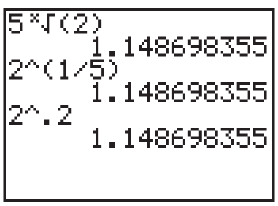
In this section

- Rational Exponents
- Using the Rules of Exponents
- Simplifying Expressions Involving Variables

calculator close-up

You can find the fifth root of 2 using radical notation or exponent notation. Note that the fractional exponent 1/5 must be in parentheses.



9.2 RATIONAL EXPONENTS

You have learned how to use exponents to express powers of numbers and radicals to express roots. In this section you will see that roots can be expressed with exponents also. The advantage of using exponents to express roots is that the rules of exponents can be applied to the expressions.

Rational Exponents

The nth root of a number can be expressed by using radical notation or the exponent $1/n$. For example, $8^{1/3}$ and $\sqrt[3]{8}$ both represent the cube root of 8, and we have

$$8^{1/3} = \sqrt[3]{8} = 2.$$

Definition of $a^{1/n}$

If n is any positive integer, then

$$a^{1/n} = \sqrt[n]{a},$$

provided that $\sqrt[n]{a}$ is a real number.

Later in this section we will see that using exponent $1/n$ for nth root is compatible with the rules for integral exponents that we already know.

EXAMPLE 1 **Radicals or exponents**

Write each radical expression using exponent notation and each exponential expression using radical notation.

a) $\sqrt[3]{35}$ **b)** $\sqrt[4]{xy}$ **c)** $5^{1/2}$ **d)** $a^{1/5}$

Solution

a) $\sqrt[3]{35} = 35^{1/3}$ **b)** $\sqrt[4]{xy} = (xy)^{1/4}$ **c)** $5^{1/2} = \sqrt{5}$ **d)** $a^{1/5} = \sqrt[5]{a}$ ■

In the next example we evaluate some exponential expressions.

EXAMPLE 2 **Finding roots**

Evaluate each expression.

a) $4^{1/2}$ **b)** $(-8)^{1/3}$ **c)** $81^{1/4}$ **d)** $(-9)^{1/2}$

Solution

a) $4^{1/2} = \sqrt{4} = 2$

b) $(-8)^{1/3} = \sqrt[3]{-8} = -2$

c) $81^{1/4} = \sqrt[4]{81} = 3$

d) Because $(-9)^{1/2}$ or $\sqrt{-9}$ is an even root of a negative number, it is not a real number. ■

We now extend the definition of exponent $1/n$ to include any rational number as an exponent. The numerator of the rational number indicates the power, and the denominator indicates the root. For example, the expression

Power

$$8^{2/3}$$ Root

represents the square of the cube root of 8. So we have

$$8^{2/3} = (8^{1/3})^2 = (2)^2 = 4.$$

helpful hint

Note that in $a^{m/n}$ we do not require m/n to be reduced. As long as the nth root of a is real, then the value of $a^{m/n}$ is the same whether or not m/n is in lowest terms.

Definition of $a^{m/n}$

If m and n are positive integers, then

$$a^{m/n} = (a^{1/n})^m,$$

provided that $a^{1/n}$ is a real number.

We define negative rational exponents just like negative integral exponents.

Definition of $a^{-m/n}$

If m and n are positive integers and $a \neq 0$, then

$$a^{-m/n} = \frac{1}{a^{m/n}},$$

provided that $a^{1/n}$ is a real number.

