51. $\dfrac{4}{a-6}, \dfrac{5}{6-a}$

52. $\dfrac{4}{x-y}, \dfrac{5x}{2y-2x}$

53. $\dfrac{x}{x^2-9}, \dfrac{5x}{x^2-6x+9}$

54. $\dfrac{5x}{x^2-1}, \dfrac{-4}{x^2-2x+1}$

55. $\dfrac{w+2}{w^2-2w-15}, \dfrac{-2w}{w^2-4w-5}$

56. $\dfrac{z-1}{z^2+6z+8}, \dfrac{z+1}{z^2+5z+6}$

57. $\dfrac{-5}{6x-12}, \dfrac{x}{x^2-4}, \dfrac{3}{2x+4}$

58. $\dfrac{3}{4b^2-9}, \dfrac{2b}{2b+3}, \dfrac{-5}{2b^2-3b}$

59. $\dfrac{2}{2q^2-5q-3}, \dfrac{3}{2q^2+9q+4}, \dfrac{4}{q^2+q-12}$

60. $\dfrac{-3}{2p^2+7p-15}, \dfrac{p}{2p^2-11p+12}, \dfrac{2}{p^2+p-20}$

GETTING MORE INVOLVED

61. ***Discussion.*** Why do we learn how to convert two rational expressions into equivalent rational expressions with the same denominator?

62. ***Discussion.*** Which expression is the LCD for

$$\frac{3x-1}{2^2 \cdot 3 \cdot x^2(x+2)} \quad \text{and} \quad \frac{2x+7}{2 \cdot 3^2 \cdot x(x+2)^2}?$$

a) $2 \cdot 3 \cdot x(x+2)$
b) $36x(x+2)$
c) $36x^2(x+2)^2$
d) $2^3 \cdot 3^3x^3(x+2)^2$

7.4 ADDITION AND SUBTRACTION

In this section

- Addition and Subtraction of Rational Numbers
- Addition and Subtraction of Rational Expressions
- Applications

In Section 7.3 you learned how to find the LCD and build up the denominators of rational expressions. In this section we will use that knowledge to add and subtract rational expressions with different denominators.

Addition and Subtraction of Rational Numbers

We can add or subtract rational numbers (or fractions) only with identical denominators according to the following definition.

Addition and Subtraction of Rational Numbers

If $b \neq 0$, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

EXAMPLE 1 **Adding or subtracting fractions with the same denominator**

Perform the indicated operations. Reduce answers to lowest terms.

a) $\frac{1}{12} + \frac{7}{12}$ **b)** $\frac{1}{4} - \frac{3}{4}$

Solution

a) $\frac{1}{12} + \frac{7}{12} = \frac{8}{12} = \frac{4 \cdot 2}{4 \cdot 3} = \frac{2}{3}$ **b)** $\frac{1}{4} - \frac{3}{4} = \frac{-2}{4} = -\frac{1}{2}$ ■

If the rational numbers have different denominators, we must convert them to equivalent rational numbers that have identical denominators and then add or subtract. Of course, it is most efficient to use the least common denominator (LCD), as in the following example.

EXAMPLE 2 **Adding or subtracting fractions with different denominators**

Find each sum or difference.

a) $\frac{3}{20} + \frac{7}{12}$ **b)** $\frac{1}{6} - \frac{4}{15}$

helpful hint

Note how all of the operations with rational expressions are performed according to the rules for fractions that we studied in Chapter 1. So keep thinking of how you perform operations with fractions and you will improve your skills with fractions and with rational expressions.

Solution

a) Because $20 = 2^2 \cdot 5$ and $12 = 2^2 \cdot 3$, the LCD is $2^2 \cdot 3 \cdot 5$, or 60. Convert each fraction to an equivalent fraction with a denominator of 60:

$$\frac{3}{20} + \frac{7}{12} = \frac{3 \cdot 3}{20 \cdot 3} + \frac{7 \cdot 5}{12 \cdot 5} \quad \text{Build up the denominators.}$$

$$= \frac{9}{60} + \frac{35}{60} \quad \text{Simplify numerators and denominators.}$$

