6.5 THE FACTORING STRATEGY

In this section

- Using Division in Factoring
- Factoring a Difference or Sum of Two Cubes
- The Factoring Strategy

In previous sections we established the general idea of factoring and some special cases. In this section we will see how division relates to factoring and see two more special cases. We will then summarize all of the factoring that we have done with a factoring strategy.

Using Division in Factoring

To find the prime factorization for a large integer such as 1001, you could divide possible factors (prime numbers) into 1001 until you find one that leaves no remainder. If you are told that 13 is a factor (or make a lucky guess), then you could divide 1001 by 13 to get the quotient 77. With this information you can factor 1001:

$$1001 = 77 \cdot 13$$

Now you can factor 77 to get the prime factorization of 1001:

$$1001 = 7 \cdot 11 \cdot 13$$

We can use this same idea with polynomials that are of higher degree than the ones we have been factoring. If we can guess a factor or if we are given a factor, we can use division to find the other factor and then proceed to factor the polynomial completely. Of course, it is harder to guess a factor of a polynomial than it is to guess a factor of an integer. In the next example we will factor a third-degree polynomial completely, given one factor.

EXAMPLE 1

Using division in factoring

Factor the polynomial $x^3 + 2x^2 - 5x - 6$ completely, given that the binomial $x + 1$ is a factor of the polynomial.

Solution

Divide the polynomial by the binomial:

$$\begin{array}{r l}
 & x^2 + x - 6 \\
x + 1 \overline{)} & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 + x^2} \\
 & x^2 - 5x \\
 & \underline{x^2 + x} \\
 & -6x - 6 \quad -5x - x = -6x \\
 & \underline{-6x - 6} \\
 & 0 \quad -6 - (-6) = 0
\end{array}$$

Because the remainder is 0, the dividend is the divisor times the quotient:

$$x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6)$$

Now we factor the remaining trinomial to get the complete factorization:

$$x^3 + 2x^2 - 5x - 6 = (x + 1)(x + 3)(x - 2)$$

study tip

Everyone has a different attention span. Start by studying 10–15 minutes at a time and then build up to longer periods over time. In your senior year, you should be able to concentrate on one task for 30–45 minutes without a break. Be realistic. When you can't remember what you have read and can no longer concentrate, take a break.

Factoring a Difference or Sum of Two Cubes

We can use division to discover that $a - b$ is a factor of $a^3 - b^3$ (a difference of two cubes) and $a + b$ is a factor of $a^3 + b^3$ (a sum of two cubes):

$$
\begin{array}{r}
a^2 + ab + b^2 \\
a - b\,\overline{)\,a^3 + 0a^2b + 0ab^2 - b^3} \\
\underline{a^3 - a^2b} \qquad\qquad\qquad \\
a^2b + 0ab^2 \qquad \\
\underline{a^2b - ab^2} \qquad \\
ab^2 - b^3 \\
\underline{ab^2 - b^3} \\
0
\end{array}
\qquad
\begin{array}{r}
a^2 - ab + b^2 \\
a + b\,\overline{)\,a^3 + 0a^2b + 0ab^2 + b^3} \\
\underline{a^3 + a^2b} \qquad\qquad\qquad \\
-a^2b + 0ab^2 \qquad \\
\underline{-a^2b - ab^2} \qquad \\
ab^2 + b^3 \\
\underline{ab^2 + b^3} \\
0
\end{array}
$$

study tip

Many schools have study skills centers that offer courses, workshops, and individual help on how to study. A search for "study skills" on the World Wide Web will turn up more information than you could possibly read. If you are not having the success in school that you would like, do something about it. What you do now will affect you the rest of your life.

So $a - b$ is a factor of $a^3 - b^3$, and $a + b$ is a factor of $a^3 + b^3$. These results give us two more factoring rules.

Factoring a Difference or Sum of Two Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Note that $a^2 + ab + b^2$ and $a^2 - ab + b^2$ are prime. Do not confuse them with $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, which are not prime because

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad \text{and} \qquad a^2 - 2ab + b^2 = (a - b)^2.$$

These similarities can help you remember the rules for factoring $a^3 - b^3$ and $a^3 + b^3$. Note also how $a^3 - b^3$ compares with $a^2 - b^2$:

$$a^2 - b^2 = (a - b)(a + b)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

EXAMPLE 2 **Factoring a difference or sum of two cubes**

Factor each polynomial.

a) $w^3 - 8$ **b)** $x^3 + 1$ **c)** $8y^3 - 27$

Solution

a) Because $8 = 2^3$, $w^3 - 8$ is a difference of two cubes. To factor $w^3 - 8$, let $a = w$ and $b = 2$ in the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$:

$$w^3 - 8 = (w - 2)(w^2 + 2w + 4)$$

b) Because $1 = 1^3$, the binomial $x^3 + 1$ is a sum of two cubes. Let $a = x$ and $b = 1$ in the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

c) $8y^3 - 27 = (2y)^3 - 3^3$ This is a difference of two cubes.