EXAMPLE 3 **Radicals or exponents**

Write each radical expression using exponent notation and each exponential expression using radical notation.

a) $\sqrt[3]{x^2}$ **b)** $\dfrac{1}{\sqrt[4]{m^3}}$

c) $5^{2/3}$ **d)** $a^{-2/5}$

Solution

a) $\sqrt[3]{x^2} = x^{2/3}$ **b)** $\dfrac{1}{\sqrt[4]{m^3}} = \dfrac{1}{m^{3/4}} = m^{-3/4}$

c) $5^{2/3} = \sqrt[3]{5^2}$ **d)** $a^{-2/5} = \dfrac{1}{\sqrt[5]{a^2}}$ ■

To evaluate an expression with a negative rational exponent, remember that the denominator indicates root, the numerator indicates power, and the negative sign indicates reciprocal:

$$a^{-m/n}$$

Root
Power
Reciprocal

The root, power, and reciprocal can be evaluated in any order. However, to evaluate $a^{-m/n}$ mentally it is usually simplest to use the following strategy.

Strategy for Evaluating $a^{-m/n}$ Mentally

1. Find the nth root of a.
2. Raise your result to the mth power.
3. Find the reciprocal.

For example, to evaluate $8^{-2/3}$ mentally, we find the cube root of 8 (which is 2), square 2 to get 4, then find the reciprocal of 4 to get $\frac{1}{4}$. In print $8^{-2/3}$ could be written for evaluation as $((8^{1/3})^2)^{-1}$ or $\frac{1}{(8^{1/3})^2}$.

EXAMPLE 4

Rational exponents

Evaluate each expression.

a) $27^{2/3}$ **b)** $4^{-3/2}$ **c)** $81^{-3/4}$ **d)** $(-8)^{-5/3}$

calculator close-up

A negative fractional exponent indicates a reciprocal, a root, and a power. To find $4^{-3/2}$ you can find the reciprocal first, the square root first, or the third power first as shown here.

```
(1/4)^(3/2)
           .125
(√(4))^-3
           .125
(4³)^(-1/2)
           .125
```

Solution

a) Because the exponent is $2/3$, we find the cube root of 27 and then square it:

$$27^{2/3} = (27^{1/3})^2 = 3^2 = 9$$

b) Because the exponent is $-3/2$, we find the square root of 4, cube it, and find the reciprocal:

$$4^{-3/2} = \frac{1}{(4^{1/2})^3} = \frac{1}{2^3} = \frac{1}{8}$$

c) Because the exponent is $-3/4$, we find the fourth root of 81, cube it, and find the reciprocal:

$$81^{-3/4} = \frac{1}{(81^{1/4})^3} = \frac{1}{3^3} = \frac{1}{27} \quad \text{Definition of negative exponent}$$

d) $(-8)^{-5/3} = \dfrac{1}{((-8)^{1/3})^5} = \dfrac{1}{(-2)^5} = \dfrac{1}{-32} = -\dfrac{1}{32}$ ■

CAUTION An expression with a negative base and a negative exponent can have a positive or a negative value. For example,

$$(-8)^{-5/3} = -\frac{1}{32} \quad \text{and} \quad (-8)^{-2/3} = \frac{1}{4}.$$

Using the Rules of Exponents

All of the rules for exponents hold for rational exponents as well as integral exponents. Of course, we cannot apply the rules of exponents to expressions that are not real numbers.

Rules for Rational Exponents

The following rules hold for any nonzero real numbers a and b and rational numbers r and s for which the expressions represent real numbers.

1. $a^r a^s = a^{r+s}$	Product rule
2. $\dfrac{a^r}{a^s} = a^{r-s}$	Quotient rule
3. $(a^r)^s = a^{rs}$	Power of a power rule
4. $(ab)^r = a^r b^r$	Power of a product rule
5. $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$	Power of a quotient rule

We can use the product rule to add rational exponents. For example,

$$16^{1/4} \cdot 16^{1/4} = 16^{2/4}.$$

The fourth root of 16 is 2, and 2 squared is 4. So $16^{2/4} = 4$. Because we also have $16^{1/2} = 4$, we see that a rational exponent can be reduced to its lowest terms. If an exponent can be reduced, it is usually simpler to reduce the exponent before we evaluate the expression. We can simplify $16^{1/4} \cdot 16^{1/4}$ as follows:

$$16^{1/4} \cdot 16^{1/4} = 16^{2/4} = 16^{1/2} = 4$$

EXAMPLE 5 **Using the product and quotient rules with rational exponents**

Simplify each expression.

