7.7 APPLICATIONS OF RATIOS AND PROPORTIONS

In this section

- Ratios
- Proportions

In this section we will use the ideas of rational expressions in ratio and proportion problems. We will solve proportions in the same way we solved equations in Section 7.6.

Ratios

In Chapter 1 we defined a rational number as the *ratio of two integers.* We will now give a more general definition of ratio. If a and b are any real numbers (not just integers), with $b \neq 0$, then the expression $\frac{a}{b}$ is called the **ratio of *a* and *b*** or the **ratio of *a* to *b*.** The ratio of a to b is also written as $a:b$. A ratio is a comparison of two numbers. Some examples of ratios are

$$\frac{3}{4},\quad \frac{4.2}{2.1},\quad \frac{\frac{1}{4}}{\frac{1}{2}},\quad \frac{3.6}{5},\quad \text{and}\quad \frac{100}{1}.$$

Ratios are treated just like fractions. We can reduce ratios, and we can build them up. We generally express ratios as ratios of integers. When possible, we will convert a ratio into an equivalent ratio of integers in lowest terms.

EXAMPLE 1 **Finding equivalent ratios**

Find an equivalent ratio of integers in lowest terms for each ratio.

a) $\dfrac{4.2}{2.1}$ **b)** $\dfrac{\frac{1}{4}}{\frac{1}{2}}$ **c)** $\dfrac{3.6}{5}$

Solution

a) Because both the numerator and the denominator have one decimal place, we will multiply the numerator and denominator by 10 to eliminate the decimals:

$$\frac{4.2}{2.1} = \frac{4.2(10)}{2.1(10)} = \frac{42}{21} = \frac{21 \cdot 2}{21 \cdot 1} = \frac{2}{1}$$

Do not omit the 1 in a ratio.

So the ratio of 4.2 to 2.1 is equivalent to the ratio 2 to 1.

b) This ratio is a complex fraction. We can simplify this expression using the LCD method as shown in Section 7.5. Multiply the numerator and denominator of this ratio by 4:

$$\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{4} \cdot 4}{\frac{1}{2} \cdot 4} = \frac{1}{2}$$

c) We can get a ratio of integers if we multiply the numerator and denominator by 10.

$$\frac{3.6}{5} = \frac{3.6(10)}{5(10)} = \frac{36}{50}$$

$$= \frac{18}{25}$$

Reduce to lowest terms.

In the next example a ratio is used to compare quantities.

EXAMPLE 2 **Nitrogen to potash**

In a 50-pound bag of lawn fertilizer there are 8 pounds of nitrogen and 12 pounds of potash. What is the ratio of nitrogen to potash?

Solution

The nitrogen and potash occur in this fertilizer in the ratio of 8 pounds to 12 pounds:

$$\frac{8}{12} = \frac{2 \cdot \cancel{4}}{3 \cdot \cancel{4}} = \frac{2}{3}$$

So the ratio of nitrogen to potash is 2 to 3. ■

EXAMPLE 3 **Males to females**

In a class of 50 students, there were exactly 20 male students. What was the ratio of males to females in this class?

Solution

Because there were 20 males in the class of 50, there were 30 females. The ratio of males to females was 20 to 30, or 2 to 3. ■

Ratios give us a means of comparing the size of two quantities. For this reason *the numbers compared in a ratio should be expressed in the same units*. For example, if one dog is 24 inches high and another is 1 foot high, then the ratio of their heights is 2 to 1, not 24 to 1.

EXAMPLE 4 **Quantities with different units**

What is the ratio of length to width for a poster with a length of 30 inches and a width of 2 feet?

Solution

Because the width is 2 feet, or 24 inches, the ratio of length to width is 30 to 24. Reduce as follows:

$$\frac{30}{24} = \frac{5 \cdot 6}{4 \cdot 6} = \frac{5}{4}$$

So the ratio of length to width is 5 to 4. ■

study tip

To get the "big picture," survey the chapter that you are studying. Read the headings to get the general idea of the chapter content. Read the chapter summary to see what is important in the chapter. Repeat this survey procedure several times while you are working in a chapter.

Proportions

A **proportion** is any statement expressing the equality of two ratios. The statement

$$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad a:b = c:d$$

is a proportion. In any proportion the numbers in the positions of a and d above are called the **extremes.** The numbers in the positions of b and c above are called the **means.** In the proportion

$$\frac{30}{24} = \frac{5}{4},$$

the means are 24 and 5, and the extremes are 30 and 4. Note that $30 \cdot 4 = 5 \cdot 24$.

