**116.** $\sqrt[4]{a^3}(\sqrt[4]{a} - \sqrt[4]{a^5})$

**117.** $\sqrt[3]{2x} \cdot \sqrt{2x}$

**118.** $\sqrt[3]{2m} \cdot \sqrt[4]{2n}$

*In Exercises 119–122, solve each problem.*

**119.** ***Area of a rectangle.*** Find the exact area of a rectangle that has a length of $\sqrt{6}$ feet and a width of $\sqrt{3}$ feet.

**120.** ***Volume of a cube.*** Find the exact volume of a cube with sides of length $\sqrt{3}$ meters.

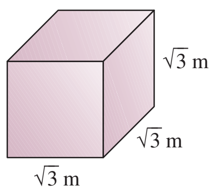


**FIGURE FOR EXERCISE 120**

**121.** ***Area of a trapezoid.*** Find the exact area of a trapezoid with a height of $\sqrt{6}$ feet and bases of $\sqrt{3}$ feet and $\sqrt{12}$ feet.

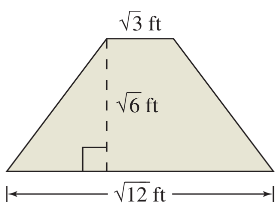


**FIGURE FOR EXERCISE 121**

**122.** ***Area of a triangle.*** Find the exact area of a triangle with a base of $\sqrt{30}$ meters and a height of $\sqrt{6}$ meters.

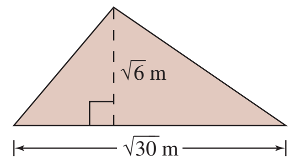


**FIGURE FOR EXERCISE 122**

## GETTING MORE INVOLVED

**123.** ***Discussion.*** Is $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ for all values of $a$ and $b$?

**124.** ***Discussion.*** Which of the following equations are identities? Explain your answers.

**a)** $\sqrt{9x} = 3\sqrt{x}$

**b)** $\sqrt{9 + x} = 3 + \sqrt{x}$

**c)** $\sqrt{x - 4} = \sqrt{x} - 2$

**d)** $\sqrt{\dfrac{x}{4}} = \dfrac{\sqrt{x}}{2}$

**125.** ***Exploration.*** Because 3 is the square of $\sqrt{3}$, a binomial such as $y^2 - 3$ is a difference of two squares.

**a)** Factor $y^2 - 3$ and $2a^2 - 7$ using radicals.

**b)** Use factoring with radicals to solve the equations $x^2 - 8 = 0$ and $3y^2 - 11 = 0$.

**c)** Assuming $a$ and $b$ are positive real numbers, solve the equations $x^2 - a = 0$ and $ax^2 - b = 0$.

# 9.4 MORE OPERATIONS WITH RADICALS

**In this section**

- Dividing Radicals
- Rationalizing the Denominator
- Powers of Radical Expressions

In this section you will continue studying operations with radicals. We learn to rationalize some denominators that are different from those rationalized in Section 9.2.

## Dividing Radicals

In Section 9.3 you learned how to add, subtract, and multiply radical expressions. To divide two radical expressions, simply write the quotient as a ratio and then simplify, as we did in Section 9.2. In general, we have

$$\sqrt[n]{a} \div \sqrt[n]{b} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},$$

provided that all expressions represent real numbers. Note that the quotient rule is applied only to radicals that have the same index.

**EXAMPLE 1** **Dividing radicals with the same index**

Divide and simplify. Assume the variables represent positive numbers.

**a)** $\sqrt{10} \div \sqrt{5}$ **b)** $(3\sqrt{2}) \div (2\sqrt{3})$ **c)** $\sqrt[3]{10x^2} \div \sqrt[3]{5x}$

**Solution**

**a)**
$$\sqrt{10} \div \sqrt{5} = \frac{\sqrt{10}}{\sqrt{5}} \qquad a \div b = \frac{a}{b}, \text{ provided that } b \neq 0.$$
$$= \sqrt{\frac{10}{5}} \qquad \text{Quotient rule for radicals}$$
$$= \sqrt{2} \qquad \text{Reduce.}$$