a) $27^{1/6} \cdot 27^{1/2}$ **b)** $\dfrac{5^{3/4}}{5^{1/4}}$

Solution

a) $27^{1/6} \cdot 27^{1/2} = 27^{1/6+1/2}$ Product rule for exponents

$= 27^{2/3}$

$= 9$

b) $\dfrac{5^{3/4}}{5^{1/4}} = 5^{3/4-1/4} = 5^{2/4} = 5^{1/2} = \sqrt{5}$ We used the quotient rule to subtract the exponents. ■

EXAMPLE 6 **Using the power rules with rational exponents**

Simplify each expression.

a) $3^{1/2} \cdot 12^{1/2}$ **b)** $(3^{10})^{1/2}$ **c)** $\left(\dfrac{2^6}{3^9}\right)^{-1/3}$

Solution

a) Because the bases 3 and 12 are different, we cannot use the product rule to add the exponents. Instead, we use the power of a product rule to place the 1/2 power outside the parentheses:

$$3^{1/2} \cdot 12^{1/2} = (3 \cdot 12)^{1/2} = 36^{1/2} = 6$$

b) Use the power of a power rule to multiply the exponents:

$$(3^{10})^{1/2} = 3^5$$

c)
$$\left(\frac{2^6}{3^9}\right)^{-1/3} = \frac{(2^6)^{-1/3}}{(3^9)^{-1/3}} \quad \text{Power of a quotient rule}$$
$$= \frac{2^{-2}}{3^{-3}} \quad \text{Power of a power rule}$$
$$= \frac{3^3}{2^2} \quad \text{Definition of negative exponent}$$
$$= \frac{27}{4}$$

Simplifying Expressions Involving Variables

helpful hint

We usually think of squaring and taking a square root as inverse operations, which they are as long as we stick to positive numbers. We can square 3 to get 9, and then find the square root of 9 to get 3—what we started with. We don't get back to where we began if we start with -3.

When simplifying expressions involving rational exponents and variables, we must be careful to write equivalent expressions. For example, in the equation

$$(x^2)^{1/2} = x$$

it looks as if we are correctly applying the power of a power rule. However, this statement is false if x is negative because the $1/2$ power on the left-hand side indicates the positive square root of x^2. For example, if $x = -3$, we get

$$[(-3)^2]^{1/2} = 9^{1/2} = 3,$$

which is not equal to -3. To write a simpler equivalent expression for $(x^2)^{1/2}$, we use absolute value as follows.

Square Root of x^2

$$(x^2)^{1/2} = |x| \text{ for any real number } x.$$

Note that $(x^2)^{1/2} = |x|$ is also written as $\sqrt{x^2} = |x|$. Both of these equations are identities.

It is also necessary to use absolute value when writing identities for other even roots of expressions involving variables.

EXAMPLE 7 **Using absolute value symbols with roots**

Simplify each expression. Assume the variables represent any real numbers and use absolute value symbols as necessary.

a) $(x^8y^4)^{1/4}$

b) $\left(\dfrac{x^9}{8}\right)^{1/3}$

Solution

a) Apply the power of a product rule to get the equation $(x^8y^4)^{1/4} = x^2y$. The left-hand side is nonnegative for any choices of x and y, but the right-hand side is negative when y is negative. So for any real values of x and y we have

$$(x^8y^4)^{1/4} = x^2|y|.$$

b) Using the power of a quotient rule, we get

$$\left(\frac{x^9}{8}\right)^{1/3} = \frac{x^3}{2}.$$

This equation is valid for every real number x, so no absolute value signs are used.

