93. $200x - 300y = 6$

95. $y = 300x - 1$

94. $300x + 5y = 20$

96. $y = 300x - 6000$

4.2 SLOPE

In this section

- Slope Concepts
- Slope Using Coordinates
- Graphing a Line Given a Point and Its Slope
- Parallel Lines
- Perpendicular Lines
- Interpreting Slope

In Section 4.1 you learned that the graph of a linear equation is a straight line. In this section, we will continue our study of lines in the coordinate plane.

Slope Concepts

If a highway rises 6 feet in a horizontal run of 100 feet, then the grade is $\frac{6}{100}$ or 6%. See Fig. 4.11. The grade of a road is a measurement of the steepness of the road. It is the rate at which the road is going upward.

FIGURE 4.11

The steepness of a line is called the **slope** of the line and it is measured like the grade of a road. As you move from (1, 1) to (4, 3) in Fig. 4.12 on page 171 the x-coordinate increases by 3 and the y-coordinate increases by 2. The line rises 2 units in a horizontal run of 3 units. So the slope of the line is $\frac{2}{3}$. The slope is the rate at which the y-coordinate is increasing. It increases 2 units for every 3-unit increase in x or it increases $\frac{2}{3}$ of a unit for every 1-unit increase in x. In general, we have the following definition of slope.

Slope

$$\text{Slope} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

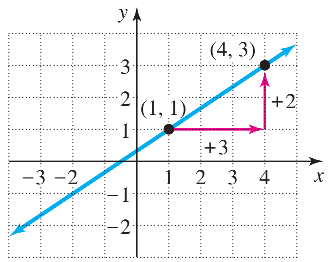


FIGURE 4.12

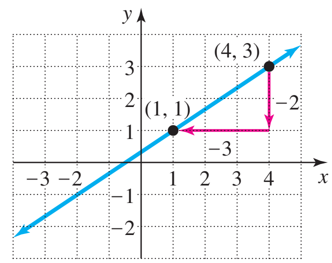


FIGURE 4.13

If we move from the point (4, 3) to the point (1, 1), there is a change of -2 in the y-coordinate and a change of -3 in the x-coordinate. See Fig. 4.13. In this case we get

$$\text{Slope} = \frac{-2}{-3} = \frac{2}{3}.$$

Note that going from (4, 3) to (1, 1) gives the same slope as going from (1, 1) to (4, 3).

We call the change in y-coordinate the **rise** and the change in x-coordinate the **run.** Moving up is a positive rise, and moving down is a negative rise. Moving to the right is a positive run, and moving to the left is a negative run. We usually use the letter m to stand for slope. So we have

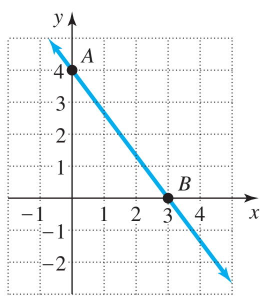
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}.$$

EXAMPLE 1

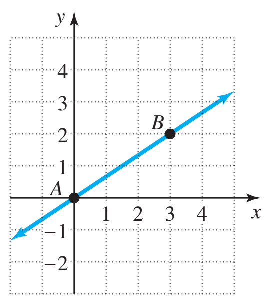
Finding the slope of a line

Find the slopes of the given lines by going from point A to point B.

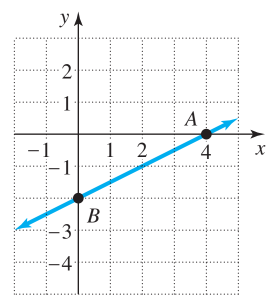
a)



b)



c)



Solution

a) The coordinates of point A are (0, 4), and the coordinates of point B are (3, 0). Going from A to B, the change in y is -4, and the change in x is $+3$. So

$$m = \frac{-4}{3} = -\frac{4}{3}.$$

b) Going from A to B, the rise is 2, and the run is 3. So

$$m = \frac{2}{3}.$$

c) Going from A to B, the rise is -2, and the run is -4. So

$$m = \frac{-2}{-4} = \frac{1}{2}.$$

study tip

Working problems 1 hour per day every day of the week is better than working problems for 7 hours on one day of the week. It is usually better to spread out your study time than to try and learn everything in one big session.

