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73. ***Discussion.*** Evaluate each expression.

a) One-half of $\frac{1}{4}$ **b)** One-third of 4

c) One-half of $\frac{4x}{3}$ **d)** One-half of $\frac{3x}{2}$

74. ***Exploration.*** Let $R = \frac{6x^2 + 23x + 20}{24x^2 + 29x - 4}$ and $H = \frac{2x + 5}{8x - 1}$.

a) Find R when $x = 2$ and $x = 3$. Find H when $x = 2$ and $x = 3$.

b) How are these values of R and H related and why?

7.3 FINDING THE LEAST COMMON DENOMINATOR

In this section

- Building Up the Denominator
- Finding the Least Common Denominator
- Converting to the LCD

Every rational expression can be written in infinitely many equivalent forms. Because we can add or subtract only fractions with identical denominators, we must be able to change the denominator of a fraction. You have already learned how to change the denominator of a fraction by reducing. In this section you will learn the opposite of reducing, which is called **building up the denominator.**

Building Up the Denominator

To convert the fraction $\frac{2}{3}$ into an equivalent fraction with a denominator of 21, we factor 21 as $21 = 3 \cdot 7$. Because $\frac{2}{3}$ already has a 3 in the denominator, multiply the numerator and denominator of $\frac{2}{3}$ by the missing factor 7 to get a denominator of 21:

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{7}{7} = \frac{14}{21}$$

For rational expressions the process is the same. To convert the rational expression

$$\frac{5}{x + 3}$$

into an equivalent rational expression with a denominator of $x^2 - x - 12$, first factor $x^2 - x - 12$:

$$x^2 - x - 12 = (x + 3)(x - 4)$$

From the factorization we can see that the denominator $x + 3$ needs only a factor of $x - 4$ to have the required denominator. So multiply the numerator and denominator by the missing factor $x - 4$:

$$\frac{5}{x + 3} = \frac{5}{x + 3} \cdot \frac{x - 4}{x - 4} = \frac{5x - 20}{x^2 - x - 12}$$

EXAMPLE 1 **Building up the denominator**

Build each rational expression into an equivalent rational expression with the indicated denominator.

a) $3 = \frac{?}{12}$ **b)** $\frac{3}{w} = \frac{?}{wx}$ **c)** $\frac{2}{3y^3} = \frac{?}{12y^8}$

Solution

a) Because $3 = \frac{3}{1}$, we get a denominator of 12 by multiplying the numerator and denominator by 12:

$$3 = \frac{3}{1} = \frac{3 \cdot 12}{1 \cdot 12} = \frac{36}{12}$$

b) Multiply the numerator and denominator by x:

$$\frac{3}{w} = \frac{3 \cdot x}{w \cdot x} = \frac{3x}{wx}$$

c) To build the denominator $3y^3$ up to $12y^8$, multiply by $4y^5$:

$$\frac{2}{3y^3} = \frac{2 \cdot 4y^5}{3y^3 \cdot 4y^5} = \frac{8y^5}{12y^8}$$

In the next example we must factor the original denominator before building up the denominator.

EXAMPLE 2 Building up the denominator

Build each rational expression into an equivalent rational expression with the indicated denominator.

a) $\frac{7}{3x - 3y} = \frac{?}{6y - 6x}$ **b)** $\frac{x - 2}{x + 2} = \frac{?}{x^2 + 8x + 12}$

Solution

a) Because $3x - 3y = 3(x - y)$, we factor -6 out of $6y - 6x$. This will give a factor of $x - y$ in each denominator:

$$3x - 3y = 3(x - y)$$
$$6y - 6x = -6(x - y) = -2 \cdot 3(x - y)$$

To get the required denominator, we multiply the numerator and denominator by -2 only:

$$\frac{7}{3x - 3y} = \frac{7(-2)}{(3x - 3y)(-2)}$$
$$= \frac{-14}{6y - 6x}$$

b) Because $x^2 + 8x + 12 = (x + 2)(x + 6)$, we multiply the numerator and denominator by $x + 6$, the missing factor:

$$\frac{x - 2}{x + 2} = \frac{(x - 2)(x + 6)}{(x + 2)(x + 6)}$$
$$= \frac{x^2 + 4x - 12}{x^2 + 8x + 12}$$

helpful hint

Notice that reducing and building up are exactly the opposite of each other. In reducing you remove a factor that is common to the numerator and denominator, and in building up you put a common factor into the numerator and denominator.

CAUTION When building up a denominator, *both* the numerator and the denominator must be multiplied by the appropriate expression, because that is how we build up fractions.