Because there are no real even roots of negative numbers, the expressions

$$a^{1/2}, \quad x^{-3/4}, \quad \text{and} \quad y^{1/6}$$

are not real numbers if the variables have negative values. To simplify matters, we sometimes assume the variables represent only positive numbers when we are working with expressions involving variables with rational exponents. That way we do not have to be concerned with undefined expressions and absolute value.

EXAMPLE 8 **Expressions involving variables with rational exponents**

Use the rules of exponents to simplify the following. Write your answers with positive exponents. Assume all variables represent *positive* real numbers.

a) $x^{2/3}x^{4/3}$ **b)** $\dfrac{a^{1/2}}{a^{1/4}}$

c) $(x^{1/2}y^{-3})^{1/2}$ **d)** $\left(\dfrac{x^2}{y^{1/3}}\right)^{-1/2}$

Solution

a) $x^{2/3}x^{4/3} = x^{6/3}$ Use the product rule to add the exponents.

$= x^2$ Reduce the exponent.

b) $\dfrac{a^{1/2}}{a^{1/4}} = a^{1/2-1/4}$ Use the quotient rule to subtract the exponents.

$= a^{1/4}$ Simplify.

c) $(x^{1/2}y^{-3})^{1/2} = (x^{1/2})^{1/2}(y^{-3})^{1/2}$ Power of a product rule

$= x^{1/4}y^{-3/2}$ Power of a power rule

$= \dfrac{x^{1/4}}{y^{3/2}}$ Definition of negative exponent

d) Because this expression is a negative power of a quotient, we can first find the reciprocal of the quotient, then apply the power of a power rule:

$$\left(\frac{x^2}{y^{1/3}}\right)^{-1/2} = \left(\frac{y^{1/3}}{x^2}\right)^{1/2} = \frac{y^{1/6}}{x} \qquad \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

WARM-UPS

True or false? Explain your answer.

1. $9^{1/3} = \sqrt[3]{9}$

2. $8^{5/3} = \sqrt[5]{8^3}$

3. $(-16)^{1/2} = -16^{1/2}$

4. $9^{-3/2} = \dfrac{1}{27}$

5. $6^{-1/2} = \dfrac{\sqrt{6}}{6}$

6. $\dfrac{2}{2^{1/2}} = 2^{1/2}$

7. $2^{1/2} \cdot 2^{1/2} = 4^{1/2}$

8. $16^{-1/4} = -2$

9. $6^{1/6} \cdot 6^{1/6} = 6^{1/3}$

10. $(2^8)^{3/4} = 2^6$

9.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. How do we indicate an nth root using exponents?

2. How do we indicate the mth power of the nth root using exponents?

3. What is the meaning of a negative rational exponent?

4. Which rules of exponents hold for rational exponents?

5. In what order must you perform the operations indicated by a negative rational exponent?

6. When is $a^{-m/n}$ a real number?

Write each radical expression using exponent notation and each exponential expression using radical notation. See Example 1.

7. $\sqrt[4]{7}$ **8.** $\sqrt[3]{cbs}$

9. $9^{1/5}$ **10.** $3^{1/2}$

11. $\sqrt{5x}$ **12.** $\sqrt{3y}$

13. $a^{1/2}$ **14.** $(-b)^{1/5}$

Evaluate each expression. See Example 2.

15. $25^{1/2}$ **16.** $16^{1/2}$

17. $(-125)^{1/3}$ **18.** $(-32)^{1/5}$

19. $16^{1/4}$ **20.** $8^{1/3}$

21. $(-4)^{1/2}$

22. $(-16)^{1/4}$

Write each radical expression using exponent notation and each exponential expression using radical notation. See Example 3.

23. $\sqrt[3]{w^7}$ **24.** $\sqrt{a^5}$

25. $\dfrac{1}{\sqrt[3]{2^{10}}}$ **26.** $\sqrt[3]{\dfrac{1}{a^2}}$

27. $w^{-3/4}$ **28.** $6^{-5/3}$

29. $(ab)^{3/2}$ **30.** $(3m)^{-1/5}$

Evaluate each expression. See Example 4.