CAUTION The change in y is always in the numerator, and the change in x is always in the denominator.

The ratio of rise to run is the ratio of the lengths of the two legs of any right triangle whose hypotenuse is on the line. As long as one leg is vertical and the other is horizontal, all such triangles for a certain line have the same shape. These triangles are similar triangles. The ratio of the length of the vertical side to the length of the horizontal side for any two such triangles is the same number. So we get the same value for the slope no matter which two points of the line are used to calculate it or in which order the points are used.

EXAMPLE 2

helpful hint

It is good to think of what the slope represents when x and y are measured quantities rather than just numbers. For example, if the change in y is 50 miles and the change in x is 2 hours, then the slope is 25 mph (or 25 miles per 1 hour). So the slope is the amount of change in y for a unit change in x.

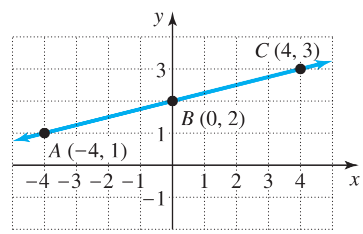
Finding slope

Find the slope of the line shown here using points A and B, points A and C, and points B and C.

Solution

Using A and B, we get

$$m = \frac{\text{rise}}{\text{run}} = \frac{1}{4}.$$



Using A and C, we get

$$m = \frac{\text{rise}}{\text{run}} = \frac{2}{8} = \frac{1}{4}.$$

Using B and C, we get

$$m = \frac{\text{rise}}{\text{run}} = \frac{1}{4}.$$

Slope Using Coordinates

One way to obtain the rise and run is from a graph. The rise and run can also be found by using the coordinates of two points on the line as shown in Fig. 4.14.

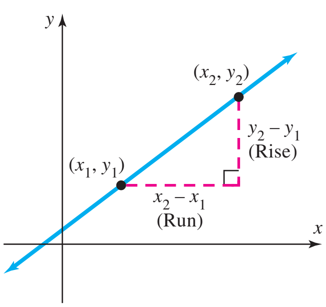


FIGURE 4.14

Coordinate Formula for Slope

The slope of the line containing the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

provided that $x_2 - x_1 \neq 0$.

EXAMPLE 3

Using coordinates to find slope

Find the slope of each of the following lines.

a) The line through (0, 5) and (6, 3)

b) The line through (−3, 4) and (−5, −2)

c) The line through (−4, 2) and the origin

Solution

a) If $(x_1, y_1) = (0, 5)$ and $(x_2, y_2) = (6, 3)$ then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{6 - 0} = \frac{-2}{6} = -\frac{1}{3}.$$

study tip

Students who have difficulty with algebra often schedule it in a class that meets one day per week so they do not have to see it as often. However, many students do better in classes that meet more often for shorter time periods. So schedule your classes to maximize your chances of success.

If $(x_1, y_1) = (6, 3)$ and $(x_2, y_2) = (0, 5)$ then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{0 - 6} = \frac{2}{-6} = -\frac{1}{3}.$$

Note that it does not matter which point is called (x_1, y_1) and which is called (x_2, y_2). In either case the slope is $-\frac{1}{3}$.

b) Let $(x_1, y_1) = (-3, 4)$ and $(x_2, y_2) = (-5, -2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{-5 - (-3)}$$

$$= \frac{-6}{-2} = 3$$

c) Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (-4, 2)$:

$$m = \frac{2 - 0}{-4 - 0} = \frac{2}{-4} = -\frac{1}{2}$$

CAUTION It does not matter which point is called (x_1, y_1) and which is called (x_2, y_2), but if you divide $y_2 - y_1$ by $x_1 - x_2$, the slope will have the wrong sign.