Finding the Least Common Denominator

We can use the idea of building up the denominator to convert two fractions with different denominators into fractions with identical denominators. For example,

$$\frac{5}{6} \quad \text{and} \quad \frac{1}{4}$$

can both be converted into fractions with a denominator of 12, since $12 = 2 \cdot 6$ and $12 = 3 \cdot 4$:

$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12} \qquad \frac{1}{4} = \frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12}$$

The smallest number that is a multiple of all of the denominators is called the **least common denominator (LCD).** The LCD for the denominators 6 and 4 is 12.

To find the LCD in a systematic way, we look at a complete factorization of each denominator. Consider the denominators 24 and 30:

$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$$
$$30 = 2 \cdot 3 \cdot 5$$

study tip

Studying in a quiet place is better than studying in a noisy place. There are very few people who can listen to music or a conversation and study at the same time.

Any multiple of 24 must have three 2's in its factorization, and any multiple of 30 must have one 2 as a factor. So a number with three 2's in its factorization will have enough to be a multiple of both 24 and 30. The LCD must also have one 3 and one 5 in its factorization. *We use each factor the maximum number of times it appears in either factorization.* So the LCD is $2^3 \cdot 3 \cdot 5$:

$$2^3 \cdot 3 \cdot 5 = \overbrace{2 \cdot 2 \cdot \underbrace{2 \cdot 3}_{} }^{24} \underbrace{\cdot\, 5}_{30} = 120$$

If we omitted any one of the factors in $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$, we would not have a multiple of both 24 and 30. That is what makes 120 the *least* common denominator. To find the LCD for two polynomials, we use the same strategy.

Strategy for Finding the LCD for Polynomials

1. Factor each denominator completely. Use exponent notation for repeated factors.
2. Write the product of all of the different factors that appear in the denominators.
3. On each factor, use the highest power that appears on that factor in any of the denominators.

EXAMPLE 3 **Finding the LCD**

If the given expressions were used as denominators of rational expressions, then what would be the LCD for each group of denominators?

a) 20, 50 **b)** x^3yz^2, x^5y^2z, xyz^5

c) $a^2 + 5a + 6, a^2 + 4a + 4$

Solution

a) First factor each number completely:

$$20 = 2^2 \cdot 5 \qquad 50 = 2 \cdot 5^2$$

The highest power of 2 is 2, and the highest power of 5 is 2. So the LCD of 20 and 50 is $2^2 \cdot 5^2$, or 100.

b) The expressions x^3yz^2, x^5y^2z, and xyz^5 are already factored. For the LCD, use the highest power of each variable. So the LCD is $x^5y^2z^5$.

c) First factor each polynomial.

$$a^2 + 5a + 6 = (a + 2)(a + 3) \qquad a^2 + 4a + 4 = (a + 2)^2$$

The highest power of $(a + 3)$ is 1, and the highest power of $(a + 2)$ is 2. So the LCD is $(a + 3)(a + 2)^2$. ■

Converting to the LCD

When adding or subtracting rational expressions, we must convert the expressions into expressions with identical denominators. To keep the computations as simple as possible, we use the least common denominator.

EXAMPLE 4

Converting to the LCD

Find the LCD for the rational expressions, and convert each expression into an equivalent rational expression with the LCD as the denominator.

a) $\dfrac{4}{9xy}, \dfrac{2}{15xz}$ **b)** $\dfrac{5}{6x^2}, \dfrac{1}{8x^3y}, \dfrac{3}{4y^2}$

helpful hint

What is the difference between LCD, GCF, CBS, and NBC? The LCD for the denominators 4 and 6 is 12. The *least* common denominator is *greater than* or equal to both numbers. The GCF for 4 and 6 is 2. The *greatest* common factor is *less than* or equal to both numbers. CBS and NBC are TV networks.

Solution

a) Factor each denominator completely:

$$9xy = 3^2xy \qquad 15xz = 3 \cdot 5xz$$

The LCD is $3^2 \cdot 5xyz$. Now convert each expression into an expression with this denominator. We must multiply the numerator and denominator of the first rational expression by $5z$ and the second by $3y$:

$$\frac{4}{9xy} = \frac{4 \cdot 5z}{9xy \cdot 5z} = \frac{20z}{45xyz}$$
$$\frac{2}{15xz} = \frac{2 \cdot 3y}{15xz \cdot 3y} = \frac{6y}{45xyz}$$