**b)**
$$(3\sqrt{2}) \div (2\sqrt{3}) = \frac{3\sqrt{2}}{2\sqrt{3}}$$
$$= \frac{3\sqrt{2}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \qquad \text{Rationalize the denominator.}$$
$$= \frac{3\sqrt{6}}{2 \cdot 3}$$
$$= \frac{\sqrt{6}}{2} \qquad \text{Note that } \sqrt{6} \div 2 \neq \sqrt{3}.$$

**c)**
$$\sqrt[3]{10x^2} \div \sqrt[3]{5x} = \frac{\sqrt[3]{10x^2}}{\sqrt[3]{5x}}$$
$$= \sqrt[3]{\frac{10x^2}{5x}} \qquad \text{Quotient rule for radicals}$$
$$= \sqrt[3]{2x} \qquad \text{Reduce.}$$

Note that in Example 1(a) we applied the quotient rule to get $\sqrt{10} \div \sqrt{5} = \sqrt{2}$. In Example 1(b) we did not use the quotient rule because 2 is not evenly divisible by 3. Instead, we rationalized the denominator to get the result in simplified form.

In Chapter 10 it will be necessary to simplify expressions of the type found in the next example.

**EXAMPLE 2** **Simplifying radical expressions**

Simplify.

**a)** $\dfrac{4 - \sqrt{12}}{4}$ **b)** $\dfrac{-6 + \sqrt{20}}{-2}$

**Solution**

**a)** First write $\sqrt{12}$ in simplified form. Then simplify the expression.

$$\frac{4 - \sqrt{12}}{4} = \frac{4 - 2\sqrt{3}}{4} \qquad \text{Simplify } \sqrt{12}.$$
$$= \frac{\cancel{2}(2 - \sqrt{3})}{\cancel{2} \cdot 2} \qquad \text{Factor.}$$
$$= \frac{2 - \sqrt{3}}{2} \qquad \text{Divide out the common factor.}$$

**helpful hint**

The expressions in Example 2 are the types of expressions that you must simplify when learning the quadratic formula in Chapter 10.

**b)** $\dfrac{-6 + \sqrt{20}}{-2} = \dfrac{-6 + 2\sqrt{5}}{-2}$

$= \dfrac{-\cancel{2}(3 - \sqrt{5})}{-\cancel{2}}$

$= 3 - \sqrt{5}$

**CAUTION** To simplify the expressions in Example 2, you must simplify the radical, factor the numerator, and then divide out the common factors. You cannot simply "cancel" the 4's in $\frac{4 - \sqrt{12}}{4}$ or the 2's in $\frac{2 - \sqrt{3}}{2}$ because they are not common factors.

## Rationalizing the Denominator

In Section 9.2 you learned that a simplified expression involving radicals does not have radicals in the denominator. If an expression such as $4 - \sqrt{3}$ appears in a denominator, we can multiply both the numerator and denominator by its conjugate $4 + \sqrt{3}$ to get a rational number in the denominator.

**EXAMPLE 3** **Rationalizing the denominator using conjugates**

Write in simplified form.

**a)** $\dfrac{2 + \sqrt{3}}{4 - \sqrt{3}}$ **b)** $\dfrac{\sqrt{5}}{\sqrt{6} + \sqrt{2}}$

**Solution**

**a)** $\dfrac{2 + \sqrt{3}}{4 - \sqrt{3}} = \dfrac{(2 + \sqrt{3})(4 + \sqrt{3})}{(4 - \sqrt{3})(4 + \sqrt{3})}$ Multiply the numerator and denominator by $4 + \sqrt{3}$.

$= \dfrac{8 + 6\sqrt{3} + 3}{13}$ $(4 - \sqrt{3})(4 + \sqrt{3}) = 16 - 3 = 13$

$= \dfrac{11 + 6\sqrt{3}}{13}$ Simplify.

**b)** $\dfrac{\sqrt{5}}{\sqrt{6} + \sqrt{2}} = \dfrac{\sqrt{5}(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}$ Multiply the numerator and denominator by $\sqrt{6} - \sqrt{2}$.

