6.6 SOLVING QUADRATIC EQUATIONS BY FACTORING

In this section

- The Zero Factor Property
- Applications

The techniques of factoring can be used to solve equations involving polynomials. These equations cannot be solved by the other methods that you have learned. After you learn to solve equations by factoring, you will use this technique to solve some new types of problems.

The Zero Factor Property

In this chapter you learned to factor polynomials such as $x^2 + x - 6$. The equation $x^2 + x - 6 = 0$ is called a *quadratic equation.*

helpful hint

The prefix "quad" means four. So why is a polynomial with three terms called quadratic? Perhaps it is because a quadratic polynomial can be factored into a product of two binomials, which contain a total of four terms.

Quadratic Equation

If a, b, and c are real numbers with $a \neq 0$, then

$$ax^2 + bx + c = 0$$

is called a **quadratic equation.**

The main idea used to solve quadratic equations is the **zero factor property,** which says that if a product is zero, then one or the other of the factors is zero.

The Zero Factor Property

The equation $a \cdot b = 0$ is equivalent to

$$a = 0 \quad \text{or} \quad b = 0.$$

In our first example, we use the zero factor property and factoring to reduce a quadratic equation into two linear equations. Solving the linear equations gives us the solutions to the quadratic equation.

EXAMPLE 1 **Using the zero factor property**

Solve $x^2 + x - 6 = 0$.

Solution

First factor the polynomial on the left side:

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0 \quad \text{Factor the left side.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{Zero factor property}$$

$$x = -3 \quad \text{or} \quad x = 2 \quad \text{Solve each equation.}$$

We now check that -3 and 2 satisfy the original equation.

For $x = -3$:

$$x^2 + x - 6 = (-3)^2 + (-3) - 6$$
$$= 9 - 3 - 6$$
$$= 0$$

For $x = 2$:

$$x^2 + x - 6 = (2)^2 + (2) - 6$$
$$= 4 + 2 - 6$$
$$= 0$$

The solutions to $x^2 + x - 6 = 0$ are -3 and 2.

A sentence such as $x = -3$ or $x = 2$, which is made up of two or more equations connected with the word "or" is called a **compound equation.** In the next example we again solve a quadratic equation by using the zero factor property to write a compound equation.

EXAMPLE 2

Using the zero factor property

Solve the equation $3x^2 = -3x$.

Solution

First rewrite the equation with 0 on the right-hand side:

$$3x^2 = -3x$$
$$3x^2 + 3x = 0 \quad \text{Add } 3x \text{ to each side.}$$
$$3x(x + 1) = 0 \quad \text{Factor the left-hand side.}$$
$$3x = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Zero factor property}$$
$$x = 0 \quad \text{or} \quad x = -1 \quad \text{Solve each equation.}$$

Check 0 and -1 in the original equation $3x^2 = -3x$.

For $x = 0$:

$$3(0)^2 = -3(0)$$
$$0 = 0$$

For $x = -1$:

$$3(-1)^2 = -3(-1)$$
$$3 = 3$$

There are two solutions to the original equation, 0 and -1.

CAUTION If in Example 2 you divide each side of $3x^2 = -3x$ by $3x$, you would get $x = -1$ but not the solution $x = 0$. For this reason we usually do not divide each side of an equation by a variable.

The basic strategy for solving an equation by factoring follows.

helpful hint

We have seen quadratic polynomials that cannot be factored. So not all quadratic equations can be solved by factoring. Methods for solving all quadratic equations are presented in Chapter 10.

Strategy for Solving an Equation by Factoring

1. Rewrite the equation with 0 on the right-hand side.
2. Factor the left-hand side completely.
3. Use the zero factor property to get simple linear equations.
4. Solve the linear equations.
5. Check the answers in the original equation.
6. State the solution(s) to the original equation.

EXAMPLE 3

Using the zero factor property

Solve $(2x + 1)(x - 1) = 14$.

study tip

Set short-term goals and reward yourself for accomplishing them. When you have solved 10 problems, take a short break and listen to your favorite music.