Same denominator

b) Factor each denominator completely:

$$6x^2 = 2 \cdot 3x^2 \qquad 8x^3y = 2^3x^3y \qquad 4y^2 = 2^2y^2$$

The LCD is $2^3 \cdot 3 \cdot x^3y^2$ or $24x^3y^2$. Now convert each expression into an expression with this denominator:

$$\frac{5}{6x^2} = \frac{5 \cdot 4xy^2}{6x^2 \cdot 4xy^2} = \frac{20xy^2}{24x^3y^2}$$
$$\frac{1}{8x^3y} = \frac{1 \cdot 3y}{8x^3y \cdot 3y} = \frac{3y}{24x^3y^2}$$
$$\frac{3}{4y^2} = \frac{3 \cdot 6x^3}{4y^2 \cdot 6x^3} = \frac{18x^3}{24x^3y^2}$$

■

EXAMPLE 5 **Converting to the LCD**

Find the LCD for the rational expressions

$$\frac{5x}{x^2 - 4} \quad \text{and} \quad \frac{3}{x^2 + x - 6}$$

and convert each into an equivalent rational expression with that denominator.

Solution

First factor the denominators:

$$x^2 - 4 = (x - 2)(x + 2)$$
$$x^2 + x - 6 = (x - 2)(x + 3)$$

The LCD is $(x - 2)(x + 2)(x + 3)$. Now we multiply the numerator and denominator of the first rational expression by $(x + 3)$ and those of the second rational expression by $(x + 2)$. Because each denominator already has one factor of $(x - 2)$, there is no reason to multiply by $(x - 2)$. We multiply each denominator by the factors in the LCD that are missing from that denominator:

$$\frac{5x}{x^2 - 4} = \frac{5x(x + 3)}{(x - 2)(x + 2)(x + 3)} = \frac{5x^2 + 15x}{(x - 2)(x + 2)(x + 3)}$$
$$\frac{3}{x^2 + x - 6} = \frac{3(x + 2)}{(x - 2)(x + 3)(x + 2)} = \frac{3x + 6}{(x - 2)(x + 2)(x + 3)}$$

Same denominator

Note that in Example 5 we multiplied the expressions in the numerators but left the denominators in factored form. The numerators are simplified because it is the numerators that must be added when we add rational expressions in Section 7.4. Because we can add rational expressions with identical denominators, there is no need to multiply the denominators.

WARM-UPS

True or false? Explain your answer.

1. To convert $\frac{2}{3}$ into an equivalent fraction with a denominator of 18, we would multiply only the denominator of $\frac{2}{3}$ by 6.
2. Factoring has nothing to do with finding the least common denominator.
3. $\frac{3}{2ab^2} = \frac{15a^2b^2}{10a^3b^4}$ for any nonzero values of a and b.
4. The LCD for the denominators $2^5 \cdot 3$ and $2^4 \cdot 3^2$ is $2^5 \cdot 3^2$.
5. The LCD for the fractions $\frac{1}{6}$ and $\frac{1}{10}$ is 60.
6. The LCD for the denominators $6a^2b$ and $4ab^3$ is $2ab$.
7. The LCD for the denominators $a^2 + 1$ and $a + 1$ is $a^2 + 1$.
8. $\frac{x}{2} = \frac{x + 7}{2 + 7}$ for any real number x.
9. The LCD for the rational expressions $\frac{1}{x - 2}$ and $\frac{3}{x + 2}$ is $x^2 - 4$.
10. $x = \frac{3x}{3}$ for any real number x.

7.3 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is building up the denominator?
2. How do we build up the denominator of a rational expression?
3. What is the least common denominator for fractions?
4. How do you find the LCD for two polynomial denominators?

Build each rational expression into an equivalent rational expression with the indicated denominator. See Example 1.

5. $\frac{1}{3} = \frac{?}{27}$
6. $\frac{2}{5} = \frac{?}{35}$
7. $7 = \frac{?}{2x}$
8. $6 = \frac{?}{4y}$
9. $\frac{5}{b} = \frac{?}{3bt}$
10. $\frac{7}{2ay} = \frac{?}{2ayz}$
11. $\frac{-9z}{2aw} = \frac{?}{8awz}$
12. $\frac{-7yt}{3x} = \frac{?}{18xyt}$
13. $\frac{2}{3a} = \frac{?}{15a^3}$
14. $\frac{7b}{12c^5} = \frac{?}{36c^8}$
15. $\frac{4}{5xy^2} = \frac{?}{10x^2y^5}$
16. $\frac{5y^2}{8x^3z} = \frac{?}{24x^5z^3}$

Build each rational expression into an equivalent rational expression with the indicated denominator. See Example 2.

