## GETTING MORE INVOLVED

**43.** ***Exploration.*** Simplify

$$\frac{1}{1+\frac{1}{2}},\quad \frac{1}{1+\frac{1}{1+\frac{1}{2}}},\quad \text{and}\quad \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}.$$

**a)** Are these fractions getting larger or smaller as the fractions become more complex?
**b)** Continuing the pattern, find the next two complex fractions and simplify them.
**c)** Now what can you say about the values of all five complex fractions?

**44.** ***Discussion.*** A complex fraction can be simplified by writing the numerator and denominator as single fractions and then dividing them or by multiplying the numerator and denominator by the LCD. Simplify the complex fraction

$$\frac{\frac{4}{xy^2}-\frac{6}{xy}}{\frac{2}{x^2}+\frac{4}{x^2y}}$$

by using each of these methods. Compare the number of steps used in each method, and determine which method requires fewer steps.

# 7.6 SOLVING EQUATIONS WITH RATIONAL EXPRESSIONS

**In this section**

- Equations with Rational Expressions
- Extraneous Solutions

Many problems in algebra can be solved by using equations involving rational expressions. In this section you will learn how to solve equations that involve rational expressions, and in Sections 7.7 and 7.8 you will solve problems using these equations.

## Equations with Rational Expressions

We solved some equations involving fractions in Section 2.3. In that section the equations had only integers in the denominators. Our first step in solving those equations was to multiply by the LCD to eliminate all of the denominators.

**EXAMPLE 1** **Integers in the denominators**

Solve $\frac{1}{2}-\frac{x-2}{3}=\frac{1}{6}$.

### Solution

The LCD for 2, 3, and 6 is 6. Multiply each side of the equation by 6:

$$\frac{1}{2}-\frac{x-2}{3}=\frac{1}{6} \qquad \text{Original equation}$$

$$6\left(\frac{1}{2}-\frac{x-2}{3}\right)=6\cdot\frac{1}{6} \qquad \text{Multiply each side by 6.}$$

$$6\cdot\frac{1}{2}-\overset{2}{\cancel{6}}\cdot\frac{x-2}{\cancel{3}}=\cancel{6}\cdot\frac{1}{\cancel{6}} \qquad \text{Distributive property}$$

$$3-2(x-2)=1 \qquad \text{Simplify. Enclose } x-2 \text{ in parentheses.}$$

$$3-2x+4=1 \qquad \text{Distributive property}$$

$$-2x=-6 \qquad \text{Subtract 7 from each side.}$$

$$x=3 \qquad \text{Divide each side by } -2.$$

**helpful hint**

Note that it is not necessary to convert each fraction into an equivalent fraction with a common denominator here. Since we can multiply both sides of an equation by any expression we choose, we choose to multiply by the LCD. This tactic eliminates the fractions in one step.

Check $x = 3$ in the original equation:

$$\frac{1}{2} - \frac{3-2}{3} = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

The solution to the equation is 3. ■

**CAUTION** When a numerator contains a binomial, as in Example 1, the numerator must be enclosed in parentheses when the denominator is eliminated.

To solve an equation involving rational expressions, we usually multiply each side of the equation by the LCD for all the denominators involved, just as we do for an equation with fractions.

**EXAMPLE 2** **Variables in the denominators**

Solve $\frac{1}{x} + \frac{1}{6} = \frac{1}{4}$.

## Solution

We multiply each side of the equation by $12x$, the LCD for 4, 6, and $x$:

$$\frac{1}{x} + \frac{1}{6} = \frac{1}{4} \quad \text{Original equation}$$

$$12x\left(\frac{1}{x} + \frac{1}{6}\right) = 12x\left(\frac{1}{4}\right) \quad \text{Multiply each side by } 12x.$$

$$12\cancel{x} \cdot \frac{1}{\cancel{x}} + \overset{2}{\cancel{12}}x \cdot \frac{1}{\cancel{6}} = \overset{3}{\cancel{12}}x \cdot \frac{1}{\cancel{4}} \quad \text{Distributive property}$$

$$12 + 2x = 3x \quad \text{Simplify.}$$

$$12 = x \quad \text{Subtract } 2x \text{ from each side.}$$

Check that 12 satisfies the original equation:

$$\frac{1}{12} + \frac{1}{6} = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} = \frac{1}{4}$$

■

**EXAMPLE 3** **An equation with two solutions**

Solve the equation $\frac{100}{x} + \frac{100}{x+5} = 9$.

