6.4 FACTORING $ax^2 + bx + c$ WITH $a \neq 1$

In this section

- The *ac* Method
- Trial and Error
- Factoring Completely

In Section 6.3 we factored trinomials with a leading coefficient of 1. In this section we will use a slightly different technique to factor trinomials with leading coefficients not equal to 1.

The *ac* Method

If the leading coefficient of a trinomial is not 1, we can again use grouping to factor the trinomial. However, the procedure is slightly different.

Consider the trinomial $2x^2 + 7x + 6$. First find the product of the leading coefficient and the constant term. In this case it is $2 \cdot 6$, or 12. Now find two numbers with a product of 12 and a sum of 7. The pairs of numbers with a product of 12 are 1 and 12, 2 and 6, and 3 and 4. Only 3 and 4 have a product of 12 and a sum of 7. Now replace $7x$ by $3x + 4x$ and factor by grouping:

$$\begin{aligned} 2x^2 + 7x + 6 &= 2x^2 + 3x + 4x + 6 && \text{Replace } 7x \text{ by } 3x + 4x. \\ &= (2x + 3)x + (2x + 3)2 && \text{Factor out the common factors.} \\ &= (2x + 3)(x + 2) && \text{Factor out } 2x + 3. \end{aligned}$$

The strategy for factoring a trinomial is summarized in the following box. The steps listed here actually work whether or not the leading coefficient is 1. This is actually the method that you learned in Section 6.3 with $a = 1$. This method is called the ***ac* method.**

Strategy for Factoring $ax^2 + bx + c$ by the *ac* Method

To factor the trinomial $ax^2 + bx + c$:

1. Find two numbers that have a product equal to ac and a sum equal to b.
2. Replace bx by two terms using the two new numbers as coefficients.
3. Factor the resulting four-term polynomial by grouping.

EXAMPLE 1 **The *ac* method**

Factor each trinomial.

a) $2x^2 + x - 6$ **b)** $10x^2 + 13x - 3$

Solution

a) Because $2 \cdot (-6) = -12$, we need two integers with a product of -12 and a sum of 1. We can list the possible pairs of integers with a product of -12:

1 and −12	2 and −6	3 and −4
−1 and 12	−2 and 6	−3 and 4

Only −3 and 4 have a sum of 1. Replace x by $-3x + 4x$ and factor by grouping:

$$\begin{aligned} 2x^2 + x - 6 &= 2x^2 - 3x + 4x - 6 && \text{Replace } x \text{ by } -3x + 4x. \\ &= (2x - 3)x + (2x - 3)2 && \text{Factor out the common factors.} \\ &= (2x - 3)(x + 2) && \text{Factor out } 2x - 3. \end{aligned}$$

Check by FOIL.

b) Because $10 \cdot (-3) = -30$, we need two integers with a product of -30 and a sum of 13. The product is negative, so the integers must have opposite signs. We can list all pairs of factors of -30 as follows:

1 and -30	2 and -15	3 and -10	5 and -6
-1 and 30	-2 and 15	-3 and 10	-5 and 6

The only pair that has a sum of 13 is -2 and 15:

$$\begin{aligned} 10x^2 + 13x - 3 &= 10x^2 - 2x + 15x - 3 && \text{Replace } 13x \text{ by } -2x + 15x. \\ &= (5x - 1)2x + (5x - 1)3 && \text{Factor out the common factors.} \\ &= (5x - 1)(2x + 3) && \text{Factor out } 5x - 1. \end{aligned}$$

Check by FOIL.

Trial and Error

After you have gained some experience at factoring by the *ac* method, you can often find the factors without going through the steps of grouping. For example, consider the polynomial

$$3x^2 + 7x - 6.$$

helpful hint

The *ac* method is more systematic than trial and error. However, trial and error can be faster and easier, especially if your first or second trial is correct.