Solution

To write the equation with 0 on the right-hand side, multiply the binomials on the left and then subtract 14 from each side:

$$
\begin{aligned}
(2x + 1)(x - 1) &= 14 && \text{Original equation} \\
2x^2 - x - 1 &= 14 && \text{Multiply the binomials.} \\
2x^2 - x - 15 &= 0 && \text{Subtract 14 from each side.} \\
(2x + 5)(x - 3) &= 0 && \text{Factor.}
\end{aligned}
$$

$$
\begin{aligned}
2x + 5 = 0 \quad &\text{or} \quad x - 3 = 0 \qquad \text{Zero factor property} \\
2x = -5 \quad &\text{or} \quad x = 3 \\
x = -\frac{5}{2} \quad &\text{or} \quad x = 3
\end{aligned}
$$

Check $-\frac{5}{2}$ and 3 in the original equation:

$$
\begin{aligned}
\left(2 \cdot -\frac{5}{2} + 1\right)\left(-\frac{5}{2} - 1\right) &= (-5 + 1)\left(-\frac{5}{2} - \frac{2}{2}\right) \\
&= (-4)\left(-\frac{7}{2}\right) \\
&= 14 \\
(2 \cdot 3 + 1)(3 - 1) &= (7)(2) \\
&= 14
\end{aligned}
$$

So the solutions are $-\frac{5}{2}$ and 3. ■

CAUTION In Example 3 we started with a product of two factors equal to 14. Because there are many pairs of factors that have a product of 14, we *cannot make any conclusion about the factors.* If the product of two factors is 0, then we can conclude that one or the other factor is 0.

If a perfect square trinomial occurs in a quadratic equation, then there are two identical factors of the trinomial. In this case it is not necessary to set both factors equal to zero. The solution can be found from one factor.

EXAMPLE 4

An equation with a repeated factor

Solve $5x^2 - 30x + 45 = 0$.

Solution

Notice that the trinomial on the left-hand side has a common factor:

$$
\begin{aligned}
5x^2 - 30x + 45 &= 0 \\
5(x^2 - 6x + 9) &= 0 && \text{Factor out the GCF.} \\
5(x - 3)^2 &= 0 && \text{Factor the perfect square trinomial.} \\
(x - 3)^2 &= 0 && \text{Divide each side by 5.} \\
x - 3 &= 0 && \text{Zero factor property} \\
x &= 3
\end{aligned}
$$

You should check that 3 satisfies the original equation. Even though $x - 3$ occurs twice as a factor, it is not necessary to write $x - 3 = 0$ or $x - 3 = 0$. The only solution to the equation is 3.

CAUTION To simplify $5(x - 3)^2 = 0$ in Example 4, we divided each side by 5. If we had used the zero factor property, we would have gotten $5 = 0$ or $(x - 3)^2 = 0$. Since $5 = 0$ has no solution, we can ignore it and continue to solve $(x - 3)^2 = 0$.

If the left-hand side of the equation has more than two factors, we can write an equivalent equation by setting each factor equal to zero.

EXAMPLE 5

An equation with three solutions

Solve $2x^3 - x^2 - 8x + 4 = 0$.

Solution

We can factor the four-term polynomial by grouping:

$$2x^3 - x^2 - 8x + 4 = 0$$

$$x^2(2x - 1) - 4(2x - 1) = 0 \quad \text{Factor out the common factors.}$$

$$(x^2 - 4)(2x - 1) = 0 \quad \text{Factor out } 2x - 1.$$

$$(x - 2)(x + 2)(2x - 1) = 0 \quad \text{Difference of two squares}$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad 2x - 1 = 0 \quad \text{Zero factor property}$$

$$x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = \frac{1}{2} \quad \text{Solve each equation.}$$

You should check that all three numbers satisfy the original equation. The solutions to this equation are -2, $\frac{1}{2}$, and 2.

helpful hint

Compare the number of solutions in Examples 1 through 5 to the degree of the polynomial. The number of real solutions to any polynomial equation is less than or equal to the degree of the polynomial. This fact is known as the fundamental theorem of algebra.

Applications

There are many problems that can be solved by equations like those we have just discussed.

EXAMPLE 6

Area of a garden

Merida's garden has a rectangular shape with a length that is 1 foot longer than twice the width. If the area of the garden is 55 square feet, then what are the dimensions of the garden?