Note that slope is not defined if $x_2 - x_1 = 0$. So slope is not defined if the x-coordinates of the two points are equal. The x-coordinates for two points are equal only for points on a vertical line. So *slope is undefined for vertical lines.*

If $y_2 - y_1 = 0$, then the points have equal y-coordinates and lie on a horizontal line. *The slope for any horizontal line is zero.*

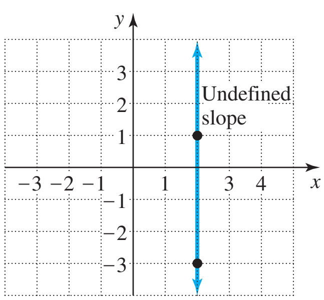
EXAMPLE 4

Slope for vertical and horizontal lines

Find the slope of the line through each pair of points.

a) (2, 1) and (2, −3)

b) (−2, 2) and (4, 2)



Vertical line

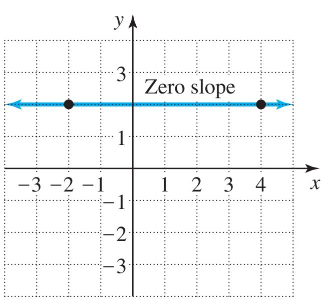
FIGURE 4.15

Solution

a) The points (2, 1) and (2, −3) are on the vertical line shown in Fig. 4.15. Since slope is undefined for vertical lines, this line does not have a slope. Using the slope formula we get

$$m = \frac{-3 - 1}{2 - 2} = \frac{-4}{0}.$$

Since division by zero is undefined, we can again conclude that slope is undefined for the vertical line through the given points.



Horizontal line

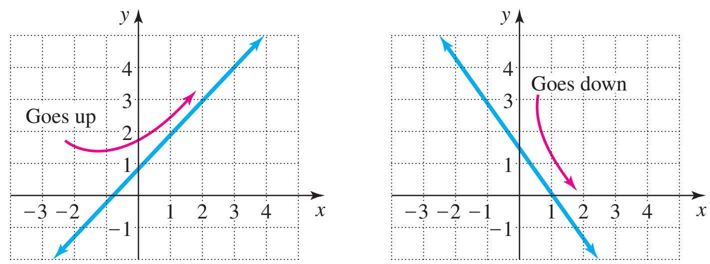
FIGURE 4.16

b) The points (−2, 2) and (4, 2) are on the horizontal line shown in Fig. 4.16. Using the slope formula we get

$$m = \frac{2 - 2}{-2 - 4} = \frac{0}{-6} = 0.$$

So the slope of the horizontal line through these points is 0.

Note that for a line with *positive slope,* the y-values increase as the x-values increase. For a line with *negative slope,* the y-values decrease as the x-values increase. See Fig. 4.17 on page 174.



Positive slope Negative slope

FIGURE 4.17

Graphing a Line Given a Point and Its Slope

We can find the slope of a line by examining its graph. We can also draw the graph of a line if we know its slope and a point on the line.

EXAMPLE 5

Graphing a line given a point and its slope

Graph each line.

a) The line through (2, 1) with slope $\frac{3}{4}$

b) The line through $(-2, 4)$ with slope -3

Solution

a) First locate the point (2, 1). Because the slope is $\frac{3}{4}$, we can find another point on the line by going up three units and to the right four units to get the point (6, 4), as shown in Fig. 4.18. Now draw the line through (2, 1) and (6, 4).

calculator close-up

When we graph a line we usually draw a graph that shows both intercepts, because they are important features of the graph. If the intercepts are not between -10 and 10, you will have to adjust the window to get a good graph. The viewing window that has x- and y-values ranging from a minimum of -10 to a maximum of 10 is called the **standard viewing window.**

