8.5 DETERMINANTS AND CRAMER'S RULE

In this section

- Determinants
- Cramer's Rule (2 × 2)
- Minors
- Evaluating a 3 × 3 Determinant
- Cramer's Rule (3 × 3)

The Gaussian elimination method of Section 8.4 can be performed the same way on every system. Another method that is applied the same way for every system is Cramer's rule, which we study in this section. Before you learn Cramer's rule, we need to introduce a new number associated with a matrix, called a *determinant*.

Determinants

The determinant of a square matrix is a real number corresponding to the matrix. For a 2 × 2 matrix the determinant is defined as follows.

Determinant of a 2 × 2 Matrix

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be the real number $ad - bc$. We write

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note that the symbol for the determinant is a pair of vertical lines similar to the absolute value symbol, while a matrix is enclosed in brackets.

EXAMPLE 1 **Using the definition of determinant**

Find the determinant of each matrix.

a) $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

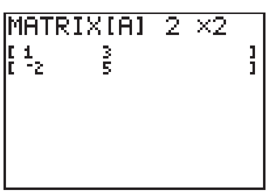
Solution

a) $$\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = 1 \cdot 5 - 3(-2) = 5 + 6 = 11$$

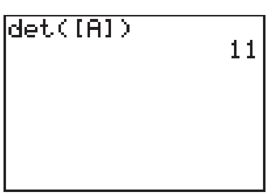
b) $$\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 2 \cdot 12 - 4 \cdot 6 = 24 - 24 = 0$$

calculator close-up

With a graphing calculator you can define matrix *A* using MATRX EDIT.



Then use the determinant function (det) found in MATRX MATH and the *A* from MATRX NAMES to find its determinant.



Cramer's Rule (2 × 2)

To understand Cramer's rule, we first solve a general system of two linear equations in two variables. Consider the system

$$(1) \quad a_1x + b_1y = c_1$$
$$(2) \quad a_2x + b_2y = c_2$$

where a_1, b_1, c_1, a_2, b_2, and c_2 represent real numbers. To eliminate y, we multiply Eq. (1) by b_2 and Eq. (2) by $-b_1$:

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 && \text{Eq. (1) multiplied by } b_2 \\ -a_2b_1x - b_1b_2y &= -c_2b_1 && \text{Eq. (2) multiplied by } -b_1 \\ a_1b_2x - a_2b_1x &= c_1b_2 - c_2b_1 && \text{Add.} \\ (a_1b_2 - a_2b_1)x &= c_1b_2 - c_2b_1 \\ x &= \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} && \text{Provided that } a_1b_2 - a_2b_1 \neq 0 \end{aligned}$$

Using similar steps to eliminate x from the system, we get

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$. These formulas for x and y can be written by using determinants. In the determinant form they are known as **Cramer's rule.**

Cramer's Rule

The solution to the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

is given by $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \qquad \text{and} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

provided that $D \neq 0$.

helpful hint

Notice that Cramer's rule gives us a precise formula for finding the solution to an independent system. The addition and substitution methods are more like guidelines under which we choose the best way to proceed.

Note that D is the determinant made up of the original coefficients of x and y. D is used in the denominator for both x and y. D_x is obtained by replacing the first (or x) column of D by the constants c_1 and c_2. D_y is found by replacing the second (or y) column of D by the constants c_1 and c_2.

EXAMPLE 2 **Solving an independent system with Cramer's rule**

Use Cramer's rule to solve the system:

$$\begin{aligned} 3x - 2y &= 4 \\ 2x + y &= -3 \end{aligned}$$

Solution

First find the determinants D, D_x, and D_y:

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7$$

$$D_x = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = 4 - 6 = -2, \qquad D_y = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -9 - 8 = -17$$

By Cramer's rule, we have

$$x = \frac{D_x}{D} = -\frac{2}{7} \qquad \text{and} \qquad y = \frac{D_y}{D} = -\frac{17}{7}.$$

Check in the original equations. The solution set is $\left\{\left(-\frac{2}{7}, -\frac{17}{7}\right)\right\}$. ■

calculator close-up

Use MATRX EDIT to define D, D_x, and D_y as A, B, and C. Now use Cramer's rule on the home screen to find x and y.

```
det([B])/det([A]
)►Frac
                -2/7
det([C])/det([A]
)►Frac
               -17/7
```

CAUTION Cramer's rule works *only* when the determinant D is *not* equal to zero. Cramer's rule solves only those systems that have a single point in their solution set. If $D = 0$, we use elimination to determine whether the solution set is empty or contains all points of a line.