The factors of $3x^2$ can only be $3x$ and x. The factors of 6 could be 2 and 3 or 1 and 6. We can list all of the possibilities that give the correct first and last terms, without regard to the signs:

$$(3x \quad 3)(x \quad 2) \qquad (3x \quad 2)(x \quad 3) \qquad (3x \quad 6)(x \quad 1) \qquad (3x \quad 1)(x \quad 6)$$

Because the factors of -6 have unlike signs, one binomial factor is a sum and the other binomial is a difference. Now we try some products to see if we get a middle term of $7x$:

$$\begin{aligned} (3x + 3)(x - 2) &= 3x^2 - 3x - 6 && \text{Incorrect.} \\ (3x - 3)(x + 2) &= 3x^2 + 3x - 6 && \text{Incorrect.} \end{aligned}$$

Actually, there is no need to try $(3x \quad 3)(x \quad 2)$ or $(3x \quad 6)(x \quad 1)$ because each contains a binomial with a common factor. As you can see from the above products, a common factor in the binomial causes a common factor in the product. But $3x^2 + 7x - 6$ has no common factor. So the factors must come from either $(3x \quad 2)(x \quad 3)$ or $(3x \quad 1)(x \quad 6)$. So we try again:

$$\begin{aligned} (3x + 2)(x - 3) &= 3x^2 - 7x - 6 && \text{Incorrect.} \\ (3x - 2)(x + 3) &= 3x^2 + 7x - 6 && \text{Correct.} \end{aligned}$$

Even though there may be many possibilities in some factoring problems, it is often possible to find the correct factors without writing down every possibility. We can use a bit of guesswork in factoring trinomials. *Try* whichever possibility you think might work. *Check* it by multiplying. If it is not right, then *try again.* That is why this method is called **trial and error.**

EXAMPLE 2 **Trial and error**

Factor each trinomial using trial and error.

a) $2x^2 + 5x - 3$ **b)** $3x^2 - 11x + 6$

study tip

Have you ever used excuses to avoid studying? ("Before I can study, I have to do my laundry and go to the bank.") Since the average attention span for one task is approximately 20 minutes, it is better to take breaks from studying to run errands and do laundry than to get everything done before you start studying.

Solution

a) Because $2x^2$ factors only as $2x \cdot x$ and 3 factors only as $1 \cdot 3$, there are only two possible ways to get the correct first and last terms, without regard to the signs:

$$(2x \quad 1)(x \quad 3) \quad \text{and} \quad (2x \quad 3)(x \quad 1)$$

Because the last term of the trinomial is negative, one of the missing signs must be $+$, and the other must be $-$. The trinomial is factored correctly as

$$2x^2 + 5x - 3 = (2x - 1)(x + 3).$$

Check by using FOIL.

b) There are four possible ways to factor $3x^2 - 11x + 6$:

$$(3x \quad 1)(x \quad 6) \qquad (3x \quad 2)(x \quad 3)$$
$$(3x \quad 6)(x \quad 1) \qquad (3x \quad 3)(x \quad 2)$$

Because the last term in $3x^2 - 11x + 6$ is positive and the middle term is negative, both signs in the factors must be negative. Because $3x^2 - 11x + 6$ has no common factor, we can rule out $(3x \quad 6)(x \quad 1)$ and $(3x \quad 3)(x \quad 2)$. So the only possibilities left are $(3x - 1)(x - 6)$ and $(3x - 2)(x - 3)$. The trinomial is factored correctly as

$$3x^2 - 11x + 6 = (3x - 2)(x - 3).$$

Check by using FOIL. ■

Factoring by trial and error is not just guessing. In fact, if the trinomial has a positive leading coefficient, we can determine in advance whether its factors are sums or differences.

Using Signs in Trial and Error

1. If the signs of the terms of a trinomial are $+ + +$, then both factors are sums: $x^2 + 5x + 6 = (x + 2)(x + 3)$.
2. If the signs are $+ - +$, then both factors are differences: $x^2 - 5x + 6 = (x - 2)(x - 3)$.
3. If the signs are $+ + -$ or $+ - -$, then one factor is a sum and the other is a difference: $x^2 + x - 6 = (x + 3)(x - 2)$ and $x^2 - x - 6 = (x - 3)(x + 2)$.