Solution

If x represents the width of the garden, then $2x + 1$ represents the length. See Fig. 6.1. Because the area of a rectangle is the length times the width, we can write the equation

$$x(2x + 1) = 55.$$

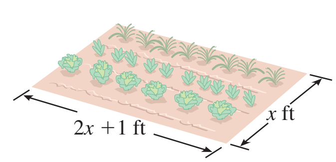
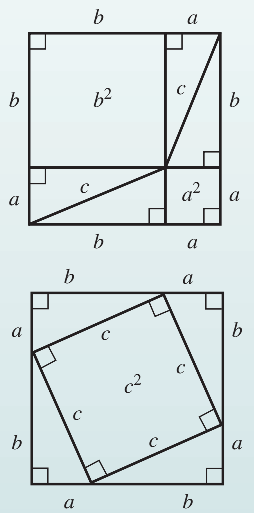


FIGURE 6.1

helpful hint

To prove the Pythagorean theorem, draw two squares with sides of length $a + b$, and partition them as shown.



Erasing the four identical triangles from each picture will subtract the same amount of area from each original square. Since we started with equal areas, we will have equal areas after erasing the triangles:

$$a^2 + b^2 = c^2$$

We must have zero on the right-hand side of the equation to use the zero factor property. So we rewrite the equation and then factor:

$$2x^2 + x - 55 = 0$$
$$(2x + 11)(x - 5) = 0 \quad \text{Factor.}$$
$$2x + 11 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{Zero factor property}$$
$$x = -\frac{11}{2} \quad \text{or} \quad x = 5$$

The width is certainly not $-\frac{11}{2}$. So we use $x = 5$ to get the length:

$$2x + 1 = 2(5) + 1 = 11$$

We check by multiplying 11 feet and 5 feet to get the area of 55 square feet. So the width is 5 ft, and the length is 11 ft. ■

The next application involves a theorem from geometry called the **Pythagorean theorem.** This theorem says that in any right triangle the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

The Pythagorean Theorem

The triangle shown in Fig. 6.2 is a right triangle if and only if

$$a^2 + b^2 = c^2.$$

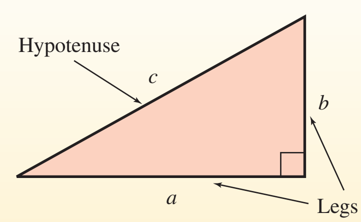


FIGURE 6.2

EXAMPLE 7

Using the Pythagorean theorem

The length of a rectangle is 1 meter longer than the width, and the diagonal measures 5 meters. What are the length and width?

Solution

If x represents the width of the rectangle, then $x + 1$ represents the length. Because the two sides are the legs of a right triangle, we can use the Pythagorean theorem to get a relationship between the length, width, and diagonal. See Fig. 6.3.

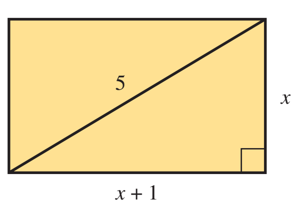


FIGURE 6.3

$$x^2 + (x + 1)^2 = 5^2 \quad \text{Pythagorean theorem}$$
$$x^2 + x^2 + 2x + 1 = 25 \quad \text{Simplify.}$$
$$2x^2 + 2x - 24 = 0$$
$$x^2 + x - 12 = 0 \quad \text{Divide each side by 2.}$$
$$(x - 3)(x + 4) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x + 4 = 0 \quad \text{Zero factor property}$$
$$x = 3 \quad \text{or} \quad x = -4 \quad \text{The length cannot be negative.}$$
$$x + 1 = 4$$

To check this answer, we compute $3^2 + 4^2 = 5^2$, or $9 + 16 = 25$. So the rectangle is 3 meters by 4 meters.

CAUTION The hypotenuse is the longest side of a right triangle. So if the lengths of the sides of a right triangle are 5 meters, 12 meters, and 13 meters, then the length of the hypotenuse is 13 meters, and $5^2 + 12^2 = 13^2$.

MATH AT WORK

INVESTMENT ADVISOR

Can you successfully invest money and at the same time be socially responsible? Geeta Bhide, president and founder of Walden Capital Management, answers with an emphatic "yes." Ms. Bhide helps clients integrate their social values with a portfolio of stocks, bonds, cash, and cash equivalents.

