$

51. $3 \le 3 - 5(x - 3) \le 8$

52. $2 \le 4 - \frac{1}{2}(x - 8) \le 10$

Write each union or intersection of intervals as a single interval if possible.

53. $(2, \infty) \cup (4, \infty)$

54. $(-3, \infty) \cup (-6, \infty)$

55. $(-\infty, 5) \cap (-\infty, 9)$

56. $(-\infty, -2) \cap (-\infty, 1)$

57. $(-\infty, 4] \cap [2, \infty)$

58. $(-\infty, 8) \cap [3, \infty)$

59. $(-\infty, 5) \cup [-3, \infty)$

60. $(-\infty, -2] \cup (2, \infty)$

61. $(3, \infty) \cap (-\infty, 3]$

62. $[-4, \infty) \cap (-\infty, -6]$

63. $(3, 5) \cap [4, 8)$

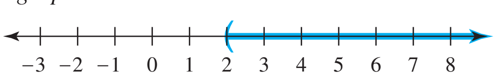
64. $[-2, 4] \cap (0, 9]$

65. $[1, 4) \cup (2, 6]$

66. $[1, 3) \cup (0, 5)$

Write either a simple or a compound inequality that has the given graph as its solution set.

67.



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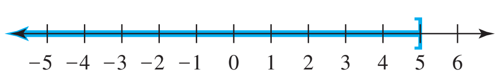

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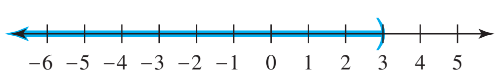

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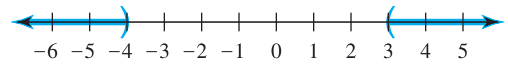

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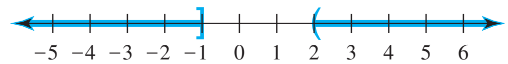

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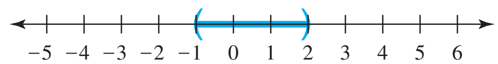

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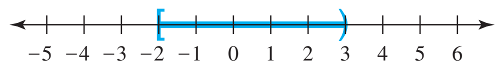

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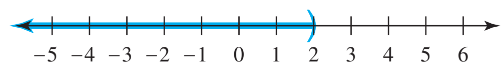

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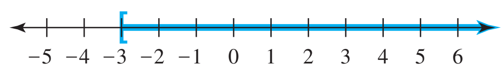

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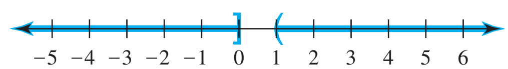


Solve each compound inequality and write the solution set using interval notation.

77. $2 < x < 7$ and $2x > 10$

78. $3 < 5 - x < 8$ or $-3x < 0$

79. $-1 < 3x + 2 \le 5$ or $\frac{3}{2}x - 6 > 9$

80. $0 < 5 - 2x \le 10$ and $-6 < 4 - x < 0$

81. $-3 < \frac{x - 1}{2} < 5$ and $-1 < \frac{1 - x}{2} < 2$

82. $-3 < \frac{3x - 1}{5} < \frac{1}{2}$ and $\frac{1}{3} < \frac{3 - 2x}{6} < \frac{9}{2}$

Solve each problem by using a compound inequality. See Example 12.

83. ***Aiming for a C.*** Professor Johnson gives only a midterm exam and a final exam. The semester average is computed by taking $\frac{1}{3}$ of the midterm exam score plus $\frac{2}{3}$ of the final exam score. To get a C, Beth must have a semester average between 70 and 79 inclusive. If Beth scored only 64 on the midterm, then for what range of scores on the final exam would Beth get a C?

84. ***Two tests only.*** Professor Davis counts his midterm as $\frac{2}{3}$ of the grade, and his final as $\frac{1}{3}$ of the grade. Jason scored only 64 on the midterm. What range of scores on the final exam would put Jason's average between 70 and 79 inclusive?

85. ***Keep on truckin'.*** Abdul is shopping for a new truck in a city with an 8% sales tax. There is also an \$84 title and license fee to pay. He wants to get a good truck and plans to spend at least \$12,000 but no more than \$15,000. What is the price range for the truck?