In the next example we factor a trinomial that has two variables.

EXAMPLE 3 **Factoring a trinomial with two variables by trial and error**

Factor $6x^2 - 7xy + 2y^2$.

Solution

We list the possible ways to factor the trinomial:

$$(3x \quad 2y)(2x \quad y) \qquad (3x \quad y)(2x \quad 2y) \qquad (6x \quad 2y)(x \quad y) \qquad (6x \quad y)(x \quad 2y)$$

Because the last term of the trinomial is positive and the middle term is negative, both factors must contain subtraction symbols. To get the middle term of $-7xy$, we use the first possibility listed:

$$6x^2 - 7xy + 2y^2 = (3x - 2y)(2x - y)$$ ■

Factoring Completely

You can use the latest factoring technique along with the techniques that you learned earlier to factor polynomials completely. Remember always to first factor out the greatest common factor (if it is not 1).

EXAMPLE 4 **Factoring completely**

Factor each polynomial completely.

a) $4x^3 + 14x^2 + 6x$ **b)** $12x^2y + 6xy + 6y$

Solution

a) $4x^3 + 14x^2 + 6x = 2x(2x^2 + 7x + 3)$ Factor out the GCF, $2x$.

$= 2x(2x + 1)(x + 3)$ Factor $2x^2 + 7x + 3$.

Check by multiplying.

b) $12x^2y + 6xy + 6y = 6y(2x^2 + x + 1)$ Factor out the GCF, $6y$.

To factor $2x^2 + x + 1$ by the *ac* method, we need two numbers with a product of 2 and a sum of 1. Because there are no such numbers, $2x^2 + x + 1$ is prime and the factorization is complete.

Usually, our first step in factoring is to factor out the greatest common factor (if it is not 1). If the first term of a polynomial has a negative coefficient, then it is better to factor out the opposite of the GCF so that the resulting polynomial will have a positive leading coefficient.

EXAMPLE 5 **Factoring out the opposite of the GCF**

Factor each polynomial completely.

a) $-18x^3 + 51x^2 - 15x$ **b)** $-3a^2 + 2a + 21$

Solution

a) The GCF is $3x$. Because the first term has a negative coefficient, we factor out $-3x$:

$$-18x^3 + 51x^2 - 15x = -3x(6x^2 - 17x + 5) \quad \text{Factor out } -3x.$$
$$= -3x(3x - 1)(2x - 5) \quad \text{Factor } 6x^2 - 17x + 5.$$

b) The GCF for $-3a^2 + 2a + 21$ is 1. Because the first term has a negative coefficient, factor out -1:

$$-3a^2 + 2a + 21 = -1(3a^2 - 2a - 21) \quad \text{Factor out } -1.$$
$$= -1(3a + 7)(a - 3) \quad \text{Factor } 3a^2 - 2a - 21.$$

WARM-UPS

True or false? Answer true if the correct factorization is given and false if the factorization is incorrect. Explain your answer.

1. $2x^2 + 3x + 1 = (2x + 1)(x + 1)$
2. $2x^2 + 5x + 3 = (2x + 1)(x + 3)$
3. $3x^2 + 10x + 3 = (3x + 1)(x + 3)$
4. $15x^2 + 31x + 14 = (3x + 7)(5x + 2)$
5. $2x^2 - 7x - 9 = (2x - 9)(x + 1)$
6. $2x^2 + 3x - 9 = (2x + 3)(x - 3)$
7. $2x^2 - 16x - 9 = (2x - 9)(2x + 1)$
8. $8x^2 - 22x - 5 = (4x - 1)(2x + 5)$
9. $9x^2 + x - 1 = (5x - 1)(4x + 1)$
10. $12x^2 - 13x + 3 = (3x - 1)(4x - 3)$

6.4 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What types of polynomials did we factor in this section?

2. What is the *ac* method of factoring?

3. How can you determine if $ax^2 + bx + c$ is prime?

4. What is the trial-and-error method of factoring?

Find the following. See Example 1.