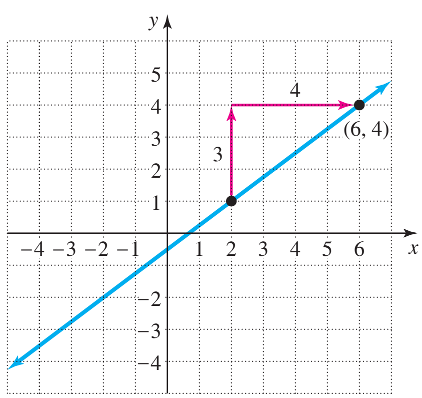


FIGURE 4.18

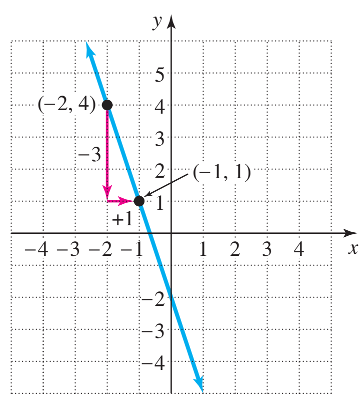


FIGURE 4.19

b) First locate the point $(-2, 4)$. Because the slope is -3, or $\frac{-3}{1}$, we can locate another point on the line by starting at $(-2, 4)$ and moving down three units and then one unit to the right to get the point $(-1, 1)$. Now draw a line through $(-2, 4)$ and $(-1, 1)$, as shown in Fig. 4.19.

Parallel Lines

Every nonvertical line has a unique slope, but there are infinitely many lines with a given slope. All lines that have a given slope are parallel.

Parallel Lines

Nonvertical lines are parallel if and only if they have equal slopes. Any two vertical lines are parallel to each other.

EXAMPLE 6

Graphing parallel lines

Draw a line through the point $(-2, 1)$ with slope $\frac{1}{2}$ and a line through $(3, 0)$ with slope $\frac{1}{2}$.

Solution

Because slope is the ratio of rise to run, a slope of $\frac{1}{2}$ means that we can locate a second point of the line by starting at $(-2, 1)$ and going up one unit and to the right two units. For the line through $(3, 0)$ we start at $(3, 0)$ and go up one unit and to the right two units. See Fig. 4.20.

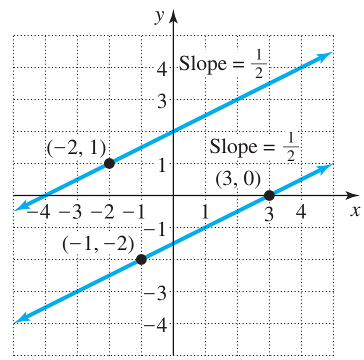


FIGURE 4.20

Perpendicular Lines

Slope can also be used to determine whether lines are perpendicular. If the slope of one line is the opposite of the reciprocal of the slope of another line, then the lines are perpendicular. For example, lines with slopes $\frac{3}{4}$ and $-\frac{4}{3}$ are perpendicular.

helpful hint

The relationship between the slopes of perpendicular lines can also be remembered as

$$m_1 \cdot m_2 = -1.$$

For example, lines with slopes -3 and $\frac{1}{3}$ are perpendicular because $-3 \cdot \frac{1}{3} = -1$.

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if

$$m_1 = -\frac{1}{m_2}.$$

Any vertical line is perpendicular to any horizontal line.

EXAMPLE 7

Graphing perpendicular lines

Draw two lines through the point $(-1, 2)$, one with slope $-\frac{1}{3}$ and the other with slope 3.

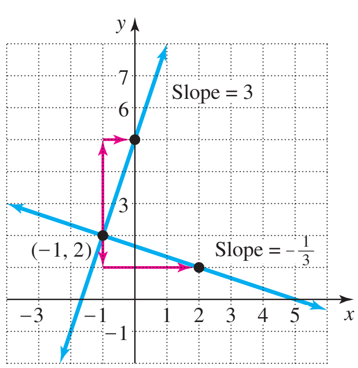


FIGURE 4.21

Solution

Because slope is the ratio of rise to run, a slope of $-\frac{1}{3}$ means that we can locate a second point on the line by starting at $(-1, 2)$ and going down one unit and to the right three units. For the line with slope 3, we start at $(-1, 2)$ and go up three units and to the right one unit. See Fig. 4.21.