Minors

To each element of a 3×3 matrix there corresponds a 2×2 matrix that is obtained by deleting the row and column of that element. The determinant of the 2×2 matrix is called the **minor** of that element.

EXAMPLE 3 **Finding minors**

Find the minors for the elements 2, 3, and -6 of the 3×3 matrix

$$\begin{bmatrix} 2 & -1 & -8 \\ 0 & -2 & 3 \\ 4 & -6 & 7 \end{bmatrix}.$$

Solution

To find the minor for 2, delete the first row and first column of the matrix:

$$\begin{bmatrix} 2 & -1 & -8 \\ 0 & -2 & 3 \\ 4 & -6 & 7 \end{bmatrix}$$

Now find the determinant of $\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}$:

$$\begin{vmatrix} -2 & 3 \\ -6 & 7 \end{vmatrix} = (-2)(7) - (-6)(3) = 4$$

The minor for 2 is 4. To find the minor for 3, delete the second row and third column of the matrix:

$$\begin{bmatrix} 2 & -1 & -8 \\ 0 & -2 & 3 \\ 4 & -6 & 7 \end{bmatrix}$$

Now find the determinant of $\begin{bmatrix} 2 & -1 \\ 4 & -6 \end{bmatrix}$:

$$\begin{vmatrix} 2 & -1 \\ 4 & -6 \end{vmatrix} = (2)(-6) - (4)(-1) = -8$$

The minor for 3 is -8. To find the minor for -6, delete the third row and the second column of the matrix:

$$\begin{bmatrix} 2 & -1 & -8 \\ 0 & -2 & 3 \\ 4 & -6 & 7 \end{bmatrix}$$

Now find the determinant of $\begin{bmatrix} 2 & -8 \\ 0 & 3 \end{bmatrix}$:

$$\begin{vmatrix} 2 & -8 \\ 0 & 3 \end{vmatrix} = (2)(3) - (0)(-8) = 6$$

The minor for -6 is 6. ■

Evaluating a 3 × 3 Determinant

study tip

Remember that everything we do in solving problems is based on principles (which are also called rules, theorems, and definitions). These principles justify the steps we take. Be sure that you understand the reasons. If you just memorize procedures without understanding, you will soon forget the procedures.

The determinant of a 3 × 3 matrix is defined in terms of the determinants of minors.

Determinant of a 3 × 3 Matrix

The determinant of a 3 × 3 matrix is defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Note that the determinants following a_1, a_2, and a_3 are the minors for a_1, a_2, and a_3, respectively. Writing the determinant of a 3 × 3 matrix in terms of minors is called **expansion by minors.** In the definition we expanded by minors about the first column. Later we will see how to expand by minors using any row or column and get the same value for the determinant.

EXAMPLE 4 **Determinant of a 3 × 3 matrix**

Find the determinant of the matrix by expansion by minors about the first column.

$$\begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

Solution

$$\begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -5 \\ 4 & 6 \end{vmatrix}$$

$$= 1 \cdot [36 - (-42)] + 2 \cdot (27 - 35) + 0 \cdot [18 - (-20)]$$
$$= 1 \cdot 78 + 2 \cdot (-8) + 0$$
$$= 78 - 16$$
$$= 62$$

In the next example we evaluate a determinant using expansion by minors about the second row. In expanding about any row or column, the signs of the coefficients of the minors alternate according to the **sign array** that follows:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The sign array is easily remembered by observing that there is a "+" sign in the upper left position and then alternating signs for all of the remaining positions.

EXAMPLE 5

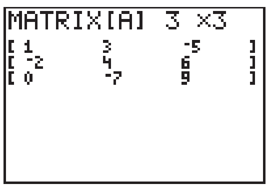
Determinant of a 3 × 3 matrix

Evaluate the determinant of the matrix by expanding by minors about the second row.

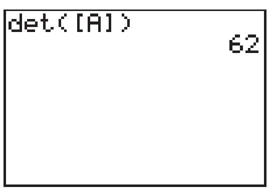
$$\begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

calculator close-up

A calculator is very useful for finding the determinant of a 3×3 matrix. Define A using MATRX EDIT.



Now use the determinant function from MATRX MATH and the A from MATRX NAMES to find the determinant.