5. Two integers that have a product of 20 and a sum of 12

6. Two integers that have a product of 36 and a sum of -20

7. Two integers that have a product of -12 and a sum of -4

8. Two integers that have a product of -8 and a sum of 7

Each of the following trinomials is in the form $ax^2 + bx + c$. For each trinomial, find two integers that have a product of ac and a sum of b. Do not factor the trinomials. See Example 1.

9. $6x^2 + 7x + 2$ **10.** $5x^2 + 17x + 6$

11. $6y^2 - 11y + 3$ **12.** $6z^2 - 19z + 10$

13. $12w^2 + w - 1$ **14.** $15t^2 - 17t - 4$

Factor each trinomial using the ac method. See Example 1.

15. $2x^2 + 3x + 1$ **16.** $2x^2 + 11x + 5$

17. $2x^2 + 9x + 4$ **18.** $2h^2 + 7h + 3$

19. $3t^2 + 7t + 2$ **20.** $3t^2 + 8t + 5$

21. $2x^2 + 5x - 3$ **22.** $3x^2 - x - 2$

23. $6x^2 + 7x - 3$ **24.** $21x^2 + 2x - 3$

25. $2x^2 - 7x + 6$ **26.** $3a^2 - 14a + 15$

27. $5b^2 - 13b + 6$ **28.** $7y^2 + 16y - 15$

29. $4y^2 - 11y - 3$ **30.** $35x^2 - 2x - 1$

31. $3x^2 + 2x + 1$ **32.** $6x^2 - 4x - 5$

33. $8x^2 - 2x - 1$ **34.** $8x^2 - 10x - 3$

35. $9t^2 - 9t + 2$ **36.** $9t^2 + 5t - 4$

37. $15x^2 + 13x + 2$ **38.** $15x^2 - 7x - 2$

39. $15x^2 - 13x + 2$ **40.** $15x^2 + x - 2$

Complete the factoring.

41. $3x^2 + 7x + 2 = (x + 2)(\quad)$

42. $2x^2 - x - 15 = (x - 3)(\quad)$

43. $5x^2 + 11x + 2 = (5x + 1)(\quad)$

44. $4x^2 - 19x - 5 = (4x + 1)(\quad)$

45. $6a^2 - 17a + 5 = (3a - 1)(\quad)$

46. $4b^2 - 16b + 15 = (2b - 5)(\quad)$

Factor each trinomial using trial and error. See Examples 2 and 3.

47. $5a^2 + 11a + 2$ **48.** $3y^2 + 10y + 7$

49. $4w^2 + 8w + 3$ **50.** $6z^2 + 13z + 5$

51. $15x^2 - x - 2$ **52.** $15x^2 + 13x - 2$

53. $8x^2 - 6x + 1$ **54.** $8x^2 - 22x + 5$

55. $15x^2 - 31x + 2$ **56.** $15x^2 + 31x + 2$

57. $2x^2 + 18x - 90$ **58.** $3x^2 + 11x + 10$

59. $3x^2 + x - 10$ **60.** $3x^2 - 17x + 10$

61. $10x^2 - 3xy - y^2$ **62.** $8x^2 - 2xy - y^2$

63. $42a^2 - 13ab + b^2$ **64.** $10a^2 - 27ab + 5b^2$

Factor each polynomial completely. See Examples 4 and 5.

65. $81w^3 - w$ **66.** $81w^3 - w^2$

67. $4w^2 + 2w - 30$ **68.** $2x^2 - 28x + 98$

69. $27 + 12x^2 + 36x$ **70.** $24y + 12y^2 + 12$

71. $6w^2 - 11w - 35$ **72.** $18x^2 - 6x + 6$

73. $3x^2z - 3zx - 18z$ **74.** $a^2b + 2ab - 15b$

75. $10x^2y^2 + xy^2 - 9y^2$
76. $2x^2y^2 + xy^2 + 3y^2$
77. $a^2 + 2ab - 15b^2$
78. $a^2b^2 - 2a^2b - 15a^2$
79. $-6t^3 - t^2 + 2t$
80. $-36t^2 - 6t + 12$
81. $12t^4 - 2t^3 - 4t^2$
82. $12t^3 + 14t^2 + 4t$
83. $4x^2y - 8xy^2 + 3y^3$
84. $9x^2 + 24xy - 9y^2$
85. $-4w^2 + 7w - 3$
86. $-30w^2 + w + 1$
87. $-12a^3 + 22a^2b - 6ab^2$
88. $-36a^2b + 21ab^2 - 3b^3$

Solve each problem.