$= \dfrac{\sqrt{30} - \sqrt{10}}{4}$ $(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2}) = 6 - 2 = 4$

**study tip**

Read the text and recite to yourself what you have read. Ask questions and answer them out loud. Listen to your answers to see if they are complete and correct. Would other students understand your answers?

## Powers of Radical Expressions

We can use the power of a product rule and the power of a power rule to simplify a radical expression raised to a power. In the next example we also use the fact that a root and a power can be found in either order.

**EXAMPLE 4** **Finding powers of rational expressions**

Simplify. Assume the variables represent positive numbers.

**a)** $(5\sqrt{2})^3$ **b)** $(2\sqrt{x^3})^4$

**c)** $(3w\sqrt[3]{2w})^3$ **d)** $(2t\sqrt[4]{3t})^3$

**Solution**

**a)** $(5\sqrt{2})^3 = 5^3(\sqrt{2})^3$ Power of a product rule
$= 125\sqrt{8}$ $(\sqrt{2})^3 = \sqrt{2^3} = \sqrt{8}$
$= 125 \cdot 2\sqrt{2}$ $\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$
$= 250\sqrt{2}$

**b)** $(2\sqrt{x^3})^4 = 2^4(\sqrt{x^3})^4$
$= 16\sqrt{x^{12}}$
$= 16x^6$

**c)** $(3w\sqrt[3]{2w})^3 = 3^3w^3(\sqrt[3]{2w})^3$
$= 27w^3(2w)$
$= 54w^4$

**d)** $(2t\sqrt[4]{3t})^3 = 2^3t^3(\sqrt[4]{3t})^3 = 8t^3\sqrt[4]{27t^3}$

## WARM-UPS

**True or false? Explain your answer.**

1. $\dfrac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$
2. $\dfrac{2}{\sqrt{2}} = \sqrt{2}$
3. $\dfrac{4 - \sqrt{10}}{2} = 2 - \sqrt{10}$
4. $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$
5. $\dfrac{8\sqrt{7}}{2\sqrt{7}} = 4\sqrt{7}$
6. $\dfrac{2(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = 4 + 2\sqrt{3}$
7. $\dfrac{\sqrt{12}}{3} = \sqrt{4}$
8. $\dfrac{\sqrt{20}}{\sqrt{5}} = 2$
9. $(2\sqrt{4})^2 = 16$
10. $(3\sqrt{5})^3 = 27\sqrt{125}$

## 9.4 EXERCISES

*All variables in the following exercises represent positive numbers.*

*Divide and simplify. See Example 1.*

1. $\sqrt{15} \div \sqrt{5}$
2. $\sqrt{14} \div \sqrt{7}$
3. $\sqrt{3} \div \sqrt{5}$
4. $\sqrt{5} \div \sqrt{7}$
5. $(3\sqrt{3}) \div (5\sqrt{6})$
6. $(2\sqrt{2}) \div (4\sqrt{10})$
7. $(2\sqrt{3}) \div (3\sqrt{6})$
8. $(5\sqrt{12}) \div (4\sqrt{6})$
9. $\sqrt[3]{20} \div \sqrt[3]{2}$
10. $\sqrt[4]{48} \div \sqrt[4]{3}$
11. $\sqrt[3]{8x^7} \div \sqrt[3]{2x}$
12. $\sqrt[4]{4a^{10}} \div \sqrt[4]{2a^2}$