Solution

For expansion using the second row we prefix the signs "$-$ $+$ $-$" from the second row of the sign array to the corresponding numbers in the second row of the matrix, -2, 4, and 6. Note that the signs from the sign array are used in addition to any signs that occur on the numbers in the second row.

From the sign array, second row

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} &= -(-2) \cdot \begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 4 \cdot \begin{vmatrix} 1 & -5 \\ 0 & 9 \end{vmatrix} - 6 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -7 \end{vmatrix} \\ &= 2(27 - 35) + 4(9 - 0) - 6(-7 - 0) \\ &= 2(-8) + 4(9) - 6(-7) \\ &= -16 + 36 + 42 \\ &= 62 \end{aligned}$$

Note that 62 is the same value that was obtained for this determinant in Example 4. ■

It can be shown that expanding by minors using any row or column prefixed by the corresponding signs from the sign array yields the same value for the determinant. Because we can use any row or column to evaluate a determinant of a 3×3 matrix, we can choose a row or column that makes the work easier. We can shorten the work considerably by picking a row or column with zeros in it.

EXAMPLE 6

Choosing the simplest row or column

Find the determinant of the matrix

$$\begin{bmatrix} 3 & -5 & 0 \\ 4 & -6 & 0 \\ 7 & 9 & 2 \end{bmatrix}$$

Solution

We choose to expand by minors about the third column of the matrix because the third column contains two zeros. Prefix the third-column entries 0, 0, 2 by the signs "$+$ $-$ $+$" from the third column of the sign array:

$$\begin{aligned} \begin{vmatrix} 3 & -5 & 0 \\ 4 & -6 & 0 \\ 7 & 9 & 2 \end{vmatrix} &= 0 \cdot \begin{vmatrix} 4 & -6 \\ 7 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 3 & -5 \\ 7 & 9 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & -5 \\ 4 & -6 \end{vmatrix} \\ &= 0 - 0 + 2[-18 - (-20)] \\ &= 4 \end{aligned}$$

■

Cramer's Rule (3 × 3)

A system of three linear equations in three variables can be solved by using determinants and Cramer's rule.

Cramer's Rule for Three Equations in Three Unknowns

The solution to the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

is given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, and $z = \frac{D_z}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

provided that $D \neq 0$.

Note that D_x, D_y, and D_z are obtained from D by replacing the x-, y-, or z-column with the constants d_1, d_2, and d_3.

EXAMPLE 7 **Solving an independent system with Cramer's rule**

Use Cramer's rule to solve the system:

$$\begin{aligned} x + y + z &= 4 \\ x - y &= -3 \\ x + 2y - z &= 0 \end{aligned}$$

Solution

We first calculate D, D_x, D_y, and D_z. To calculate D, expand by minors about the third column because the third column has a zero in it:

$$\begin{aligned} D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 1 \cdot [2 - (-1)] - 0 + (-1)[-1 - 1] \\ &= 3 - 0 + 2 \\ &= 5 \end{aligned}$$

calculator close-up

When you see the amount of arithmetic required to solve the system in Example 7 by Cramer's rule, you can understand why computers and calculators have been programmed to perform this method. Some calculators can find determinants for matrices as large as 10 × 10. Try to solve Example 7 with a graphing calculator that has determinants.

For D_x, expand by minors about the first column:

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ -3 & -1 & 0 \\ 0 & 2 & -1 \end{vmatrix} = 4 \cdot \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}$$

$$= 4 \cdot (1 - 0) + 3 \cdot (-1 - 2) + 0$$

$$= 4 - 9 + 0 = -5$$

For D_y, expand by minors about the third row:

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 1 \\ -3 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 4 \\ 1 & -3 \end{vmatrix}$$

$$= 1 \cdot 3 - 0 + (-1)(-7) = 10$$

To get D_z, expand by minors about the third row:

$$D_z = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & -3 \\ 1 & 2 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 4 \\ -1 & -3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 4 \\ 1 & -3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1 \cdot 1 - 2(-7) + 0 = 15$$

Now, by Cramer's rule,

$$x = \frac{D_x}{D} = \frac{-5}{5} = -1, \qquad y = \frac{D_y}{D} = \frac{10}{5} = 2, \qquad \text{and} \qquad z = \frac{D_z}{D} = \frac{15}{5} = 3.$$

Check $(-1, 2, 3)$ in the original equations. The solution set is $\{(-1, 2, 3)\}$. ■

If $D = 0$, Cramer's rule does not apply. Cramer's rule provides the solution only to a system of three equations with three variables that has a single point in the solution set. If $D = 0$, then the solution set either is empty or consists of infinitely many points, and we can use the methods discussed in Sections 8.3 or 8.4 to find the solution.