17. $\frac{5}{2x + 2} = \frac{?}{-8x - 8}$
18. $\frac{3}{m - n} = \frac{?}{2n - 2m}$
19. $\frac{8a}{5b^2 - 5b} = \frac{?}{20b^2 - 20b^3}$
20. $\frac{5x}{-6x - 9} = \frac{?}{18x^2 + 27x}$
21. $\frac{3}{x + 2} = \frac{?}{x^2 - 4}$
22. $\frac{a}{a + 3} = \frac{?}{a^2 - 9}$
23. $\frac{3x}{x + 1} = \frac{?}{x^2 + 2x + 1}$
24. $\frac{-7x}{2x - 3} = \frac{?}{4x^2 - 12x + 9}$
25. $\frac{y - 6}{y - 4} = \frac{?}{y^2 + y - 20}$
26. $\frac{z - 6}{z + 3} = \frac{?}{z^2 - 2z - 15}$

If the given expressions were used as denominators of rational expressions, then what would be the LCD for each group of denominators? See Example 3.

27. 12, 16
28. 28, 42
29. 12, 18, 20
30. 24, 40, 48
31. $6a^2$, $15a$
32. $18x^2$, $20xy$
33. $2a^4b$, $3ab^6$, $4a^3b^2$
34. $4m^3nw$, $6mn^5w^8$, $9m^6nw$
35. $x^2 - 16$, $x^2 + 8x + 16$
36. $x^2 - 9$, $x^2 + 6x + 9$
37. x, $x + 2$, $x - 2$
38. y, $y - 5$, $y + 2$
39. $x^2 - 4x$, $x^2 - 16$, $2x$
40. y, $y^2 - 3y$, $3y$

Find the LCD for the given rational expressions, and convert each rational expression into an equivalent rational expression with the LCD as the denominator. See Example 4.

41. $\frac{1}{6}, \frac{3}{8}$
42. $\frac{5}{12}, \frac{3}{20}$
43. $\frac{3}{84a}, \frac{5}{63b}$
44. $\frac{4b}{75a}, \frac{6}{105ab}$
45. $\frac{1}{3x^2}, \frac{3}{2x^5}$
46. $\frac{3}{8a^3b^9}, \frac{5}{6a^2c}$
47. $\frac{x}{9y^5z}, \frac{y}{12x^3}, \frac{1}{6x^2y}$
48. $\frac{5}{12a^6b}, \frac{3b}{14a^3}, \frac{1}{2ab^3}$

In Exercises 49–60, find the LCD for the given rational expressions, and convert each rational expression into an equivalent rational expression with the LCD as the denominator. See Example 5.

49. $\frac{2x}{x - 3}, \frac{5x}{x + 2}$
50. $\frac{2a}{a - 5}, \frac{3a}{a + 2}$

51. $\dfrac{4}{a-6}, \dfrac{5}{6-a}$

52. $\dfrac{4}{x-y}, \dfrac{5x}{2y-2x}$

53. $\dfrac{x}{x^2-9}, \dfrac{5x}{x^2-6x+9}$

54. $\dfrac{5x}{x^2-1}, \dfrac{-4}{x^2-2x+1}$

55. $\dfrac{w+2}{w^2-2w-15}, \dfrac{-2w}{w^2-4w-5}$

56. $\dfrac{z-1}{z^2+6z+8}, \dfrac{z+1}{z^2+5z+6}$

57. $\dfrac{-5}{6x-12}, \dfrac{x}{x^2-4}, \dfrac{3}{2x+4}$

58. $\dfrac{3}{4b^2-9}, \dfrac{2b}{2b+3}, \dfrac{-5}{2b^2-3b}$

59. $\dfrac{2}{2q^2-5q-3}, \dfrac{3}{2q^2+9q+4}, \dfrac{4}{q^2+q-12}$

60. $\dfrac{-3}{2p^2+7p-15}, \dfrac{p}{2p^2-11p+12}, \dfrac{2}{p^2+p-20}$

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61. ***Discussion.*** Why do we learn how to convert two rational expressions into equivalent rational expressions with the same denominator?

62. ***Discussion.*** Which expression is the LCD for

$$\frac{3x-1}{2^2 \cdot 3 \cdot x^2(x+2)} \quad \text{and} \quad \frac{2x+7}{2 \cdot 3^2 \cdot x(x+2)^2}?$$

a) $2 \cdot 3 \cdot x(x+2)$
b) $36x(x+2)$
c) $36x^2(x+2)^2$
d) $2^3 \cdot 3^3x^3(x+2)^2$

7.4 ADDITION AND SUBTRACTION

In this section

- Addition and Subtraction of Rational Numbers
- Addition and Subtraction of Rational Expressions
- Applications

In Section 7.3 you learned how to find the LCD and build up the denominators of rational expressions. In this section we will use that knowledge to add and subtract rational expressions with different denominators.

Addition and Subtraction of Rational Numbers

We can add or subtract rational numbers (or fractions) only with identical denominators according to the following definition.

Addition and Subtraction of Rational Numbers

If $b \neq 0$, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$