If we multiply each side of the proportion

$$\frac{a}{b} = \frac{c}{d}$$

by the LCD, *bd*, we get

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$$

or

$$a \cdot d = b \cdot c.$$

We can express this result by saying that *the product of the extremes is equal to the product of the means.* We call this fact the **extremes-means property** or **cross-multiplying.**

Extremes-Means Property (Cross-Multiplying)

Suppose a, b, c, and d are real numbers with $b \neq 0$ and $d \neq 0$. If

$$\frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

We use the extremes-means property to solve proportions.

EXAMPLE 5

Using the extremes-means property

Solve the proportion $\frac{3}{x} = \frac{5}{x + 5}$ for x.

helpful hint

The extremes-means property or cross-multiplying is nothing new. You can accomplish the same thing by multiplying each side of the equation by the LCD.

Solution

Instead of multiplying each side by the LCD, we use the extremes-means property:

$$\frac{3}{x} = \frac{5}{x + 5} \quad \text{Original proportion}$$

$$3(x + 5) = 5x \quad \text{Extremes-means property}$$

$$3x + 15 = 5x \quad \text{Distributive property}$$

$$15 = 2x$$

$$\frac{15}{2} = x$$

Check:

$$\frac{3}{\frac{15}{2}} = 3 \cdot \frac{2}{15} = \frac{2}{5}$$

$$\frac{5}{\frac{15}{2} + 5} = \frac{5}{\frac{25}{2}} = 5 \cdot \frac{2}{25} = \frac{2}{5}$$

So $\frac{15}{2}$ is the solution to the equation or the solution to the proportion. ■

EXAMPLE 6 **The capture-recapture proportion**

To estimate the number of catfish in her pond, a catfish farmer caught, tagged, and released 30 of them. Later, only 3 tagged catfish were found in a sample of 500. Estimate the number of catfish in the pond.

Solution

Let x be the number of catfish in the pond. The ratio $\frac{30}{x}$ is the ratio of tagged catfish to the total population. The ratio $\frac{3}{500}$ is the ratio of tagged catfish in the sample to the sample size. If the tagged catfish are well-mixed and the sample is truly random, then these ratios should be equal:

$$\frac{30}{x} = \frac{3}{500}$$

$$3x = 15{,}000 \quad \text{Extremes-means property}$$

$$x = 5000$$

So there are approximately 5000 catfish in the pond. ■

Note that any proportion can be solved by multiplying each side by the LCD as we did when we solved other equations involving rational expressions. The extremes-means property gives us a shortcut for solving proportions.

EXAMPLE 7 **Solving a proportion**

In a conservative portfolio the ratio of the amount invested in bonds to the amount invested in stocks should be 3 to 1. A conservative investor invested $2850 more in bonds than she did in stocks. How much did she invest in each category?

Solution

Because the ratio of the amount invested in bonds to the amount invested in stocks is 3 to 1, we have

$$\frac{\text{Amount invested in bonds}}{\text{Amount invested in stocks}} = \frac{3}{1}.$$

If x represents the amount invested in stocks and $x + 2850$ represents the amount invested in bonds, then we can write and solve the following proportion:

$$\frac{x + 2850}{x} = \frac{3}{1}$$

$$3x = x + 2850 \quad \text{Extremes-means property}$$

$$2x = 2850$$

$$x = 1425$$

$$x + 2850 = 4275$$

So she invested $4275 in bonds and $1425 in stocks. Note that these amounts are in the ratio of 3 to 1. ■

The next example shows how conversions from one unit of measurement to another can be done by using proportions.

EXAMPLE 8 **Converting measurements**

There are 3 feet in 1 yard. How many feet are there in 12 yards?