31. $125^{2/3}$ **32.** $1000^{2/3}$

33. $25^{3/2}$ **34.** $16^{3/2}$

35. $27^{-4/3}$ **36.** $16^{-3/4}$

37. $16^{-3/2}$ **38.** $25^{-3/2}$

39. $(-27)^{-1/3}$ **40.** $(-8)^{-4/3}$

41. $(-16)^{-1/4}$

42. $(-100)^{-3/2}$

Use the rules of exponents to simplify each expression. See Examples 5 and 6.

43. $3^{1/3}3^{1/4}$ **44.** $2^{1/2}\,2^{1/3}$

45. $3^{1/3}3^{-1/3}$ **46.** $5^{1/4}5^{-1/4}$

47. $\dfrac{8^{1/3}}{8^{2/3}}$ **48.** $\dfrac{27^{-2/3}}{27^{-1/3}}$

49. $4^{3/4} \div 4^{1/4}$ **50.** $9^{1/4} \div 9^{3/4}$

51. $18^{1/2}2^{1/2}$ **52.** $8^{1/2}2^{1/2}$

53. $(2^6)^{1/3}$ **54.** $(3^{10})^{1/5}$

55. $(3^8)^{1/2}$ **56.** $(3^{-6})^{1/3}$

57. $(2^{-4})^{1/2}$ **58.** $(5^4)^{1/2}$

59. $\left(\dfrac{3^4}{2^6}\right)^{1/2}$ **60.** $\left(\dfrac{5^4}{3^6}\right)^{1/2}$

Simplify each expression. Assume the variables represent any real numbers and use absolute value as necessary. See Example 7.

61. $(x^4)^{1/4}$ **62.** $(y^6)^{1/6}$

63. $(a^8)^{1/2}$ **64.** $(b^{10})^{1/2}$

65. $(y^3)^{1/3}$ **66.** $(w^9)^{1/3}$

67. $(9x^6y^2)^{1/2}$ **68.** $(16a^8b^4)^{1/4}$

69. $\left(\dfrac{81x^{12}}{y^{20}}\right)^{1/4}$ **70.** $\left(\dfrac{144a^8}{9y^{18}}\right)^{1/2}$

Simplify. Assume all variables represent positive numbers. Write answers with positive exponents only. See Example 8.

71. $x^{1/2}x^{1/4}$ **72.** $y^{1/3}y^{1/3}$

73. $(x^{1/2}y)(x^{-3/4}y^{1/2})$ **74.** $(a^{1/2}b^{-1/3})(ab)$

75. $\dfrac{w^{1/3}}{w^3}$ **76.** $\dfrac{a^{1/2}}{a^2}$

77. $(144x^{16})^{1/2}$ **78.** $(125a^8)^{1/3}$

79. $\left(\dfrac{a^{-1/2}}{b^{-1/4}}\right)^{-4}$ **80.** $\left(\dfrac{2a^{1/2}}{b^{1/3}}\right)^{6}$

Simplify each expression. Write your answers with positive exponents. Assume that all variables represent positive real numbers.

81. $(9^2)^{1/2}$ **82.** $(4^{16})^{1/2}$

83. $-16^{-3/4}$

84. $-25^{-3/2}$

85. $125^{-4/3}$

86. $27^{-2/3}$

87. $2^{1/2}2^{-1/4}$

88. $9^{-1}9^{1/2}$

89. $3^{0.26}3^{0.74}$

90. $2^{1.5}2^{0.5}$

91. $3^{1/4}27^{1/4}$

92. $3^{2/3}9^{2/3}$

93. $\left(-\frac{8}{27}\right)^{2/3}$

94. $\left(-\frac{8}{27}\right)^{-1/3}$

95. $\left(-\frac{1}{16}\right)^{-3/4}$

96. $\left(\frac{9}{16}\right)^{-1/2}$

97. $(9x^9)^{1/2}$

98. $(-27x^9)^{1/3}$

99. $(3a^{-2/3})^{-3}$

100. $(5x^{-1/2})^{-2}$

101. $(a^{1/2}b)^{1/2}(ab^{1/2})$

102. $(m^{1/4}n^{1/2})^2(m^2n^3)^{1/2}$

103. $(km^{1/2})^3(k^3m^5)^{1/2}$

104. $(tv^{1/3})^2(t^2v^{-3})^{-1/2}$

Use a scientific calculator with a power key (x^y) to find the decimal value of each expression. Round answers to four decimal places.