$$= \frac{44}{60} \quad \text{Add the fractions.}$$

$$= \frac{4 \cdot 11}{4 \cdot 15} \quad \text{Factor.}$$

$$= \frac{11}{15} \quad \text{Reduce.}$$

b) Because $6 = 2 \cdot 3$ and $15 = 3 \cdot 5$, the LCD is $2 \cdot 3 \cdot 5$ or 30:

$$\frac{1}{6} - \frac{4}{15} = \frac{1}{2 \cdot 3} - \frac{4}{3 \cdot 5} \quad \text{Factor the denominators.}$$

$$= \frac{1 \cdot 5}{2 \cdot 3 \cdot 5} - \frac{4 \cdot 2}{3 \cdot 5 \cdot 2} \quad \text{Build up the denominators.}$$

$$= \frac{5}{30} - \frac{8}{30} \quad \text{Simplify the numerators and denominators.}$$

$$= \frac{-3}{30} \quad \text{Subtract.}$$

$$= \frac{-1 \cdot 3}{10 \cdot 3} \quad \text{Factor.}$$

$$= -\frac{1}{10} \quad \text{Factor.}$$

■

Addition and Subtraction of Rational Expressions

Rational expressions are added or subtracted just like rational numbers. We can add or subtract only rational expressions that have identical denominators.

EXAMPLE 3 **Rational expressions with the same denominator**

Perform the indicated operations and reduce answers to lowest terms.

a) $\dfrac{2}{3y} + \dfrac{4}{3y}$ **b)** $\dfrac{2x}{x+2} + \dfrac{4}{x+2}$

c) $\dfrac{x^2 + 2x}{(x-1)(x+3)} - \dfrac{2x+1}{(x-1)(x+3)}$

study tip

Eliminate the obvious distractions when you study. Disconnect the telephone and put away newspapers, magazines, and unfinished projects. Even the sight of a textbook from another class might keep reminding you of how far behind you are in that class.

Solution

a) $\dfrac{2}{3y} + \dfrac{4}{3y} = \dfrac{6}{3y}$ Add the fractions.

$= \dfrac{2}{y}$ Reduce.

b) $\dfrac{2x}{x+2} + \dfrac{4}{x+2} = \dfrac{2x+4}{x+2}$ Add the fractions.

$= \dfrac{2(x+2)}{x+2}$ Factor the numerator.

$= 2$ Reduce.

c) $\dfrac{x^2 + 2x}{(x-1)(x+3)} - \dfrac{2x+1}{(x-1)(x+3)} = \dfrac{x^2 + 2x - (2x+1)}{(x-1)(x+3)}$ Subtract the fractions.

$= \dfrac{x^2 + 2x - 2x - 1}{(x-1)(x+3)}$ Remove parentheses.

$= \dfrac{x^2 - 1}{(x-1)(x+3)}$ Combine like terms.

$= \dfrac{(x-1)(x+1)}{(x-1)(x+3)}$ Factor.

$= \dfrac{x+1}{x+3}$ Reduce. ■

CAUTION When subtracting a numerator containing more than one term, be sure to enclose it in parentheses, as in Example 3(c). Because that numerator is a binomial, the sign of each of its terms must be changed for the subtraction.

In the next example the rational expressions have different denominators.

EXAMPLE 4 **Rational expressions with different denominators**

Perform the indicated operations.

a) $\dfrac{5}{2x} + \dfrac{2}{3}$ **b)** $\dfrac{4}{x^3y} + \dfrac{2}{xy^3}$ **c)** $\dfrac{a+1}{6} - \dfrac{a-2}{8}$

Solution

a) The LCD for $2x$ and 3 is $6x$:

$$\frac{5}{2x} + \frac{2}{3} = \frac{5 \cdot 3}{2x \cdot 3} + \frac{2 \cdot 2x}{3 \cdot 2x} \quad \text{Build up both denominators to } 6x.$$

$$= \frac{15}{6x} + \frac{4x}{6x} \quad \text{Simplify numerators and denominators.}$$

$$= \frac{15 + 4x}{6x} \quad \text{Add the rational expressions.}$$

b) The LCD is x^3y^3.