Solution

Let x represent the number of feet in 12 yards. There are two proportions that we can write to solve the problem:

$$\frac{3 \text{ feet}}{x \text{ feet}} = \frac{1 \text{ yard}}{12 \text{ yards}} \qquad \frac{3 \text{ feet}}{1 \text{ yard}} = \frac{x \text{ feet}}{12 \text{ yards}}$$

The ratios in the second proportion violate the rule of comparing only measurements that are expressed in the same units. Note that each side of the second proportion is actually the ratio 1 to 1, since 3 feet $=$ 1 yard and x feet $=$ 12 yards. For doing conversions we can use ratios like this to compare measurements in different units. Applying the extremes-means property to either proportion gives

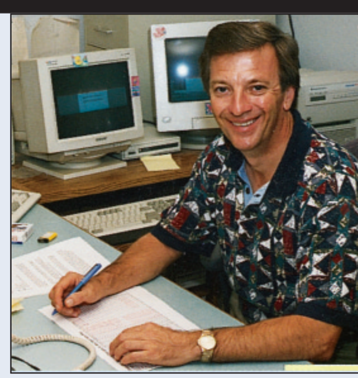
$$3 \cdot 12 = x \cdot 1,$$

or

$$x = 36.$$

So there are 36 feet in 12 yards. ■

MATH AT WORK



SALES ANALYST

Did you ever wonder how your local store calculates how much of your favorite cosmetic to stock on the shelf? Mike Pittman, National Account Manager for a major cosmetic company, is responsible for providing more than 2000 stores across the United States with personal care products such as skin lotions, fragrances, and cosmetics.

Data on what has been sold is transmitted from the point of sale across a number of satellite dishes and computers to Mr. Pittman. The data usually includes size, color, and other pertinent facts. The information is then combined with demographics for certain geographic areas and movement data, as well as advertising and promotional information to answer questions such as: What color is selling best? Is it time to stock sunscreen? Is this a trend-setting area of the country? On the basis of his analysis of these and many other questions, Mr. Pittman recommends changes in packaging, promotional programs, and the quantities of products to be shipped.

Mr. Pittman's job requires a unique blend of sales, marketing, and quantitative skills. Of course, knowledge of computers and an understanding of people help. So the next time you see a whole aisle of personal care and cosmetic products, think of all the information that has been analyzed to put it there.

In Exercise 63 of this section you will see how Mr. Pittman uses a proportion to determine the quantity of mascara needed in a warehouse.

WARM-UPS

True or false? Explain your answer.

1. The ratio of 40 men to 30 women can be expressed as the ratio 4 to 3.
2. The ratio of 3 feet to 2 yards can be expressed as the ratio 3 to 2.
3. If the ratio of men to women in the Chamber of Commerce is 3 to 2 and there are 20 men, then there must be 30 women.
4. The ratio of 1.5 to 2 is equivalent to the ratio of 3 to 4.
5. A statement that two ratios are equal is called a proportion.
6. The product of the extremes is equal to the product of the means.
7. If $\frac{2}{x} = \frac{3}{5}$, then $5x = 6$.
8. The ratio of the height of a 12-inch cactus to the height of a 3-foot cactus is 4 to 1.
9. If 30 out of 100 lawyers preferred aspirin and the rest did not, then the ratio of lawyers that preferred aspirin to those who did not is 30 to 100.
10. If $\frac{x+5}{x} = \frac{2}{3}$, then $3x + 15 = 2x$.

7.7 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a ratio?
2. What are the different ways of expressing a ratio?
3. What are equivalent ratios?
4. What is a proportion?
5. What are the means and what are the extremes?
6. What is the extremes-means property?

For each ratio, find an equivalent ratio of integers in lowest terms. See Example 1.

7. $\frac{2.5}{3.5}$
8. $\frac{4.8}{1.2}$
9. $\frac{0.32}{0.6}$
10. $\frac{0.05}{0.8}$
11. $\frac{35}{10}$
12. $\frac{88}{33}$
13. $\frac{4.5}{7}$
14. $\frac{3}{2.5}$
15. $\frac{\frac{1}{2}}{\frac{1}{5}}$
16. $\frac{\frac{2}{3}}{\frac{3}{4}}$
17. $\frac{5}{\frac{1}{3}}$
18. $\frac{4}{\frac{1}{4}}$

Find a ratio for each of the following, and write it as a ratio of integers in lowest terms. See Examples 2–4.

19. ***Men and women.*** Find the ratio of men to women in a bowling league containing 12 men and 8 women.
20. ***Coffee drinkers.*** Among 100 coffee drinkers, 36 said that they preferred their coffee black and the rest did not prefer their coffee black. Find the ratio of those who prefer black coffee to those who prefer nonblack coffee.

FIGURE FOR EXERCISE 20

21. ***Smokers.*** A life insurance company found that among its last 200 claims, there were six dozen smokers. What is the ratio of smokers to nonsmokers in this group of claimants?

22. ***Hits and misses.*** A woman threw 60 darts and hit the target a dozen times. What is her ratio of hits to misses?