WARM-UPS

True or false? Explain your answer.

1. $\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} = -1$ **2.** $\begin{vmatrix} 2 & 4 \\ -4 & 8 \end{vmatrix} = 0$

3. Cramer's rule solves any system of two linear equations in two variables.

4. The determinant of a 2×2 matrix is a real number.

5. If $D = 0$, then there might be no solution to the system.

6. Cramer's rule is used to solve systems of linear equations only.

7. If the graphs of a pair of linear equations intersect at exactly one point, then this point can be found by using Cramer's rule.

8. The determinant of a 3×3 matrix is found by using minors.

9. Expansion by minors about any row or any column gives the same value for the determinant of a 3×3 matrix.

10. The sign array is used in evaluating the determinant of a 3×3 matrix.

8.5 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a determinant?

2. What is Cramer's rule used for?

3. Which systems can be solved using Cramer's rule?

4. What is a minor?

5. How do you find the minor for an element of a 3×3 matrix?

6. What is the purpose of the sign array?

Find the value of each determinant. See Example 1.

7. $\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$

8. $\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}$

9. $\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix}$

10. $\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix}$

11. $\begin{vmatrix} -3 & -2 \\ -4 & 2 \end{vmatrix}$

12. $\begin{vmatrix} -2 & 2 \\ -3 & -5 \end{vmatrix}$

13. $\begin{vmatrix} 0.05 & 0.06 \\ 10 & 20 \end{vmatrix}$

14. $\begin{vmatrix} 0.02 & -0.5 \\ 30 & 50 \end{vmatrix}$

Solve each system using Cramer's rule. See Example 2.

15. $2x - y = 5$
$3x + 2y = -3$

16. $3x + y = -1$
$x + 2y = 8$

17. $3x - 5y = -2$
$2x + 3y = 5$

18. $x - y = 1$
$3x - 2y = 0$

19. $4x - 3y = 5$
$2x + 5y = 7$

20. $2x - y = 2$
$3x - 2y = 1$

21. $0.5x + 0.2y = 8$
$0.4x - 0.6y = -5$

22. $0.6x + 0.5y = 18$
$0.5x - 0.25y = 7$

23. $\frac{1}{2}x + \frac{1}{4}y = 5$
$\frac{1}{3}x - \frac{1}{2}y = -1$

24. $\frac{1}{2}x + \frac{2}{3}y = 4$
$\frac{3}{4}x + \frac{1}{3}y = -2$

Find the indicated minors using the following matrix. See Example 3.

$$\begin{bmatrix} 3 & -2 & 5 \\ 4 & -3 & 7 \\ 0 & 1 & -6 \end{bmatrix}$$

25. Minor for 3

26. Minor for -2

27. Minor for 5

28. Minor for -3

29. Minor for 7

30. Minor for 0

31. Minor for 1

32. Minor for -6

Find the determinant of each 3×3 matrix by using expansion by minors about the first column. See Example 4.

33. $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 5 \end{bmatrix}$

34. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 4 & 6 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & 3 & 0 \end{bmatrix}$

37. $\begin{bmatrix} -2 & 1 & 2 \\ -3 & 3 & 1 \\ -5 & 4 & 0 \end{bmatrix}$

38. $\begin{bmatrix} -2 & 1 & 3 \\ -1 & 4 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 1 & 5 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 9 \end{bmatrix}$

Evaluate the determinant of each 3×3 matrix using expansion by minors about the row or column of your choice. See Examples 5 and 6.

41. $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 0 & 6 \\ 4 & 0 & 1 \end{bmatrix}$

42. $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 5 \\ 3 & 0 & 0 \end{bmatrix}$

43. $\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & -4 & -3 \end{bmatrix}$

44. $\begin{bmatrix} -2 & 0 & 1 \\ -3 & 2 & -5 \\ 4 & -2 & 6 \end{bmatrix}$

45. $\begin{bmatrix} -2 & -3 & 0 \\ 4 & -1 & 0 \\ 0 & 3 & 5 \end{bmatrix}$

46. $\begin{bmatrix} -2 & 6 & 3 \\ 0 & 4 & 0 \\ -1 & -4 & 5 \end{bmatrix}$

47. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 5 \\ 5 & 0 & 4 \end{bmatrix}$

48. $\begin{bmatrix} 2 & 3 & 0 \\ 6 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Use Cramer's rule to solve each system. See Example 7.