$$\frac{4}{x^3y} + \frac{2}{xy^3} = \frac{4 \cdot y^2}{x^3y \cdot y^2} + \frac{2 \cdot x^2}{xy^3 \cdot x^2} \quad \text{Build up both denominators to the LCD.}$$

$$= \frac{4y^2}{x^3y^3} + \frac{2x^2}{x^3y^3} \quad \text{Simplify numerators and denominators.}$$

$$= \frac{4y^2 + 2x^2}{x^3y^3} \quad \text{Add the rational expressions.}$$

c) Because $6 = 2 \cdot 3$ and $8 = 2^3$, the LCD is $2^3 \cdot 3$, or 24:

$$\frac{a+1}{6} - \frac{a-2}{8} = \frac{(a+1)4}{6 \cdot 4} - \frac{(a-2)3}{8 \cdot 3} \quad \text{Build up both denominators to the LCD 24.}$$

$$= \frac{4a+4}{24} - \frac{3a-6}{24} \quad \text{Simplify numerators and denominators.}$$

$$= \frac{4a + 4 - (3a - 6)}{24} \quad \text{Subtract the rational expressions.}$$

$$= \frac{4a + 4 - 3a + 6}{24} \quad \text{Remove the parentheses.}$$

$$= \frac{a + 10}{24} \quad \text{Combine like terms.}$$

helpful hint

You can remind yourself of the difference between addition and multiplication of fractions with a simple example: If you and your spouse each own $\frac{1}{7}$ of Microsoft, then together you own $\frac{2}{7}$ of Microsoft. If you own $\frac{1}{7}$ of Microsoft, and give $\frac{1}{7}$ of your stock to your child, then your child owns $\frac{1}{49}$ of Microsoft.

EXAMPLE 5 **Rational expressions with different denominators**

Perform the indicated operations:

a) $\frac{1}{x^2 - 9} + \frac{2}{x^2 + 3x}$ **b)** $\frac{4}{5 - a} - \frac{2}{a - 5}$

Solution

a)

$$\frac{1}{x^2 - 9} + \frac{2}{x^2 + 3x} = \underbrace{\frac{1}{(x-3)(x+3)}}_{\text{Needs } x} + \underbrace{\frac{2}{x(x+3)}}_{\text{Needs } x - 3} \quad \text{The LCD is } x(x-3)(x+3).$$

$$= \frac{1 \cdot x}{(x-3)(x+3)x} + \frac{2(x-3)}{x(x+3)(x-3)}$$

$$= \frac{x}{x(x-3)(x+3)} + \frac{2x - 6}{x(x-3)(x+3)}$$

$$= \frac{3x - 6}{x(x-3)(x+3)} \quad \text{We usually leave the denominator in factored form.}$$

helpful hint

Once the denominators are factored as in Example 5(a), you can simply look at each denominator and ask, "What factor does the other denominator(s) have that is missing from this one." Then use the missing factor to build up the denominator. Repeat until all denominators are identical and you will have the LCD.

b) Because $-1(5 - a) = a - 5$, we can get identical denominators by multiplying only the first expression by -1 in the numerator and denominator:

$$\frac{4}{5 - a} - \frac{2}{a - 5} = \frac{4(-1)}{(5 - a)(-1)} - \frac{2}{a - 5}$$

$$= \frac{-4}{a - 5} - \frac{2}{a - 5}$$

$$= \frac{-6}{a - 5} \quad -4 - 2 = -6$$

$$= -\frac{6}{a - 5}$$

In the next example we combine three rational expressions by addition and subtraction.

EXAMPLE 6 **Rational expressions with different denominators**

Perform the indicated operations.

$$\frac{x + 1}{x^2 + 2x} + \frac{2x + 1}{6x + 12} - \frac{1}{6}$$

Solution

The LCD for $x(x + 2)$, $6(x + 2)$, and 6 is $6x(x + 2)$.