With the client's consent, Ms. Bhide might invest in bonds backed by the Department of Housing and Urban Development for housing in specific inner-city areas. Other choices might be environmentally conscious companies or companies that have a proven record of equal employment practices. In many cases, categorizing a particular company as socially conscious is a judgment call. For example, many oil companies provide a good return on investment, but they might not have an unblemished record on oil spills. In this instance, picking the best of the worst might be the correct choice. Because such trade-offs are necessary, clients are encouraged to define both their investment goals and the social ideals to which they subscribe.

As any investment advisor would, Ms. Bhide tries to minimize risk and maximize reward, or return on investment. In Exercises 81 and 82 of this section you will see how solving a quadratic equation by factoring can be used to find the average annual return on an investment.

WARM-UPS

True or false? Explain your answer.

1. The equation $x(x + 2) = 3$ is equivalent to $x = 3$ or $x + 2 = 3$.
2. Equations solved by factoring always have two different solutions.
3. The equation $a \cdot d = 0$ is equivalent to $a = 0$ or $d = 0$.
4. If x is the width in feet of a rectangular room and the length is 5 feet longer than the width, then the area is $x^2 + 5x$ square feet.
5. Both 1 and -4 are solutions to the equation $(x - 1)(x + 4) = 0$.
6. If a, b, and c are the sides of any triangle, then $a^2 + b^2 = c^2$.
7. If the perimeter of a rectangular room is 50 feet, then the sum of the length and width is 25 feet.
8. Equations solved by factoring may have more than two solutions.
9. Both 0 and 2 are solutions to the equation $x(x - 2) = 0$.
10. The solutions to $3(x - 2)(x + 5) = 0$ are 3, 2, and -5.

6.6 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a quadratic equation?

2. What is a compound equation?

3. What is the zero factor property?

4. What method is used to solve quadratic equations in this section?

5. Why don't we usually divide each side of an equation by a variable?

6. What is the Pythagorean theorem?

Solve each equation. See Example 1.

7. $(x + 5)(x + 4) = 0$

8. $(a + 6)(a + 5) = 0$

9. $(2x + 5)(3x - 4) = 0$

10. $(3k - 8)(4k + 3) = 0$

11. $w^2 - 9w + 14 = 0$

12. $t^2 + 6t - 27 = 0$

13. $y^2 - 2y - 24 = 0$

14. $q^2 + 3q - 18 = 0$

15. $2m^2 + m - 1 = 0$

16. $2h^2 - h - 3 = 0$

Solve each equation. See Examples 2 and 3.

17. $m^2 = -7m$

18. $h^2 = -5h$

19. $a^2 + a = 20$

20. $p^2 + p = 42$

21. $2x^2 + 5x = 3$

22. $3x^2 - 10x = -7$

23. $(x + 2)(x + 6) = 12$

24. $(x + 2)(x - 6) = 20$

25. $(a + 3)(2a - 1) = 15$

26. $(b - 3)(3b + 4) = 10$

27. $2(4 - 5h) = 3h^2$

28. $2w(4w + 1) = 1$

Solve each equation. See Examples 4 and 5.

29. $2x^2 + 50 = 20x$

30. $3x^2 + 48 = 24x$

31. $4m^2 - 12m + 9 = 0$

32. $25y^2 + 20y + 4 = 0$

33. $x^3 - 9x = 0$

34. $25x - x^3 = 0$

35. $w^3 + 4w^2 - 4w = 16$

36. $a^3 + 2a^2 - a = 2$

37. $n^3 - 3n^2 + 3 = n$

38. $w^3 + w^2 - 25w = 25$

39. $y^3 - 9y^2 + 20y = 0$

40. $m^3 + 2m^2 - 3m = 0$

Solve each equation.

41. $x^2 - 16 = 0$

42. $x^2 - 36 = 0$

43. $x^2 = 9$

44. $x^2 = 25$

45. $a^3 = a$

46. $x^3 = 4x$

47. $3x^2 + 15x + 18 = 0$

48. $-2x^2 - 2x + 24 = 0$

49. $z^2 + \frac{11}{2}z = -6$

50. $m^2 + \frac{8}{3}m = 1$

51. $(t - 3)(t + 5) = 9$

52. $3x(2x + 1) = 18$

53. $(x - 2)^2 + x^2 = 10$

54. $(x - 3)^2 + (x + 2)^2 = 17$

55. $\frac{1}{16}x^2 + \frac{1}{8}x = \frac{1}{2}$

56. $\frac{1}{18}h^2 - \frac{1}{2}h + 1 = 0$

57. $a^3 + 3a^2 - 25a = 75$

58. $m^4 + m^3 = 100m^2 + 100m$

Solve each problem. See Examples 6 and 7.