*Simplify. See Example 2.*

13. $\dfrac{6 + \sqrt{45}}{3}$
14. $\dfrac{10 + \sqrt{50}}{5}$

**15.** $\dfrac{-2 + \sqrt{12}}{-2}$ **16.** $\dfrac{-6 + \sqrt{72}}{-6}$

*Simplify each expression by rationalizing the denominator. See Example 3.*

**17.** $\dfrac{1 + \sqrt{2}}{\sqrt{3} - 1}$

**18.** $\dfrac{2 - \sqrt{3}}{\sqrt{2} + \sqrt{6}}$

**19.** $\dfrac{\sqrt{2}}{\sqrt{6} + \sqrt{3}}$

**20.** $\dfrac{5}{\sqrt{7} - \sqrt{5}}$

**21.** $\dfrac{2\sqrt{3}}{3\sqrt{2} - \sqrt{5}}$

**22.** $\dfrac{3\sqrt{5}}{5\sqrt{2} + \sqrt{6}}$

**23.** $\dfrac{1 + 3\sqrt{2}}{2\sqrt{6} + 3\sqrt{10}}$

**24.** $\dfrac{3\sqrt{3} + 1}{4 - 5\sqrt{3}}$

*Simplify. See Example 4.*

**25.** $(2\sqrt{2})^5$ **26.** $(3\sqrt{3})^4$

**27.** $(\sqrt{x})^5$ **28.** $(2\sqrt{y})^3$

**29.** $(-3\sqrt{x^3})^3$ **30.** $(-2\sqrt{x^3})^4$

**31.** $(2x\sqrt[3]{x^2})^3$ **32.** $(2y\sqrt[3]{4y})^3$

**33.** $(-2\sqrt[3]{5})^2$ **34.** $(-3\sqrt[3]{4})^2$

**35.** $(\sqrt[3]{x^2})^6$ **36.** $(2\sqrt[4]{y^3})^3$

*In Exercises 37–74, simplify.*

**37.** $\dfrac{\sqrt{3}}{\sqrt{2}} + \dfrac{2}{\sqrt{2}}$ **38.** $\dfrac{2}{\sqrt{7}} + \dfrac{5}{\sqrt{7}}$

**39.** $\dfrac{\sqrt{3}}{\sqrt{2}} + \dfrac{3\sqrt{6}}{2}$ **40.** $\dfrac{\sqrt{3}}{2\sqrt{2}} + \dfrac{\sqrt{5}}{3\sqrt{2}}$

**41.** $\dfrac{\sqrt{6}}{2} \cdot \dfrac{1}{\sqrt{3}}$ **42.** $\dfrac{\sqrt{6}}{\sqrt{7}} \cdot \dfrac{\sqrt{14}}{\sqrt{3}}$