$$\frac{x + 1}{x^2 + 2x} + \frac{2x + 1}{6x + 12} - \frac{1}{6} = \frac{x + 1}{x(x + 2)} + \frac{2x + 1}{6(x + 2)} - \frac{1}{6} \quad \text{Factor denominators.}$$

$$= \frac{6(x + 1)}{6x(x + 2)} + \frac{x(2x + 1)}{6x(x + 2)} - \frac{1x(x + 2)}{6x(x + 2)} \quad \text{Build up to the LCD.}$$

$$= \frac{6x + 6}{6x(x + 2)} + \frac{2x^2 + x}{6x(x + 2)} - \frac{x^2 + 2x}{6x(x + 2)} \quad \text{Simplify numerators.}$$

$$= \frac{6x + 6 + 2x^2 + x - x^2 - 2x}{6x(x + 2)} \quad \text{Combine the numerators.}$$

$$= \frac{x^2 + 5x + 6}{6x(x + 2)} \quad \text{Combine like terms.}$$

$$= \frac{(x + 3)(x + 2)}{6x(x + 2)} \quad \text{Factor.}$$

$$= \frac{x + 3}{6x} \quad \text{Reduce.}$$

Applications

Rational expressions occur in problems involving rates. In the next two examples we have situations where we must add rational expressions.

EXAMPLE 7 **Adding time**

Alice drove at a rate of x miles per hour for the first 20 miles and then increased her speed to $x + 10$ miles per hour for the next 40 miles. Write a rational expression for the total time of the trip.

Solution

Time is distance divided by rate. So her time for the first part of the trip was $20/x$ hours and her time for the last part of the trip was $40/(x + 10)$ hours. Find her total time as follows.

$$\frac{20}{x} + \frac{40}{x + 10} = \frac{20(x + 10)}{x(x + 10)} + \frac{40 \cdot x}{(x + 10)x} = \frac{60x + 200}{x(x + 10)}$$

So the total time is $\frac{60x + 200}{x(x + 10)}$ hours.

EXAMPLE 8

Adding work

Harry takes twice as long as Lucy to proofread a manuscript. Write a rational expression for the amount of work they do in 3 hours working together on a manuscript.

Solution

Let x = the number of hours it would take Lucy to complete the manuscript alone and $2x$ = the number of hours it would take Harry to complete the manuscript alone. Make a table showing rate, time, and work completed:

	Rate	Time	Work
Lucy	$\frac{1}{x}\frac{\text{msp}}{\text{hr}}$	3 hr	$\frac{3}{x}$ msp
Harry	$\frac{1}{2x}\frac{\text{msp}}{\text{hr}}$	3 hr	$\frac{3}{2x}$ msp

Now find the sum of each person's work.

$$\begin{aligned}\frac{3}{x} + \frac{3}{2x} &= \frac{2 \cdot 3}{2 \cdot x} + \frac{3}{2x} \\ &= \frac{6}{2x} + \frac{3}{2x} \\ &= \frac{9}{2x}\end{aligned}$$

So in 3 hours working together they will complete $\frac{9}{2x}$ of the manuscript.

WARM-UPS

True or false? Explain your answer.

1. $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$
2. $\frac{7}{12} - \frac{1}{12} = \frac{1}{2}$
3. $\frac{3}{5} + \frac{4}{3} = \frac{29}{15}$
4. $\frac{4}{5} - \frac{5}{7} = \frac{3}{35}$
5. $\frac{5}{20} + \frac{3}{4} = 1$
6. $\frac{2}{x} + 1 = \frac{3}{x}$ for any nonzero value of x.

WARM-UPS

(continued)

7. $1 + \frac{1}{a} = \frac{a + 1}{a}$ for any nonzero value of a.

8. $a - \frac{1}{4} = \frac{3}{4}a$ for any value of a.

9. $\frac{a}{2} + \frac{b}{3} = \frac{3a + 2b}{6}$ for any values of a and b.

10. The LCD for the rational expressions $\frac{1}{x}$ and $\frac{3x}{x - 1}$ is $x^2 - 1$.

7.4 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. How do you add or subtract rational numbers?

2. How do you add or subtract rational expressions?

3. What is the least common denominator?

4. Why do we use the *least* common denominator when adding rational expressions?

Perform the indicated operation. Reduce each answer to lowest terms. See Example 1.

5. $\frac{1}{10} + \frac{1}{10}$

6. $\frac{1}{8} + \frac{3}{8}$

7. $\frac{7}{8} - \frac{1}{8}$

8. $\frac{4}{9} - \frac{1}{9}$

9. $\frac{1}{6} - \frac{5}{6}$

10. $-\frac{3}{8} - \frac{7}{8}$

11. $-\frac{7}{8} + \frac{1}{8}$

12. $-\frac{9}{20} + \left(-\frac{3}{20}\right)$

Perform the indicated operation. Reduce each answer to lowest terms. See Example 2.