## Solution

The LCD for the denominators $x$ and $x + 5$ is $x(x + 5)$:

$$\frac{100}{x} + \frac{100}{x+5} = 9 \quad \text{Original equation}$$

$$x(x+5)\frac{100}{x} + x(x+5)\frac{100}{x+5} = x(x+5)9 \quad \text{Multiply each side by } x(x+5).$$

$$(x+5)100 + x(100) = (x^2 + 5x)9 \quad \text{All denominators are eliminated.}$$

$$100x + 500 + 100x = 9x^2 + 45x \quad \text{Simplify.}$$

$$500 + 200x = 9x^2 + 45x$$

$$0 = 9x^2 - 155x - 500 \quad \text{Get 0 on one side.}$$

$$0 = (9x + 25)(x - 20) \quad \text{Factor.}$$

$$9x + 25 = 0 \quad \text{or} \quad x - 20 = 0 \quad \text{Zero factor property}$$

$$x = -\frac{25}{9} \quad \text{or} \quad x = 20$$

A check will show that both $-\frac{25}{9}$ and 20 satisfy the original equation. ■

**study tip**

Your mood for studying should match the mood in which you are tested. Being too relaxed during studying will not match the increased level of activation you attain during a test. Likewise, if you get too tensed-up during a test, you will not do well because your test-taking mood will not match your studying mood.

## Extraneous Solutions

In a rational expression we can replace the variable only by real numbers that do not cause the denominator to be 0. When solving equations involving rational expressions, we must check every solution to see whether it causes 0 to appear in a denominator. If a number causes the denominator to be 0, then it cannot be a solution to the equation. A number that appears to be a solution but causes 0 in a denominator is called an **extraneous solution.**

**EXAMPLE 4** **An equation with an extraneous solution**

Solve the equation $\frac{1}{x-2} = \frac{x}{2x-4} + 1$.

### Solution

Because the denominator $2x - 4$ factors as $2(x - 2)$, the LCD is $2(x - 2)$.

$$2(x-2)\frac{1}{x-2} = 2(x-2)\frac{x}{2(x-2)} + 2(x-2)\cdot 1 \quad \text{Multiply each side of the original equation by } 2(x-2).$$

$$2 = x + 2x - 4 \quad \text{Simplify.}$$

$$2 = 3x - 4$$

$$6 = 3x$$

$$2 = x$$

Check 2 in the original equation:

$$\frac{1}{2-2} = \frac{2}{2\cdot 2 - 4} + 1$$

The denominator $2 - 2$ is 0. So 2 does not satisfy the equation, and it is an extraneous solution. The equation has no solutions.

**EXAMPLE 5** **Another extraneous solution**

Solve the equation $\frac{1}{x} + \frac{1}{x-3} = \frac{x-2}{x-3}$.

### Solution

The LCD for the denominators $x$ and $x - 3$ is $x(x - 3)$:

$$\frac{1}{x} + \frac{1}{x-3} = \frac{x-2}{x-3} \quad \text{Original equation}$$

$$x(x-3)\cdot\frac{1}{x} + x(x-3)\cdot\frac{1}{x-3} = x(x-3)\cdot\frac{x-2}{x-3} \quad \text{Multiply each side by } x(x-3).$$

$$x - 3 + x = x(x-2)$$

$$2x - 3 = x^2 - 2x$$

$$0 = x^2 - 4x + 3$$

$$0 = (x-3)(x-1)$$

$$x - 3 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 3 \quad \text{or} \quad x = 1$$

If $x = 3$, then the denominator $x - 3$ has a value of 0. If $x = 1$, the original equation is satisfied. The only solution to the equation is 1.

**CAUTION** Always be sure to check your answers in the original equation to determine whether they are extraneous solutions.