**43.** $(2\sqrt{w}) \div (3\sqrt{w})$ **44.** $2 \div (3\sqrt{a})$

**45.** $\dfrac{8 - \sqrt{32}}{20}$ **46.** $\dfrac{4 - \sqrt{28}}{6}$

**47.** $\dfrac{5 + \sqrt{75}}{10}$ **48.** $\dfrac{3 + \sqrt{18}}{6}$

**49.** $\sqrt{a}(\sqrt{a} - 3)$ **50.** $3\sqrt{m}(2\sqrt{m} - 6)$

**51.** $4\sqrt{a}(a + \sqrt{a})$ **52.** $\sqrt{3ab}(\sqrt{3a} + \sqrt{3})$

**53.** $(2\sqrt{3m})^2$ **54.** $(-3\sqrt{4y})^2$

**55.** $(-2\sqrt{xy^2z})^2$ **56.** $(5a\sqrt{ab})^2$

**57.** $\sqrt[3]{m}(\sqrt[3]{m^2} - \sqrt[3]{m^5})$ **58.** $\sqrt[4]{w}(\sqrt[4]{w^3} - \sqrt[4]{w^7})$

**59.** $\sqrt[3]{8x^4} + \sqrt[3]{27x^4}$ **60.** $\sqrt[3]{16a^4} + a\sqrt[3]{2a}$

**61.** $(2m\sqrt[4]{2m^2})^3$ **62.** $(-2t\sqrt[6]{2t^2})^5$

**63.** $\dfrac{4}{2 + \sqrt{8}}$ **64.** $\dfrac{6}{3 - \sqrt{18}}$

**65.** $\dfrac{5}{\sqrt{2} - 1} + \dfrac{3}{\sqrt{2} + 1}$

**66.** $\dfrac{\sqrt{3}}{\sqrt{6} - 1} - \dfrac{\sqrt{3}}{\sqrt{6} + 1}$

**67.** $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$

**68.** $\dfrac{4}{2\sqrt{3}} + \dfrac{1}{\sqrt{5}}$

**69.** $\dfrac{3}{\sqrt{2} - 1} + \dfrac{4}{\sqrt{2} + 1}$

**70.** $\dfrac{3}{\sqrt{5} - \sqrt{3}} - \dfrac{2}{\sqrt{5} + \sqrt{3}}$

**71.** $\dfrac{\sqrt{x}}{\sqrt{x} + 2} + \dfrac{3\sqrt{x}}{\sqrt{x} - 2}$

**72.** $\dfrac{\sqrt{5}}{3 - \sqrt{y}} - \dfrac{\sqrt{5y}}{3 + \sqrt{y}}$

**73.** $\dfrac{1}{\sqrt{x}} + \dfrac{1}{1 - \sqrt{x}}$

**74.** $\dfrac{\sqrt{x}}{\sqrt{x} - 3} + \dfrac{5}{\sqrt{x}}$

*Replace the question mark by an expression that makes the equation correct. Equations involving variables are to be identities.*

**75.** $\dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{?}$

**76.** $\dfrac{2}{?} = \sqrt{2}$

**77.** $\dfrac{1}{\sqrt{2} - 1} = \dfrac{\sqrt{2} + 1}{?}$

**78.** $\dfrac{\sqrt{6}}{\sqrt{6} + 2} = \dfrac{?}{2}$

**79.** $\dfrac{1}{\sqrt{x} - 1} = \dfrac{?}{x - 1}$

**80.** $\dfrac{5}{3 - \sqrt{x}} = \dfrac{?}{9 - x}$

**81.** $\dfrac{3}{\sqrt{2} + x} = \dfrac{?}{2 - x^2}$

**82.** $\dfrac{4}{2\sqrt{3} + a} = \dfrac{?}{12 - a^2}$

*Use a calculator to find a decimal approximation for each radical expression. Round your answers to three decimal places.*

**83.** $\sqrt{3} + \sqrt{5}$

**84.** $\sqrt{5} + \sqrt{7}$

**85.** $2\sqrt{3} + 5\sqrt{3}$

**86.** $7\sqrt{3}$

**87.** $(2\sqrt{3})(3\sqrt{2})$

**88.** $6\sqrt{6}$

**89.** $\sqrt{5}(\sqrt{5} + \sqrt{3})$

**90.** $5 + \sqrt{15}$

**91.** $\dfrac{-1 + \sqrt{6}}{2}$

**92.** $\dfrac{-1 - \sqrt{6}}{2}$

**93.** $\dfrac{4 - \sqrt{10}}{-2}$

**94.** $\dfrac{4 + \sqrt{10}}{-2}$

## GETTING MORE INVOLVED

**95.** ***Exploration.*** A polynomial is prime if it cannot be factored by using integers, but many prime polynomials can be factored if we use radicals.

**a)** Find the product $(x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + \sqrt[3]{4})$.

**b)** Factor $x^3 + 5$ using radicals.

**c)** Find the product

$$(\sqrt[3]{5} - \sqrt[3]{2})(\sqrt[3]{25} + \sqrt[3]{10} + \sqrt[3]{4}).$$

**d)** Use radicals to factor $a + b$ as a sum of two cubes and $a - b$ as a difference of two cubes.

**96.** ***Discussion.*** Which one of the following expressions is not equivalent to the others?

**a)** $(\sqrt[3]{x})^4$ **b)** $\sqrt[4]{x^3}$ **c)** $\sqrt[3]{x^4}$

**d)** $x^{4/3}$ **e)** $(x^{1/3})^4$

# 9.5 SOLVING EQUATIONS WITH RADICALS AND EXPONENTS

**In this section**

- The Odd-Root Property
- The Even-Root Property
- Raising Each Side to a Power
- Equations Involving Rational Exponents
- Summary of Methods
- The Distance Formula

One of our goals in algebra is to keep increasing our knowledge of solving equations because the solutions to equations can give us the answers to various applied questions. In this section we will apply our knowledge of radicals and exponents to solving some new types of equations.

## The Odd-Root Property

Because $(-2)^3 = -8$ and $2^3 = 8$, the equation $x^3 = 8$ is equivalent to $x = 2$. The equation $x^3 = -8$ is equivalent to $x = -2$. Because there is only one real odd root of each real number, there is a simple rule for writing an equivalent equation in this situation.