13. $\frac{1}{3} + \frac{2}{9}$

14. $\frac{1}{4} + \frac{5}{6}$

15. $\frac{7}{16} + \frac{5}{18}$

16. $\frac{7}{6} + \frac{4}{15}$

17. $\frac{1}{8} - \frac{9}{10}$

18. $\frac{2}{15} - \frac{5}{12}$

19. $-\frac{1}{6} - \left(-\frac{3}{8}\right)$

20. $-\frac{1}{5} - \left(-\frac{1}{7}\right)$

Perform the indicated operation. Reduce each answer to lowest terms. See Example 3.

21. $\frac{3}{2w} + \frac{7}{2w}$

22. $\frac{5x}{3y} + \frac{7x}{3y}$

23. $\frac{3a}{a + 5} + \frac{15}{a + 5}$

24. $\frac{a + 7}{a - 4} + \frac{9 - 5a}{a - 4}$

25. $\frac{q - 1}{q - 4} - \frac{3q - 9}{q - 4}$

26. $\frac{3 - a}{3} - \frac{a - 5}{3}$

27. $\frac{4h - 3}{h(h + 1)} - \frac{h - 6}{h(h + 1)}$

28. $\frac{2t - 9}{t(t - 3)} - \frac{t - 9}{t(t - 3)}$

29. $\frac{x^2 - x - 5}{(x + 1)(x + 2)} + \frac{1 - 2x}{(x + 1)(x + 2)}$

30. $\frac{2x - 5}{(x - 2)(x + 6)} + \frac{x^2 - 2x + 1}{(x - 2)(x + 6)}$

Perform the indicated operation. Reduce each answer to lowest terms. See Example 4.

31. $\frac{3}{2a} + \frac{1}{5a}$

32. $\frac{5}{6y} - \frac{3}{8y}$

33. $\frac{w - 3}{9} - \frac{w - 4}{12}$

34. $\frac{y + 4}{10} - \frac{y - 2}{14}$

35. $\frac{b^2}{4a} - c$

36. $y + \frac{3}{7b}$

37. $\frac{2}{wz^2} + \frac{3}{w^2z}$

38. $\frac{1}{a^5b} - \frac{5}{ab^3}$

Perform the indicated operation. Reduce each answer to lowest terms. See Examples 5 and 6.

39. $\frac{2}{x + 1} - \frac{3}{x}$

40. $\frac{1}{a - 1} - \frac{2}{a}$

41. $\frac{2}{a - b} + \frac{1}{a + b}$

42. $\frac{3}{x + 1} + \frac{2}{x - 1}$

43. $\frac{3}{x^2 + x} - \frac{4}{5x + 5}$

44. $\frac{3}{a^2 + 3a} - \frac{2}{5a + 15}$

45. $\frac{2a}{a^2 - 9} + \frac{a}{a - 3}$

46. $\frac{x}{x^2 - 1} + \frac{3}{x - 1}$

47. $\frac{4}{a - b} + \frac{4}{b - a}$

48. $\frac{2}{x - 3} + \frac{3}{3 - x}$

49. $\frac{3}{2a - 2} - \frac{2}{1 - a}$

50. $\frac{5}{2x - 4} - \frac{3}{2 - x}$

51. $\frac{1}{x^2 - 4} - \frac{3}{x^2 - 3x - 10}$

52. $\frac{2x}{x^2 - 9} + \frac{3x}{x^2 + 4x + 3}$

53. $\frac{3}{x^2 + x - 2} + \frac{4}{x^2 + 2x - 3}$

54. $\frac{x - 1}{x^2 - x - 12} + \frac{x + 4}{x^2 + 5x + 6}$

55. $\frac{2}{x} - \frac{1}{x - 1} + \frac{1}{x + 2}$

56. $\frac{1}{a} - \frac{2}{a + 1} + \frac{3}{a - 1}$

57. $\frac{5}{3a - 9} - \frac{3}{2a} + \frac{4}{a^2 - 3a}$

58. $\frac{3}{4c + 2} - \frac{c - 4}{2c^2 + c} - \frac{5}{6c}$

Solve each problem. See Examples 7 and 8.