59. ***Dimensions of a rectangle.*** The perimeter of a rectangle is 34 feet, and the diagonal is 13 feet long. What are the length and width of the rectangle?

60. ***Address book.*** The perimeter of the cover of an address book is 14 inches, and the diagonal measures 5 inches. What are the length and width of the cover?

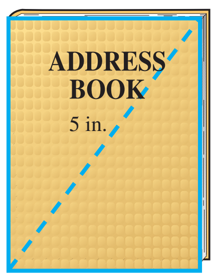


FIGURE FOR EXERCISE 60

61. ***Viola's bathroom.*** The length of Viola's bathroom is 2 feet longer than twice the width. If the diagonal measures 13 feet, then what are the length and width?

62. ***Rectangular stage.*** One side of a rectangular stage is 2 meters longer than the other. If the diagonal is 10 meters, then what are the lengths of the sides?

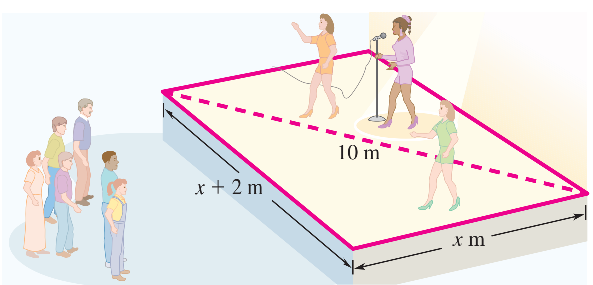


FIGURE FOR EXERCISE 62

63. ***Consecutive integers.*** The sum of the squares of two consecutive integers is 13. Find the integers.

64. ***Consecutive integers.*** The sum of the squares of two consecutive even integers is 52. Find the integers.

65. ***Two numbers.*** The sum of two numbers is 11, and their product is 30. Find the numbers.

66. ***Missing ages.*** Molly's age is twice Anita's. If the sum of the squares of their ages is 80, then what are their ages?

67. ***Skydiving.*** If there were no air resistance, then the height (in feet) above the earth for a sky diver t seconds after jumping from an airplane at 10,000 feet would be given by

$$h(t) = -16t^2 + 10{,}000.$$

a) Find the time that it would take to fall to earth with no air resistance, that is, find t for which $h(t) = 0$. A sky diver actually gets about twice as much free fall time due to air resistance.

b) Use the accompanying graph to determine whether the sky diver (with no air resistance) falls farther in the first 5 seconds or the last 5 seconds of the fall.

c) Is the sky diver's velocity increasing or decreasing as she falls?

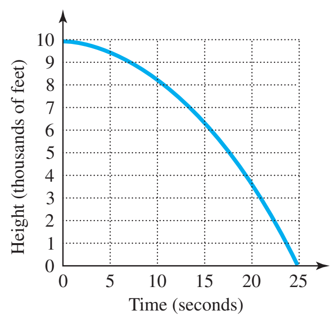


FIGURE FOR EXERCISE 67

68. ***Skydiving.*** If a sky diver jumps from an airplane at a height of 8256 feet, then for the first 5 seconds, her height above the earth is approximated by the formula $h = -16t^2 + 8256$. How many seconds does it take her to reach 8000 feet.

69. ***Throwing a sandbag.*** If a balloonist throws a sandbag downward at 24 feet per second from an altitude of 720 feet, then its height (in feet) above the ground after t seconds is given by $S = -16t^2 - 24t + 720$. How long does it take for the sandbag to reach the earth? (On the ground, $S = 0$.)

70. ***Throwing a sandbag.*** If the balloonist of the previous exercise throws his sandbag downward from an altitude of 128 feet with an initial velocity of 32 feet per second, then its altitude after t seconds is given by the formula $S = -16t^2 - 32t + 128$. How long does it take for the sandbag to reach the earth?

71. ***Glass prism.*** One end of a glass prism is in the shape of a triangle with a height that is 1 inch longer than twice the base. If the area of the triangle is 39 square inches, then how long are the base and height?