23. ***Violence and kindness.*** While watching television for one week, a consumer group counted 1240 acts of violence and 40 acts of kindness. What is the violence to kindness ratio for television, according to this group?

24. ***Length to width.*** What is the ratio of length to width for the rectangle shown below?

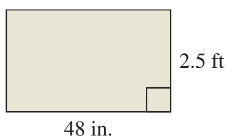


FIGURE FOR EXERCISE 24

25. ***Rise to run.*** What is the ratio of rise to run for the stairway shown in the accompanying figure?

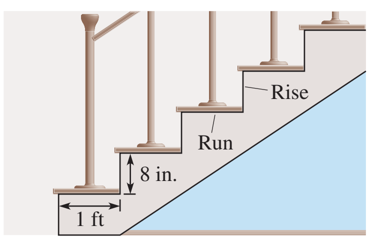


FIGURE FOR EXERCISE 25

26. ***Rise and run.*** If the rise is $\frac{3}{2}$ and the run is 5, then what is the ratio of the rise to the run?

Solve each proportion. See Example 5.

27. $\frac{4}{x} = \frac{2}{3}$

28. $\frac{9}{x} = \frac{3}{2}$

29. $\frac{a}{2} = \frac{-1}{5}$

30. $\frac{b}{3} = \frac{-3}{4}$

31. $-\frac{5}{9} = \frac{3}{x}$

32. $-\frac{3}{4} = \frac{5}{x}$

33. $\frac{x + 2}{x} = \frac{6}{5}$

34. $\frac{x}{x - 3} = \frac{2}{3}$

35. $\frac{7}{5} = \frac{2x + 1}{x + 2}$

36. $\frac{5}{8} = \frac{3x + 1}{2x + 10}$

37. $\frac{10}{x} = \frac{34}{x + 12}$

38. $\frac{x}{3} = \frac{x + 1}{2}$

39. $\frac{a}{a + 1} = \frac{a + 3}{a}$

40. $\frac{c + 3}{c - 1} = \frac{c + 2}{c - 3}$

41. $\frac{m - 1}{m - 2} = \frac{m - 3}{m + 4}$

42. $\frac{h}{h - 3} = \frac{h}{h - 9}$

Use a proportion to solve each problem. See Examples 6–8.

43. ***New shows and reruns.*** The ratio of new shows to reruns on cable TV is 2 to 27. If Frank counted only eight new shows one evening, then how many reruns were there?

44. ***Fast food.*** If four out of five doctors prefer fast food, then at a convention of 445 doctors, how many prefer fast food?

45. ***Voting.*** If 220 out of 500 voters surveyed said that they would vote for the incumbent, then how many votes could the incumbent expect out of the 400,000 voters in the state?

FIGURE FOR EXERCISE 45

46. ***New product.*** A taste test with 200 randomly selected people found that only three of them said that they would buy a box of new Sweet Wheats cereal. How many boxes could the manufacturer expect to sell in a country of 280 million people?

47. ***Basketball blowout.*** As the final buzzer signaled the end of the basketball game, the Lions were 34 points ahead of the Tigers. If the Lions scored 5 points for every 3 scored by the Tigers, then what was the final score?

48. ***The golden ratio.*** The ancient Greeks thought that the most pleasing shape for a rectangle was one for which the ratio of the length to the width was 8 to 5, the golden ratio. If the length of a rectangular painting is 2 ft longer than its width, then for what dimensions would the length and width have the golden ratio?

49. ***Automobile sales.*** The ratio of sports cars to luxury cars sold in Wentworth one month was 3 to 2. If 20 more sports cars were sold than luxury cars, then how many of each were sold that month?

50. ***Foxes and rabbits.*** The ratio of foxes to rabbits in the Deerfield Forest Preserve is 2 to 9. If there are 35 fewer foxes than rabbits, then how many of each are there?

51. ***Inches and feet.*** If there are 12 inches in 1 foot, then how many inches are there in 7 feet?

52. ***Feet and yards.*** If there are 3 feet in 1 yard, then how many yards are there in 28 feet?

53. ***Minutes and hours.*** If there are 60 minutes in 1 hour, then how many minutes are there in 0.25 hour?

54. ***Meters and kilometers.*** If there are 1000 meters in 1 kilometer, then how many meters are there in 2.33 kilometers.

55. ***Miles and hours.*** If Alonzo travels 230 miles in 3 hours, then how many miles does he travel in 7 hours?