59. ***Perimeter of a rectangle.*** Suppose that the length of a rectangle is $\frac{3}{x}$ feet and its width is $\frac{5}{2x}$ feet. Find a rational expression for the perimeter of the rectangle.

60. ***Perimeter of a triangle.*** The lengths of the sides of a triangle are $\frac{1}{x}$, $\frac{1}{2x}$, and $\frac{2}{3x}$ meters. Find a rational expression for the perimeter of the triangle.

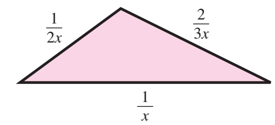

FIGURE FOR EXERCISE 60

61. ***Traveling time.*** Janet drove 120 miles at x mph before 6:00 A.M. After 6:00 A.M., she increased her speed by 5 mph and drove 195 additional miles. Use the fact that $T = \frac{D}{R}$ to complete the following table.

	Rate	Time	Distance
Before	$x \frac{\text{mi}}{\text{hr}}$		120 mi
After	$x + 5 \frac{\text{mi}}{\text{hr}}$		195 mi

Write a rational expression for her total traveling time. Evaluate the expression for $x = 60$.

62. ***Traveling time.*** After leaving Moose Jaw, Hanson drove 200 kilometers at x km/hr and then decreased his speed by 20 km/hr and drove 240 additional kilometers. Make a table like the one in Exercise 61. Write a rational expression for his total traveling time. Evaluate the expression for $x = 100$.

63. ***House painting.*** Kent can paint a certain house by himself in x days. His helper Keith can paint the same house by himself in $x + 3$ days. Suppose that they work together on the job for 2 days. To complete the table, use the fact that the work completed is the product of the rate and the time. Write a rational expression for the

	Rate	Time	Work
Kent	$\frac{1}{x} \frac{\text{job}}{\text{day}}$	2 days	
Keith	$\frac{1}{x + 3} \frac{\text{job}}{\text{day}}$	2 days	

fraction of the house that they complete by working together for 2 days. Evaluate the expression for $x = 6$.

64. ***Barn painting.*** Melanie can paint a certain barn by herself in x days. Her helper Melissa can paint the same barn by herself in $2x$ days. Write a rational expression for the fraction of the barn that they complete in one day by working together. Evaluate the expression for $x = 5$.

FIGURE FOR EXERCISE 64

GETTING MORE INVOLVED

65. ***Writing.*** Write a step-by-step procedure for adding rational expressions.

66. ***Writing.*** Explain why fractions must have the same denominator to be added. Use real-life examples.

7.5 COMPLEX FRACTIONS

In this section

- Complex Fractions
- Using the LCD to Simplify Complex Fractions
- Applications

In this section we will use the idea of least common denominator to simplify complex fractions. Also we will see how complex fractions can arise in applications.

Complex Fractions

A **complex fraction** is a fraction having rational expressions in the numerator, denominator, or both. Consider the following complex fraction:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{1}{4}-\frac{5}{8}} \quad \begin{array}{l}\leftarrow \text{Numerator of complex fraction}\\ \leftarrow \text{Denominator of complex fraction}\end{array}$$

To simplify it, we can combine the fractions in the numerator as follows:

$$\frac{1}{2}+\frac{2}{3}=\frac{1\cdot 3}{2\cdot 3}+\frac{2\cdot 2}{3\cdot 2}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$$

We can combine the fractions in the denominator as follows:

$$\frac{1}{4}-\frac{5}{8}=\frac{1\cdot 2}{4\cdot 2}-\frac{5}{8}=\frac{2}{8}-\frac{5}{8}=-\frac{3}{8}$$

Now divide the numerator by the denominator:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{1}{4}-\frac{5}{8}}=\frac{\frac{7}{6}}{-\frac{3}{8}}=\frac{7}{6}\div\left(-\frac{3}{8}\right)$$

$$=\frac{7}{6}\cdot\left(-\frac{8}{3}\right)$$

$$=-\frac{56}{18}$$

$$=-\frac{28}{9}$$