89. ***Height of a ball.*** If a ball is thrown upward at 40 feet per second from a rooftop 24 feet above the ground, then its height above the ground t seconds after it is thrown is given by $h = -16t^2 + 40t + 24$. Rewrite this formula with the polynomial on the right-hand side factored completely. Use the factored version of the formula to find h when $t = 3$.

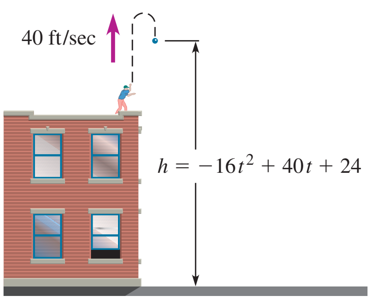

FIGURE FOR EXERCISE 89

90. ***Worker efficiency.*** In a study of worker efficiency at Wong Laboratories it was found that the number of components assembled per hour by the average worker t hours after starting work could be modeled by the function

$$N(t) = -3t^3 + 23t^2 + 8t.$$

a) Rewrite the formula by factoring the right-hand side completely.

b) Use the factored version of the formula to find $N(3)$.

c) Use the accompanying graph to estimate the time at which the workers are most efficient.

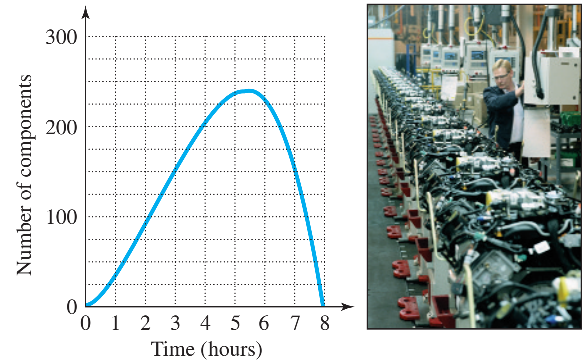

FIGURE FOR EXERCISE 90

d) Use the accompanying graph to estimate the maximum number of components assembled per hour during an 8-hour shift.

GETTING MORE INVOLVED

91. ***Exploration.*** Find all positive and negative integers b for which each polynomial can be factored.
a) $x^2 + bx + 3$
b) $3x^2 + bx + 5$
c) $2x^2 + bx - 15$

92. ***Exploration.*** Find two integers c (positive or negative) for which each polynomial can be factored. Many answers are possible.
a) $x^2 + x + c$
b) $x^2 - 2x + c$
c) $2x^2 - 3x + c$

93. ***Cooperative learning.*** Working in groups, cut two large squares, three rectangles, and one small square out of paper that are exactly the same size as shown in the accompanying figure. Then try to place the six figures next to one another so that they form a large rectangle. Do not overlap the pieces or leave any gaps. Explain how factoring $2x^2 + 3x + 1$ can help you solve this puzzle.

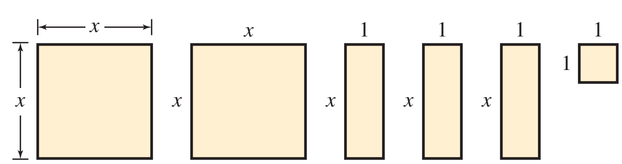

FIGURE FOR EXERCISE 93

94. ***Cooperative learning.*** Working in groups, cut four squares and eight rectangles out of paper as in the previous exercise to illustrate the trinomial $4x^2 + 7x + 3$. Select one group to demonstrate how to arrange the 12 pieces to form a large rectangle. Have another group explain how factoring the trinomial can help you solve this puzzle.