$= (2y - 3)(4y^2 + 6y + 9)$ Let $a = 2y$ and $b = 3$ in the formula. ■

CAUTION The polynomial $(a - b)^3$ is not equivalent to $a^3 - b^3$ because if $a = 2$ and $b = 1$, then

$$(a - b)^3 = (2 - 1)^3 = 1^3 = 1$$

and

$$a^3 - b^3 = 2^3 - 1^3 = 8 - 1 = 7.$$

Likewise, $(a + b)^3$ is not equivalent to $a^3 + b^3$.

The Factoring Strategy

The following is a summary of the ideas that we use to factor a polynomial completely.

Strategy for Factoring Polynomials Completely

1. If there are any common factors, factor them out first.
2. When factoring a binomial, check to see whether it is a difference of two squares, a difference of two cubes, or a sum of two cubes. *A sum of two squares does not factor.*
3. When factoring a trinomial, check to see whether it is a perfect square trinomial.
4. When factoring a trinomial that is not a perfect square, use the *ac* method or the trial-and-error method.
5. If the polynomial has four terms, try factoring by grouping.
6. Check to see whether any of the factors can be factored again.

We will use the factoring strategy in the next two examples.

EXAMPLE 3 **Factoring polynomials**

Factor each polynomial completely.

a) $2a^2b - 24ab + 72b$

b) $3x^3 + 6x^2 - 75x - 150$

Solution

a) $2a^2b - 24ab + 72b = 2b(a^2 - 12a + 36)$ First factor out the GCF, $2b$.

$= 2b(a - 6)^2$ Factor the perfect square trinomial.

b) $3x^3 + 6x^2 - 75x - 150 = 3[x^3 + 2x^2 - 25x - 50]$ Factor out the GCF, 3.

$= 3[x^2(x + 2) - 25(x + 2)]$ Factor out common factors.

$= 3(x^2 - 25)(x + 2)$ Factor by grouping.

$= 3(x + 5)(x - 5)(x + 2)$ Factor the difference of two squares.

EXAMPLE 4 **Factoring polynomials**

Factor each polynomial completely.

a) $ax^3 + ax$ **b)** $by^3 + b$ **c)** $8m^3 + 22m^2 - 6m$

Solution

a) $ax^3 + ax = ax(x^2 + 1)$ The sum of two squares is prime.

b) $by^3 + b = b(y^3 + 1)$ Factor out the GCF.

$= b(y + 1)(y^2 - y + 1)$ Factor the sum of two cubes.

c) $8m^3 + 22m^2 - 6m = 2m(4m^2 + 11m - 3)$ Factor out the GCF.

$= 2m(4m - 1)(m + 3)$ Trial and error

WARM-UPS

True or false? Explain your answer.

1. $x^2 - 4 = (x - 2)^2$ for any real number x.
2. The trinomial $4x^2 + 6x + 9$ is a perfect square trinomial.
3. The polynomial $4y^2 + 25$ is a prime polynomial.
4. $3y + ay + 3x + ax = (x + y)(3 + a)$ for any values of the variables.
5. The polynomial $3x^2 + 51$ cannot be factored.
6. If the GCF is not 1, then you should factor it out first.
7. $x^2 + 9 = (x + 3)^2$ for any real number x.
8. The polynomial $x^2 - 3x - 5$ is a prime polynomial.
9. The polynomial $y^2 - 5y - my + 5m$ can be factored by grouping.
10. The polynomial $x^2 + ax - 3x + 3a$ can be factored by grouping.

6.5 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the relationship between division and factoring?
2. How do we know that $a - b$ is a factor of $a^3 - b^3$?
3. How do we know that $a + b$ is a factor of $a^3 + b^3$?
4. How do you recognize if a polynomial is a sum of two cubes?
5. How do you factor a sum of two cubes?
6. How do you factor a difference of two cubes?

Factor each polynomial completely, given that the binomial following it is a factor of the polynomial. See Example 1.