Interpreting Slope

Slope of a line is the ratio of the rise and the run. If the rise is measured in dollars and the run in days, then the slope is measured in dollars per day or dollars/day.

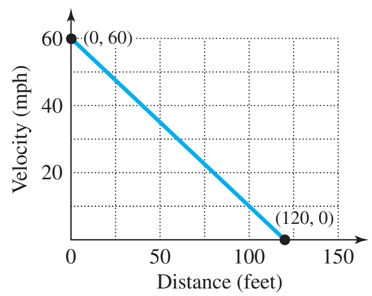
The slope of a line is the rate at which the dependent variable is increasing or decreasing.

EXAMPLE 8 **Interpreting slope**

A car goes from 60 mph to 0 mph in 120 feet after applying the brakes.

a) Find and interpret the slope of the line shown here.

b) What is the velocity at a distance of 80 feet?



Solution

a) Find the slope of the line through (0, 60) and (120, 0):

$$m = \frac{60 - 0}{0 - 120} = -0.5$$

Because the vertical axis is miles per hour and the horizontal axis is feet, the slope is -0.5 mph/ft, which means the car is losing 0.5 mph of velocity for every foot it travels after the brakes are applied.

b) If the velocity is decreasing 0.5 mph for every foot the car travels, then in 80 feet the velocity goes down 0.5(80) or 40 mph. So the velocity at 80 feet is $60 - 40$ or 20 mph.

WARM-UPS

True or false? Explain your answer.

1. Slope is a measurement of the steepness of a line.
2. Slope is rise divided by run.
3. Every line in the coordinate plane has a slope.
4. The line through the point (1, 1) and the origin has slope 1.
5. Slope can never be negative.
6. A line with slope 2 is perpendicular to any line with slope -2.
7. The slope of the line through (0, 3) and (4, 0) is $\frac{3}{4}$.
8. Two different lines cannot have the same slope.
9. The line through (1, 3) and $(-5, 3)$ has zero slope.
10. Slope can have units such as feet per second.

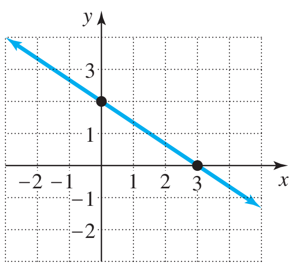
4.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the slope of a line?
2. What is the difference between rise and run?
3. For which lines is slope undefined?
4. Which lines have zero slope?
5. What is the difference between lines with positive slope and lines with negative slope?
6. What is the relationship between the slopes of perpendicular lines?

In Exercises 7–18, find the slope of each line. See Examples 1 and 2.

7.



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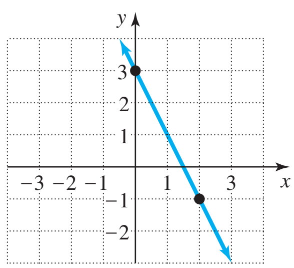

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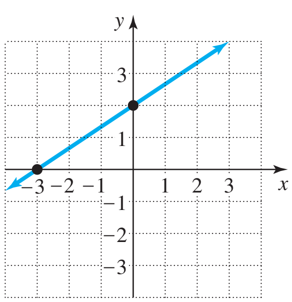

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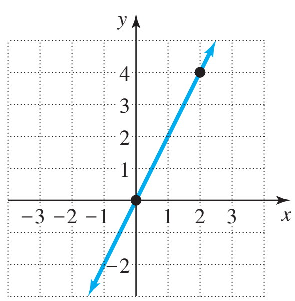

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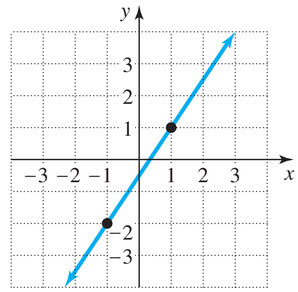