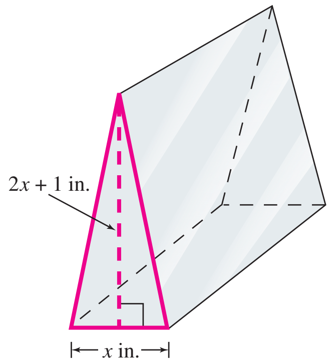


FIGURE FOR EXERCISE 71

72. ***Areas of two circles.*** The radius of a circle is 1 meter longer than the radius of another circle. If their areas differ by 5π square meters, then what is the radius of each?

73. ***Changing area.*** Last year Otto's garden was square. This year he plans to make it smaller by shortening one side 5 feet and the other 8 feet. If the area of the smaller garden will be 180 square feet, then what was the size of Otto's garden last year?

74. ***Dimensions of a box.*** Rosita's Christmas present from Carlos is in a box that has a width that is 3 inches shorter than the height. The length of the base is 5 inches longer than the height. If the area of the base is 84 square inches, then what is the height of the package?

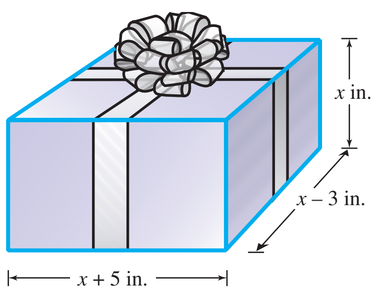


FIGURE FOR EXERCISE 74

75. ***Flying a kite.*** Imelda and Gordon have designed a new kite. While Imelda is flying the kite, Gordon is standing directly below it. The kite is designed so that its altitude is always 20 feet larger than the distance between Imelda and Gordon. What is the altitude of the kite when it is 100 feet from Imelda?

76. ***Avoiding a collision.*** A car is traveling on a road that is perpendicular to a railroad track. When the car is 30 meters from the crossing, the car's new collision detector warns the driver that there is a train 50 meters from the car and heading toward the same crossing. How far is the train from the crossing?

77. ***Carpeting two rooms.*** Virginia is buying carpet for two square rooms. One room is 3 yards wider than the other. If she needs 45 square yards of carpet, then what are the dimensions of each room?

78. ***Winter wheat.*** While finding the amount of seed needed to plant his three square wheat fields, Hank observed that the side of one field was 1 kilometer longer than the side of the smallest field and that the side of the largest field was 3 kilometers longer than the side of the smallest field. If the total area of the three fields is 38 square kilometers, then what is the area of each field?

79. ***Sailing to Miami.*** At point A the captain of a ship determined that the distance to Miami was 13 miles. If she sailed north to point B and then west to Miami, the distance would be 17 miles. If the distance from point A to point B is greater than the distance from point B to Miami, then how far is it from point A to point B?

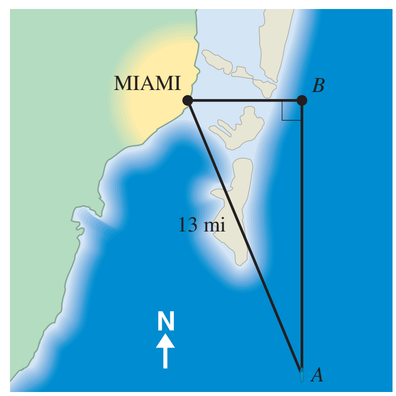


FIGURE FOR EXERCISE 79

80. ***Buried treasure.*** Ahmed has half of a treasure map, which indicates that the treasure is buried in the desert $2x + 6$ paces from Castle Rock. Vanessa has the other half of the map. Her half indicates that to find the treasure, one must get to Castle Rock, walk x paces to the north, and then walk $2x + 4$ paces to the east. If they share their information, then they can find x and save a lot of digging. What is x?

81. ***Emerging markets.*** Catarina's investment of \$16,000 in an emerging market fund grew to \$25,000 in 2 years. Find the average annual rate of return by solving the equation $16{,}000(1 + r)^2 = 25{,}000$.

82. ***Venture capital.*** Henry invested \$12,000 in a new restaurant. When the restaurant was sold 2 years later, he received \$27,000. Find his average annual return by solving the equation $12{,}000(1 + r)^2 = 27{,}000$.