## WARM-UPS

**True or false? Explain your answers.**

1. The LCD is not used in solving equations with rational expressions.
2. To solve the equation $x^2 = 8x$, we divide each side by $x$.
3. An extraneous solution is an irrational number.

Use the following equations for Questions 4–10.

**a)** $\frac{3}{x} + \frac{5}{x-2} = \frac{2}{3}$ **b)** $\frac{1}{x} + \frac{1}{2} = \frac{3}{4}$ **c)** $\frac{1}{x-1} + 2 = \frac{1}{x+1}$

4. To solve Eq. (a), we must add the expressions on the left-hand side.
5. Both 0 and 2 satisfy Eq. (a).
6. To solve Eq. (a), we multiply each side by $3x^2 - 6x$.
7. The only solution to Eq. (b) is 4.
8. Equation (b) is equivalent to $4 + 2x = 3x$.
9. To solve Eq. (c), we multiply each side by $x^2 - 1$.
10. The numbers 1 and $-1$ do not satisfy Eq. (c).

## 7.6 EXERCISES

***Reading and Writing*** *After reading this section, write out the answers to these questions. Use complete sentences.*

**1.** What is the typical first step for solving an equation involving rational expressions?

**2.** What is the difference in procedure for solving an equation involving rational expressions and adding rational expressions?