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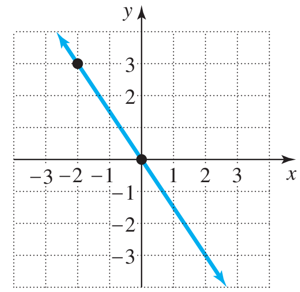

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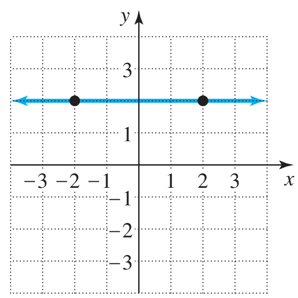

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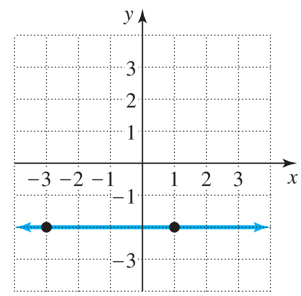

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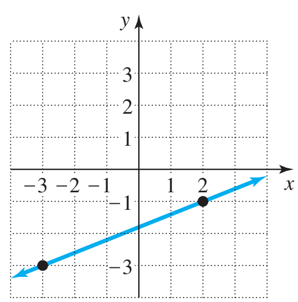

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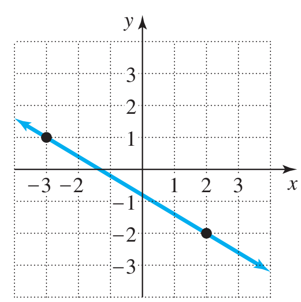

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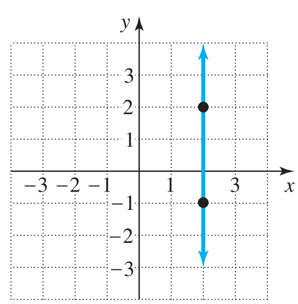

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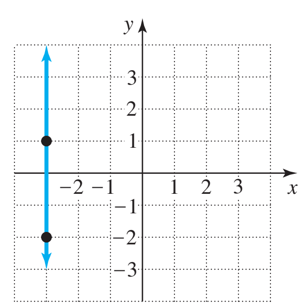

Find the slope of the line that goes through each pair of points. See Examples 3 and 4.

19. $(1, 2), (3, 6)$

20. $(2, 5), (6, 10)$

21. $(2, 4), (5, -1)$

22. $(3, 1), (6, -2)$

23. $(-2, 4), (5, 9)$

24. $(-1, 3), (3, 5)$

25. $(-2, -3), (-5, 1)$

26. $(-6, -3), (-1, 1)$

27. $(-3, 4), (3, -2)$

28. $(-1, 3), (5, -2)$

29. $\left(\frac{1}{2}, 2\right), \left(-1, \frac{1}{2}\right)$

30. $\left(\frac{1}{3}, 2\right), \left(-\frac{1}{3}, 1\right)$

31. $(2, 3), (2, -9)$

32. $(-3, 6), (8, 6)$

33. $(-2, -5), (9, -5)$

34. $(4, -9), (4, 6)$

35. $(0.3, 0.9), (-0.1, -0.3)$

36. $(-0.1, 0.2), (0.5, 0.8)$

Graph the line with the given point and slope. See Example 5.

37. The line through $(1, 1)$ with slope $\frac{2}{3}$

38. The line through $(2, 3)$ with slope $\frac{1}{2}$

39. The line through $(-2, 3)$ with slope -2

40. The line through $(-2, 5)$ with slope -1

41. The line through $(0, 0)$ with slope $-\frac{2}{5}$

42. The line through $(-1, 4)$ with slope $-\frac{2}{3}$

Solve each problem. See Examples 6 and 7.