7. $x^3 + 3x^2 - 10x - 24,\ x + 4$
8. $x^3 - 7x + 6,\ x - 1$
9. $x^3 + 4x^2 + x - 6,\ x - 1$
10. $x^3 - 5x^2 - 2x + 24,\ x + 2$
11. $x^3 - 8,\ x - 2$
12. $x^3 + 27,\ x + 3$
13. $x^3 + 4x^2 - 3x + 10,\ x + 5$
14. $2x^3 - 5x^2 - x - 6,\ x - 3$
15. $x^3 + 2x^2 + 2x + 1,\ x + 1$
16. $x^3 + 2x^2 - 5x - 6,\ x + 3$

Factor each difference or sum of cubes. See Example 2.

17. $m^3 - 1$
18. $z^3 - 27$

19. $x^3 + 8$
20. $y^3 + 27$
21. $8w^3 + 1$
22. $125m^3 + 1$
23. $8t^3 - 27$
24. $125n^3 - 8$
25. $x^3 - y^3$
26. $m^3 + n^3$
27. $8t^3 + y^3$
28. $u^3 - 125v^3$

Factor each polynomial completely. If a polynomial is prime, say so. See Examples 3 and 4.

29. $2x^2 - 18$
30. $3x^3 - 12x$
31. $4x^2 + 8x - 60$
32. $3x^2 + 18x + 27$
33. $x^3 + 4x^2 + 4x$
34. $a^3 - 5a^2 + 6a$
35. $5max^2 + 20ma$
36. $3bmw^2 - 12bm$
37. $9x^2 + 6x + 1$
38. $9x^2 + 6x + 3$
39. $6x^2y + xy - 2y$
40. $5x^2y^2 - xy^2 - 6y^2$
41. $y^2 + 10y - 25$
42. $8b^2 + 24b + 18$
43. $16m^2 - 4m - 2$
44. $32a^2 + 4a - 6$
45. $9a^2 + 24a + 16$
46. $3x^2 - 18x - 48$
47. $24x^2 - 26x + 6$
48. $4x^2 - 6x - 12$
49. $3a^2 - 27a$
50. $a^2 - 25a$
51. $8 - 2x^2$
52. $x^3 + 6x^2 + 9x$
53. $6x^3 - 5x^2 + 12x$
54. $x^3 + 2x^2 - x - 2$
55. $a^3b - 4ab$
56. $2m^2 - 1800$
57. $x^3 + 2x^2 - 4x - 8$
58. $m^2a + 2ma^2 + a^3$
59. $2w^4 - 16w$
60. $m^4n + mn^4$
61. $3a^2w - 18aw + 27w$
62. $8a^3 + 4a$
63. $5x^2 - 500$
64. $25x^2 - 16y^2$
65. $2m + 2n - wm - wn$
66. $aw - 5b - bw + 5a$
67. $3x^4 + 3x$
68. $3a^5 - 81a^2$
69. $4w^2 + 4w - 4$
70. $4w^2 + 8w - 5$
71. $a^4 + 7a^3 - 30a^2$
72. $2y^5 + 3y^4 - 20y^3$
73. $4aw^3 - 12aw^2 + 9aw$
74. $9bn^3 + 15bn^2 - 14bn$
75. $t^2 + 6t + 9$
76. $t^3 + 12t^2 + 36t$

Solve each problem.

77. ***Increasing cube.*** Each of the three dimensions of a cube with a volume of x^3 cubic centimeters is increased by a whole number of centimeters. If the new volume is $x^3 + 10x^2 + 31x + 30$ cubic centimeters and the new height is $x + 2$ centimeters, then what are the new length and width?

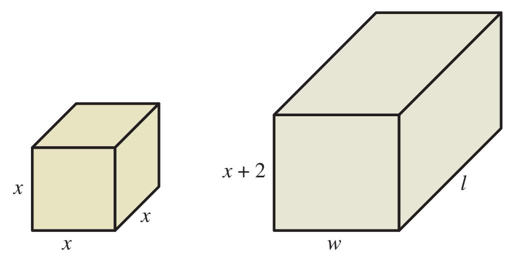

FIGURE FOR EXERCISE 77

78. ***Decreasing cube.*** Each of the three dimensions of a cube with a volume of y^3 cubic centimeters is decreased by a whole number of centimeters. If the new volume is $y^3 - 13y^2 + 54y - 72$ cubic centimeters and the new width is $y - 6$ centimeters, then what are the new length and height?

GETTING MORE INVOLVED

79. ***Discussion.*** Are there any values for a and b for which $(a + b)^3 = a^3 + b^3$? Find a pair of values for a and b for which $(a + b)^3 \neq a^3 + b^3$. Is $(a + b)^3$ equivalent to $a^3 + b^3$? Explain your answers.

80. ***Writing.*** Explain why $a^2 + ab + b^2$ and $a^2 - ab + b^2$ are prime polynomials.