49. $$\begin{aligned} x + y + z &= 6 \\ x - y + z &= 2 \\ 2x + y + z &= 7 \end{aligned}$$

50. $$\begin{aligned} x + y + z &= 2 \\ x - y - 2z &= -3 \\ 2x - y + z &= 7 \end{aligned}$$

51. $$\begin{aligned} x - 3y + 2z &= 0 \\ x + y + z &= 2 \\ x - y + z &= 0 \end{aligned}$$

52. $$\begin{aligned} 3x + 2y + 2z &= 0 \\ x - y + z &= 1 \\ x + y - z &= 3 \end{aligned}$$

53. $$\begin{aligned} x + y \qquad &= -1 \\ 2y - z &= 3 \\ x + y + z &= 0 \end{aligned}$$

54. $$\begin{aligned} x - y \qquad &= 8 \\ x \quad - 2z &= 0 \\ x + y - z &= 1 \end{aligned}$$

55. $$\begin{aligned} x + y - z &= 0 \\ 2x + 2y + z &= 6 \\ x - 3y \qquad &= 0 \end{aligned}$$

56. $$\begin{aligned} x + y + z &= 1 \\ 5x - y \qquad &= 0 \\ 3x + y + 2z &= 0 \end{aligned}$$

57. $$\begin{aligned} x + y + z &= 0 \\ 2y + 2z &= 0 \\ 3x - y \qquad &= -1 \end{aligned}$$

58. $$\begin{aligned} x \quad + z &= 0 \\ x - 3y \qquad &= 1 \\ 4y - 3z &= 3 \end{aligned}$$

Solve each problem by using two equations in two variables and Cramer's rule.

59. ***Peas and beets.*** One serving of canned peas contains 3 grams of protein and 11 grams of carbohydrates. One serving of canned beets contains 1 gram of protein and 8 grams of carbohydrates. A dietitian wants to determine the number of servings of each that would provide 38 grams of protein and 187 grams of carbohydrates.

a) Use the accompanying graph to estimate the number of servings of each.

b) Use Cramer's rule to find the number of servings of each.

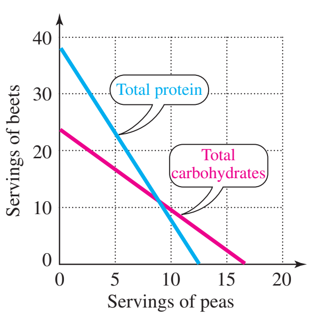


FIGURE FOR EXERCISE 59

60. ***Protein and carbohydrates.*** One serving of Cornies breakfast cereal contains 2 grams of protein and 25 grams of carbohydrates. One serving of Oaties breakfast cereal contains 4 grams of protein and 20 grams of carbohydrates. How many servings of each would provide exactly 24 grams of protein and 210 grams of carbohydrates?

61. ***Milk and a magazine.*** Althia bought a gallon of milk and a magazine for a total of \$4.65, excluding tax. Including the tax, the bill was \$4.95. If there is a 5% sales tax on milk and an 8% sales tax on magazines, then what was the price of each item?

62. ***Washing machines and refrigerators.*** A truck carrying 3600 cubic feet of cargo consisting of washing machines and refrigerators was hijacked. The washing machines are worth \$300 each and are shipped in 36-cubic-foot cartons. The refrigerators are worth \$900 each and are shipped in 45-cubic-foot cartons. If the total value of the cargo was \$51,000, then how many of each were there on the truck?

63. ***Singles and doubles.*** Windy's Hamburger Palace sells singles and doubles. Toward the end of the evening, Windy himself noticed that he had on hand only 32 patties and 34 slices of tomatoes. If a single takes 1 patty and 2 slices, and a double takes 2 patties and 1 slice, then how many more singles and doubles must Windy sell to use up all of his patties and tomato slices?

64. ***Valuable wrenches.*** Carmen has a total of 28 wrenches, all of which are either box wrenches or open-end wrenches. For insurance purposes she values the box wrenches at \$3.00 each and the open-end wrenches at \$2.50 each. If the value of her wrench collection is \$78, then how many of each type does she have?

65. ***Gary and Harry.*** Gary is 5 years older than Harry. Twenty-nine years ago, Gary was twice as old as Harry. How old are they now?