43. Draw line l_1 through $(1, -2)$ with slope $\frac{1}{2}$ and line l_2 through $(-1, 1)$ with slope $\frac{1}{2}$.

44. Draw line l_1 through $(0, 3)$ with slope 1 and line l_2 through $(0, 0)$ with slope 1.

45. Draw l_1 through (1, 2) with slope $\frac{1}{2}$, and draw l_2 through (1, 2) with slope -2.

46. Draw l_1 through $(-2, 1)$ with slope $\frac{2}{3}$, and draw l_2 through $(-2, 1)$ with slope $-\frac{3}{2}$.

47. Draw any line l_1 with slope $\frac{3}{4}$. What is the slope of any line perpendicular to l_1? Draw any line l_2 perpendicular to l_1.

48. Draw any line l_1 with slope -1. What is the slope of any line perpendicular to l_1? Draw any line l_2 perpendicular to l_1.

49. Draw l_1 through $(-2, -3)$ and (4, 0). What is the slope of any line parallel to l_1? Draw l_2 through (1, 2) so that it is parallel to l_1.

50. Draw l_1 through $(-4, 0)$ and (0, 6). What is the slope of any line parallel to l_1? Draw l_2 through the origin and parallel to l_1.

51. Draw l_1 through $(-2, 4)$ and $(3, -1)$. What is the slope of any line perpendicular to l_1? Draw l_2 through (1, 3) so that it is perpendicular to l_1.

52. Draw l_1 through $(0, -3)$ and (3, 0). What is the slope of any line perpendicular to l_1? Draw l_2 through the origin so that it is perpendicular to l_1.

Solve each problem. See Example 8.

53. ***Super cost.*** The average cost of a 30-second ad during the 1995 super bowl was \$1 million, and in 1998 it was \$1.3 million (*Detroit Free Press,* January 6, 1998, www.freep.com).

a) Find the slope of the line through (95, 1,000,000) and (98, 1,300,000) and interpret your result.

b) Use the accompanying graph to estimate the average cost of an ad in 1997.

c) What do you think the average cost will be in 2005?

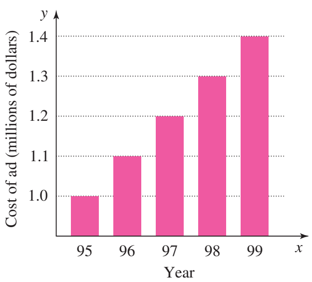

FIGURE FOR EXERCISE 53

54. ***Retirement pay.*** The annual Social Security benefit of a retiree depends on the age at the time of retirement. The accompanying graph gives the annual benefit for persons retiring at ages 62 through 70 in the year 2005 or later (Source: Social Security Administration). What is the annual benefit for a person who retires at age 64? At what retirement age does a person receive an annual benefit of \$11,600? Find the slope of each line segment on the graph, and interpret your results. Why do people who postpone retirement until 70 years of age get the highest benefit?

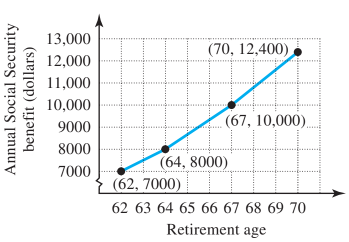

FIGURE FOR EXERCISE 54

55. ***Increasing training.*** The accompanying graph shows the percentage of U.S. workers receiving training by their employers. The percentage went from 5% in 1981 to 16% in 1995. Find the slope of this line. Interpret your result.

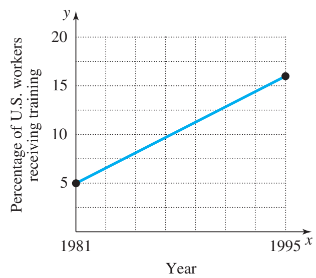

FIGURE FOR EXERCISE 55

56. ***Saving for retirement.*** Financial advisors at Fidelity Investments, Boston, use the accompanying table as a measure of whether a client is on the road to a comfortable retirement.

Age (a)	Years of Salary saved (y)
35	0.5
40	1.0
45	1.5
50	2.0

a) Graph these points and draw a line through them.

b) What is the slope of the line?

c) By what percentage of your salary should you be increasing your savings every year?