86. ***Selling-price range.*** Renee wants to sell her car through a broker who charges a commission of 10% of the selling price. The book value of the car is \$14,900, but Renee still owes \$13,104 on it. Although the car is in only fair condition and will not sell for more than the book value, Renee must get enough to at least pay off the loan. What is the range of the selling price?

87. ***Hazardous to her health.*** Trying to break her smoking habit, Jane calculates that she smokes only three full cigarettes a day, one after each meal. The rest of the time she smokes on the run and smokes only half of the cigarette. She estimates that she smokes the equivalent of 5 to 12 cigarettes per day. How many times a day does she light up on the run?

88. ***Possible width.*** The length of a rectangle is 20 meters longer than the width. The perimeter must be between 80 and 100 meters. What are the possible values for the width of the rectangle?

89. ***Higher education.*** The annual numbers of bachelor's and master's degrees awarded can be approximated using the formulas

$$B = 16.45n + 980.20$$

and $$M = 7.79n + 287.87,$$

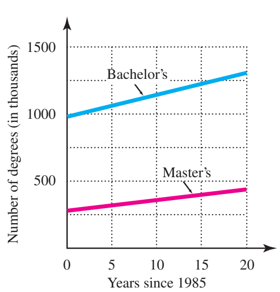
where n is the number of years after 1985 (National Center for Education Statistics, www.nces.ed.gov). For example, $n = 2$ gives the numbers in 1987.

a) How many bachelor's degrees were awarded in 1995?

b) In what year will the number of bachelor's degrees that are awarded reach 1.26 million?

c) What is the first year in which both B is greater than 1.3 million and M is greater than 0.5 million?

d) What is the first year in which either B is greater than 1.3 million or M is greater than 0.5 million?



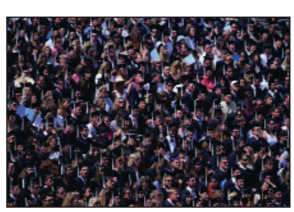


FIGURE FOR EXERCISE 89

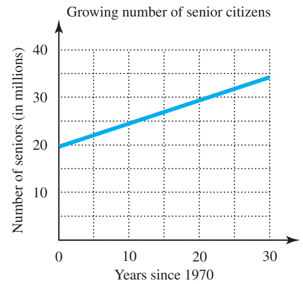


FIGURE FOR EXERCISE 90

90. ***Senior citizens.*** The number of senior citizens (65 years old and over) in the United States in millions in the year $1970 + n$ can be estimated by using the formula

$$S = 0.48n + 19.71$$

(U.S. Bureau of the Census, www.census.gov). The percentage of senior citizens living below the poverty level in the year $1970 + n$ can be estimated by using the formula

$$p = -0.72n + 24.2.$$

The variable n is the number of years after 1970.

a) How many senior citizens were there in 1998?

b) In what year did the percentage of seniors living below the poverty level reach 2.6%?

c) What is the first year in which we can expect both the number of seniors to be greater than 36 million and fewer than 2.6% living below the poverty level?

GETTING MORE INVOLVED

91. ***Discussion.*** If $-x$ is between a and b, then what can you say about x?

92. ***Discussion.*** For which of the inequalities is the notation used correctly?

a) $-2 \le x < 3$

b) $-4 \ge x < 7$

c) $-1 \le x > 0$

d) $6 < x \le -8$

e) $5 \ge x \ge -9$

93. ***Discussion.*** In each case, write the resulting set of numbers in interval notation. Explain your answers.

a) Every number in $(3, 8)$ is multiplied by 4.

b) Every number in $[-2, 4)$ is multiplied by -5.

c) Three is added to every number in $(-3, 6)$.

d) Every number in $[3, 9]$ is divided by -3.

94. ***Discussion.*** Write the solution set using interval notation for each of the following inequalities in terms of s and t. State any restrictions on s and t. For what values of s and t is the solution set empty?

a) $x > s$ and $x < t$

b) $x > s$ and $x > t$