105. $2^{1/3}$

106. $5^{1/2}$

107. $-2^{1/2}$

108. $(-3)^{1/3}$

109. $1024^{1/10}$

110. $7776^{0.2}$

111. $8^{0.33}$

112. $289^{0.5}$

113. $\left(\frac{64}{15{,}625}\right)^{-1/6}$

114. $\left(\frac{32}{243}\right)^{-3/5}$

Simplify each expression. Assume a and b are positive real numbers and m and n are rational numbers.

115. $a^{m/2} \cdot a^{m/4}$

116. $b^{n/2} \cdot b^{-n/3}$

117. $\frac{a^{-m/5}}{a^{-m/3}}$

118. $\frac{b^{-n/4}}{b^{-n/3}}$

119. $(a^{-1/m}b^{-1/n})^{-mn}$

120. $(a^{-m/2}b^{-n/3})^{-6}$

121. $\left(\frac{a^{-3m}b^{-6n}}{a^{9m}}\right)^{-1/3}$

122. $\left(\frac{a^{-3/m}b^{6/n}}{a^{-6/m}b^{9/n}}\right)^{-1/3}$

In Exercises 123–130, solve each problem.

123. ***Diagonal of a box.*** The length of the diagonal of a box can be found from the formula

$$D = (L^2 + W^2 + H^2)^{1/2},$$

where L, W, and H represent the length, width, and height of the box, respectively. If the box is 12 inches long, 4 inches wide, and 3 inches high, then what is the length of the diagonal?

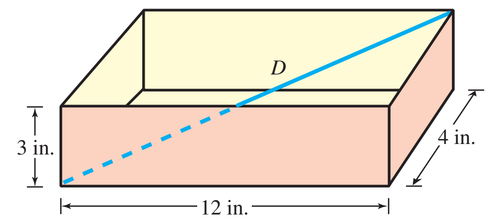


FIGURE FOR EXERCISE 123

124. ***Radius of a sphere.*** The radius of a sphere is a function of its volume, given by the formula

$$r = \left(\frac{0.75V}{\pi}\right)^{1/3}.$$

Find the radius of a spherical tank that has a volume of $\frac{32\pi}{3}$ cubic meters.

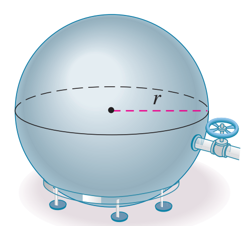


FIGURE FOR EXERCISE 124

125. ***Maximum sail area.*** According to the new International America's Cup Class Rules, the maximum sail area in square meters for a yacht in the America's Cup race is given by

$$S = (13.0368 + 7.84D^{1/3} - 0.8L)^2,$$

where D is the displacement in cubic meters (m^3), and L is the length in meters (m). (*Scientific American,* May 1992). Find the maximum sail area for a boat that has a displacement of 18.42 m^3 and a length of 21.45 m.