56. ***Hiking time.*** If Evangelica can hike 19 miles in 2 days on the Appalachian Trail, then how many days will it take her to hike 63 miles?

57. ***Force on basketball shoes.*** The designers of Converse shoes know that the force exerted on shoe soles in a jump shot is proportional to the weight of the person jumping. If a 70-pound boy exerts a force of 980 pounds on his shoe soles when he returns to the court after a jump, then what force does a 6 ft 8 in. professional ball player weighing 280 pounds exert on the soles of his shoes when he returns to the court after a jump? Use the accompanying graph to estimate the force for a 150-pound player.

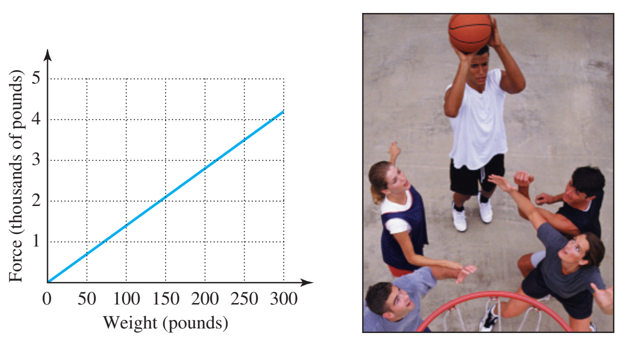


FIGURE FOR EXERCISE 57

58. ***Force on running shoes.*** The designers of Converse shoes know that the ratio of the force on the shoe soles to the weight of a runner is 3 to 1. What force does a 130-pound jogger exert on the soles of her shoes.

59. ***Capture-recapture.*** To estimate the number of trout in Trout Lake, rangers used the capture-recapture method. They caught, tagged, and released 200 trout. One week later, they caught a sample of 150 trout and found that 5 of them were tagged. Assuming that the ratio of tagged trout to the total number of trout in the lake is the same as the ratio of tagged trout in the sample to the number of trout in the sample, find the number of trout in the lake.

60. ***Bear population.*** To estimate the size of the bear population on the Keweenaw Peninsula, conservationists captured, tagged, and released 50 bears. One year later, a random sample of 100 bears included only 2 tagged bears. What is the conservationist's estimate of the size of the bear population?

61. ***Fast-food waste.*** The accompanying figure shows the typical distribution of waste at a fast-food restaurant (U.S. Environmental Protection Agency, www.epa.gov).

a) What is the ratio of customer waste to food waste?

b) If a typical McDonald's generates 67 more pounds of food waste than customer waste per day, then how many pounds of customer waste does it generate?

WASTE GENERATION AT A FAST-FOOD RESTAURANT

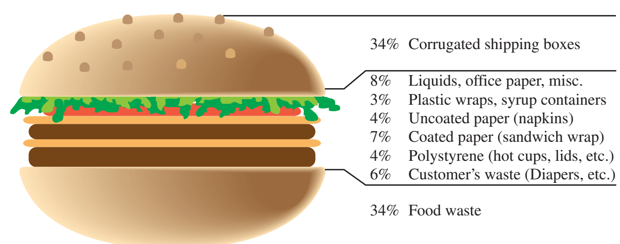


FIGURE FOR EXERCISES 61 AND 62

62. ***Corrugated waste.*** Use the accompanying figure to find the ratio of waste from corrugated shipping boxes to waste not from corrugated shipping boxes. If a typical McDonald's generates 81 pounds of waste per day from corrugated shipping boxes, then how many pounds of waste per day does it generate that is not from corrugated shipping boxes?

63. ***Mascara needs.*** In determining warehouse needs for a particular mascara for a chain of 2000 stores, Mike Pittman first determines a need B based on sales figures for the past 52 weeks. He then determines the actual need A from the equation $\frac{A}{B} = k$, where

$$k = 1 + V + C + X - D.$$

He uses $V = 0.22$ if there is a national TV ad and $V = 0$ if not, $C = 0.26$ if there is a national coupon and $C = 0$ if not, $X = 0.36$ if there is a chain-specific ad and $X = 0$ if not, and $D = 0.29$ if there is a special display in the chain and $D = 0$ if not. (D is subtracted because less product is needed in the warehouse when more is on display in the store.) If $B = 4200$ units and there is a special display and a national coupon but no national TV ad and no chain-specific ad, then what is the value of A?