66. ***Acute angles.*** One acute angle of a right triangle is 3° more than twice the other acute angle. What are the sizes of the acute angles?

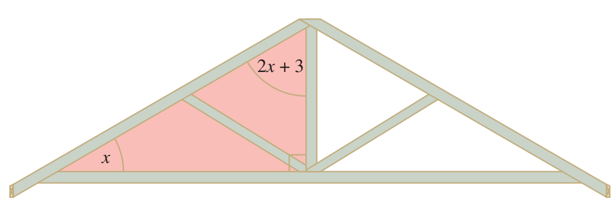


FIGURE FOR EXERCISE 66

67. ***Equal perimeters.*** A rope of length 80 feet is to be cut into two pieces. One piece will be used to form a square, and the other will be used to form an equilateral triangle.

If the figures are to have equal perimeters, then what should be the length of a side of each?

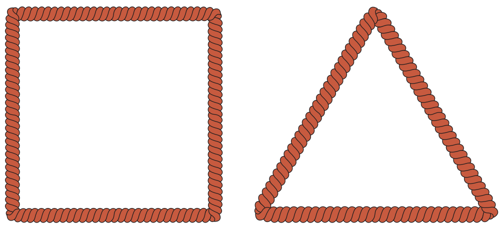


FIGURE FOR EXERCISE 67

68. ***Coffee and doughnuts.*** For a cup of coffee and a doughnut, Thurrel spent \$2.25, including a tip. Later he spent \$4.00 for two coffees and three doughnuts, including a tip. If he always tips \$1.00, then what is the price of a cup of coffee?

69. ***Chlorine mixture.*** A 10% chlorine solution is to be mixed with a 25% chlorine solution to obtain 30 gallons of 20% solution. How many gallons of each must be used?

70. ***Safe drivers.*** Emily and Camille started from the same city and drove in opposite directions on the freeway. After 3 hours they were 354 miles apart. If they had gone in the same direction, they would have been only 18 miles apart. How fast did each woman drive?

Write a system of three equations in three variables for each word problem. Use Cramer's rule to solve each system.

71. ***Weighing dogs.*** Cassandra wants to determine the weights of her two dogs, Mimi and Mitzi. However, neither dog will sit on the scale by herself. Cassandra, Mimi, and Mitzi altogether weigh 175 pounds. Cassandra and Mimi together weigh 143 pounds. Cassandra and Mitzi together weigh 139 pounds. How much does each weigh individually?

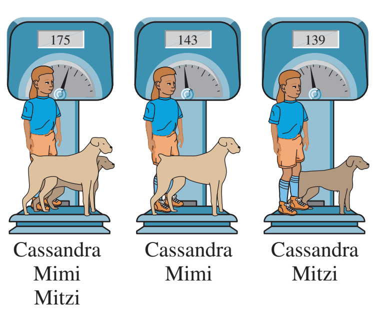


FIGURE FOR EXERCISE 71

72. ***Nickels, dimes, and quarters.*** Bernard has 41 coins consisting of nickels, dimes, and quarters, and they are worth a total of \$4.00. If the number of dimes plus the number of quarters is one more than the number of nickels, then how many of each does he have?

73. ***Finding three angles.*** If the two acute angles of a right triangle differ by 12°, then what are the measures of the three angles of this triangle?

74. ***Two acute and one obtuse.*** The obtuse angle of a triangle is twice as large as the sum of the two acute angles. If the smallest angle is only one-eighth as large as the sum of the other two, then what is the measure of each angle?

GETTING MORE INVOLVED

75. ***Writing.*** Explain what to do when you are trying to use Cramer's rule and $D = 0$.

76. ***Exploration.*** For what value of a does the system

$$\begin{aligned} ax - y &= 3 \\ x + 2y &= 1 \end{aligned}$$

have a single solution?

77. ***Exploration.*** Can Cramer's rule be used to solve the following system? Explain.

$$\begin{aligned} 2x^2 - y &= 3 \\ 3x^2 + 2y &= 22 \end{aligned}$$

78. ***Writing.*** For what values of a, b, c, and d is the determinant of the matrix

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ b & a & 0 \end{bmatrix}$$

equal to zero? Explain your answer.

GRAPHING CALCULATOR EXERCISES

79. Use the determinant feature on your graphing calculator to find the determinants in Exercises 7–14 and 33–40 of this section.

80. Solve the systems in Exercises 15–24 and 49–58 of this section by using your graphing calculator to find the necessary determinants.