**3.** What is an extraneous solution?

**4.** Why do extraneous solutions sometimes occur for equations with rational expressions?

*Solve each equation. See Example 1.*

**5.** $\frac{x}{3} - 5 = \frac{x}{2} - 7$

**6.** $\frac{x}{3} - \frac{x}{2} = \frac{x}{5} - 11$

**7.** $\frac{y}{5} - \frac{2}{3} = \frac{y}{6} + \frac{1}{3}$

**8.** $\frac{z}{6} + \frac{5}{4} = \frac{z}{2} - \frac{3}{4}$

**9.** $\frac{3}{4} - \frac{t-4}{3} = \frac{t}{12}$

**10.** $\frac{4}{5} - \frac{v-1}{10} = \frac{v-5}{30}$

**11.** $\frac{1}{5} - \frac{w+10}{15} = \frac{1}{10} - \frac{w+1}{6}$

**12.** $\frac{q}{5} - \frac{q-1}{2} = \frac{13}{20} - \frac{q+1}{4}$

*Solve each equation. See Example 2.*

**13.** $\frac{1}{x} + \frac{1}{2} = \frac{3}{4}$

**14.** $\frac{3}{x} + \frac{1}{4} = \frac{5}{8}$

**15.** $\frac{2}{3x} + \frac{1}{2x} = \frac{7}{24}$

**16.** $\frac{1}{6x} - \frac{1}{8x} = \frac{1}{72}$

**17.** $\frac{1}{2} + \frac{a-2}{a} = \frac{a+2}{2a}$

**18.** $\frac{1}{b} + \frac{1}{5} = \frac{b-1}{5b} + \frac{3}{10}$

**19.** $\frac{1}{3} - \frac{k+3}{6k} = \frac{1}{3k} - \frac{k-1}{2k}$

**20.** $\frac{3}{p} - \frac{p+3}{3p} = \frac{2p-1}{2p} - \frac{5}{6}$

*Solve each equation. See Example 3.*

**21.** $\frac{x}{2} = \frac{5}{x + 3}$

**22.** $\frac{x}{3} = \frac{4}{x + 1}$

**23.** $\frac{2}{x + 1} = \frac{1}{x} + \frac{1}{6}$

**24.** $\frac{1}{w + 1} - \frac{1}{2w} = \frac{3}{40}$

**25.** $\frac{a - 1}{a^2 - 4} + \frac{1}{a - 2} = \frac{a + 4}{a + 2}$

**26.** $\frac{b + 17}{b^2 - 1} - \frac{1}{b + 1} = \frac{b - 2}{b - 1}$

*Solve each equation. Watch for extraneous solutions. See Examples 4 and 5.*

**27.** $\frac{1}{x - 1} + \frac{2}{x} = \frac{x}{x - 1}$

**28.** $\frac{4}{x} + \frac{3}{x - 3} = \frac{x}{x - 3} - \frac{1}{3}$

**29.** $\frac{5}{x + 2} + \frac{2}{x - 3} = \frac{x - 1}{x - 3}$

**30.** $\frac{6}{y - 2} + \frac{7}{y - 8} = \frac{y - 1}{y - 8}$

**31.** $1 + \frac{3y}{y - 2} = \frac{6}{y - 2}$

**32.** $\frac{5}{y - 3} = \frac{y + 7}{2y - 6} + 1$

**33.** $\frac{z}{z + 1} - \frac{1}{z + 2} = \frac{2z + 5}{z^2 + 3z + 2}$

**34.** $\frac{z}{z - 2} - \frac{1}{z + 5} = \frac{7}{z^2 + 3z - 10}$

*In Exercises 35–56, solve each equation.*

**35.** $\frac{a}{4} = \frac{5}{2}$

**36.** $\frac{y}{3} = \frac{6}{5}$

**37.** $\frac{w}{6} = \frac{3w}{11}$

**38.** $\frac{2m}{3} = \frac{3m}{2}$

**39.** $\frac{5}{x} = \frac{x}{5}$

**40.** $\frac{-3}{x} = \frac{x}{-3}$

**41.** $\frac{x - 3}{5} = \frac{x - 3}{x}$

**42.** $\frac{a + 4}{2} = \frac{a + 4}{a}$

**43.** $\frac{1}{x + 2} = \frac{x}{x + 2}$

**44.** $\frac{-3}{w + 2} = \frac{w}{w + 2}$

**45.** $\frac{1}{2x - 4} + \frac{1}{x - 2} = \frac{3}{2}$

**46.** $\frac{7}{3x - 9} - \frac{1}{x - 3} = \frac{4}{3}$

**47.** $\frac{3}{a^2 - a - 6} = \frac{2}{a^2 - 4}$

**48.** $\frac{8}{a^2 + a - 6} = \frac{6}{a^2 - 9}$

**49.** $\frac{4}{c - 2} - \frac{1}{2 - c} = \frac{25}{c + 6}$

**50.** $\frac{3}{x + 1} - \frac{1}{1 - x} = \frac{10}{x^2 - 1}$

**51.** $\frac{1}{x^2 - 9} + \frac{3}{x + 3} = \frac{4}{x - 3}$

**52.** $\frac{3}{x - 2} - \frac{5}{x + 3} = \frac{1}{x^2 + x - 6}$

**53.** $\frac{3}{2x + 4} - \frac{1}{x + 2} = \frac{1}{3x + 1}$

**54.** $\frac{5}{2m + 6} - \frac{1}{m + 1} = \frac{1}{m + 3}$

**55.** $\frac{2t - 1}{3t + 3} + \frac{3t - 1}{6t + 6} = \frac{t}{t + 1}$

**56.** $\frac{4w - 1}{3w + 6} - \frac{w - 1}{3} = \frac{w - 1}{w + 2}$

*Solve each problem.*

**57.** ***Lens equation.*** The focal length $f$ for a camera lens is related to the object distance $o$ and the image distance $i$ by the formula

$$\frac{1}{f} = \frac{1}{o} + \frac{1}{i}.$$

See the accompanying figure. The image is in focus at distance $i$ from the lens. For an object that is 600 mm from a 50-mm lens, use $f = 50$ mm and $o = 600$ mm to find $i$.

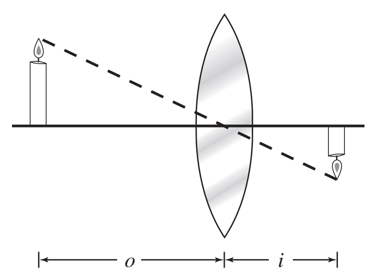


**FIGURE FOR EXERCISE 57**

**58.** ***Telephoto lens.*** Use the formula from Exercise 57 to find the image distance $i$ for an object that is 2,000,000 mm from a 250-mm telephoto lens.

**FIGURE FOR EXERCISE 58**