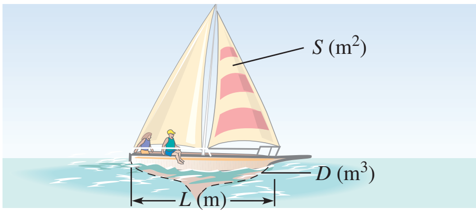
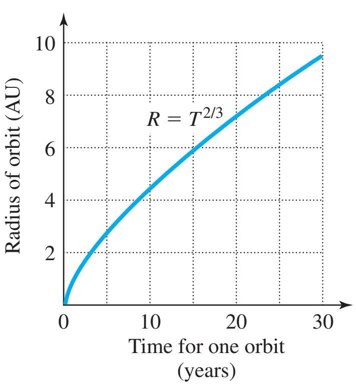


FIGURE FOR EXERCISE 125

126. ***Orbits of the planets.*** According to Kepler's third law of planetary motion, the average radius R of the orbit of a planet around the sun is determined by $R = T^{2/3}$, where T is the number of years for one orbit and R is measured in astronomical units or AUs (Windows to the Universe, www.windows.umich.edu).

a) It takes Mars 1.881 years to make one orbit of the sun. What is the average radius (in AUs) of the orbit of Mars?

b) The average radius of the orbit of Saturn is 9.05 AU. Use the accompanying graph to estimate the number of years it takes Saturn to make one orbit of the sun.



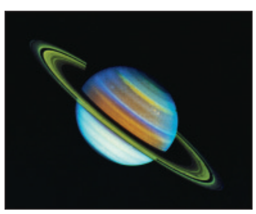


FIGURE FOR EXERCISE 126

127. ***Best stock fund.*** The average annual return for an investment is given by the formula

$$r = \left(\frac{S}{P}\right)^{1/n} - 1,$$

where P is the initial investment and S is the amount it is worth after n years. The top mutual fund for 1997 in the 3-year category was Fidelity Select-Energy Services (Money Guide to Mutual Funds, 1998), in which an investment of \$10,000 grew to \$31,895.06 from 1994 to 1997. Find the 3-year average annual return for this fund.

128. ***Best bond fund.*** The top bond fund for 1997 in the 5-year category was GT Global High Income B. An investment of \$10,000 in 1992 grew to \$21,830.95 in 1997. Use the formula from the previous exercise to find the 5-year average annual return for this fund.

129. ***Overdue loan payment.*** In 1777 a wealthy Pennsylvania merchant, Jacob DeHaven, lent \$450,000 to the Continental Congress to rescue the troops at Valley Forge. The loan was not repaid. In 1990 DeHaven's descendants filed suit for \$141.6 billion (*New York Times,* May 27, 1990). What average annual rate of return were they using to calculate the value of the debt after 213 years? (See Exercise 127.)

130. ***California growin'.*** The population of California grew from 19.9 million in 1970 to 32.5 million in 2000 (U.S. Census Bureau, www.census.gov). Find the average annual rate of growth for that time period. (Use the formula from Exercise 127 with P being the initial population and S being the population n years later.)

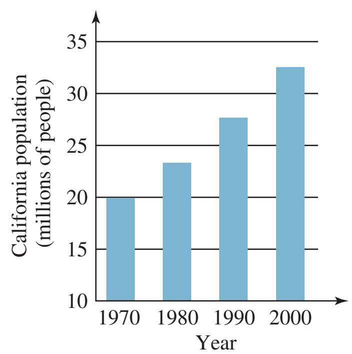


FIGURE FOR EXERCISE 130

GETTING MORE INVOLVED

131. ***Discussion.*** If we use the product rule to simplify $(-1)^{1/2} \cdot (-1)^{1/2}$, we get

$$(-1)^{1/2} \cdot (-1)^{1/2} = (-1)^1 = -1.$$

If we use the power of a product rule, we get

$$(-1)^{1/2} \cdot (-1)^{1/2} = (-1 \cdot -1)^{1/2} = 1^{1/2} = 1.$$

Which of these computations is incorrect? Explain your answer.

132. ***Discussion.*** Determine whether each equation is an identity. Explain.

a) $(w^2x^2)^{1/2} = |w| \cdot |x|$

b) $(w^2x^2)^{1/2} = |wx|$

c) $(w^2x^2)^{1/2} = w|x|$