Venus to complete one orbit of the sun? Use the information in Exercise 109.

Use a calculator to find approximate solutions to the following equations. Round your answers to three decimal places.

111. $x^2 = 3.24$

112. $(x + 4)^3 = 7.51$

113. $\sqrt{x - 2} = 1.73$

114. $\sqrt[3]{x - 5} = 3.7$

115. $x^{2/3} = 8.86$

116. $(x - 1)^{-3/4} = 7.065$

GETTING MORE INVOLVED

117. ***Cooperative learning.*** Work in a small group to write a formula that gives the side of a cube in terms of the volume of the cube and explain the formula to the other groups.

118. ***Cooperative learning.*** Work in a small group to write a formula that gives the side of a square in terms of the diagonal of the square and explain the formula to the other groups.

9.6 COMPLEX NUMBERS

In this section

- Definition
- Addition, Subtraction, and Multiplication
- Division of Complex Numbers
- Square Roots of Negative Numbers
- Imaginary Solutions to Equations

In Chapter 1 we discussed the real numbers and the various subsets of the real numbers. In this section we define a set of numbers that has the real numbers as a subset.

Definition

The equation $2x = 1$ has no solution in the set of integers, but in the set of rational numbers, $2x = 1$ has a solution. The situation is similar for the equation $x^2 = -4$. It has no solution in the set of real numbers because the square of every real number is nonnegative. However, in the set of complex numbers $x^2 = -4$ has two solutions. The complex numbers were developed so that equations such as $x^2 = -4$ would have solutions.

The complex numbers are based on the symbol $\sqrt{-1}$. In the real number system this symbol has no meaning. In the set of complex numbers this symbol is given meaning. We call it i. We make the definition that

$$i = \sqrt{-1} \quad \text{and} \quad i^2 = -1.$$

Complex Numbers

The set of **complex numbers** is the set of all numbers of the form

$$a + bi,$$

where a and b are real numbers, $i = \sqrt{-1}$, and $i^2 = -1$.

In the complex number $a + bi$, a is called the **real part** and b is called the **imaginary part.** If $b \neq 0$, the number $a + bi$ is called an **imaginary number.**

In dealing with complex numbers, we treat $a + bi$ as if it were a binomial, with i being a variable. Thus we would write $2 + (-3)i$ as $2 - 3i$. We agree that $2 + i3$, $3i + 2$, and $i3 + 2$ are just different ways of writing $2 + 3i$ (the standard form). Some examples of complex numbers are

$$-3 - 5i, \quad \frac{2}{3} - \frac{3}{4}i, \quad 1 + i\sqrt{2}, \quad 9 + 0i, \quad \text{and} \quad 0 + 7i.$$

study tip

Make sure that you know what your instructor expects from you. You can determine what your instructor feels is important by looking at the examples that your instructor works in class and the homework assignments. When in doubt, ask your instructor what you will be responsible for and write down the answer.

For simplicity we write only $7i$ for $0 + 7i$. The complex number $9 + 0i$ is the real number 9, and $0 + 0i$ is the real number 0. Any complex number with $b = 0$ is a real number. For any real number a,

$$a + 0i = a.$$

The set of real numbers is a subset of the set of complex numbers. See Fig. 9.3.

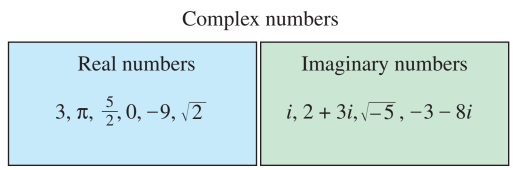

FIGURE 9.3

Addition, Subtraction, and Multiplication

Addition and subtraction of complex numbers are performed as if the complex numbers were algebraic expressions with i being a variable.

EXAMPLE 1 **Addition and subtraction of complex numbers**

Find the sums and differences.

a) $(2 + 3i) + (6 + i)$ **b)** $(-2 + 3i) + (-2 - 5i)$

c) $(3 + 5i) - (1 + 2i)$ **d)** $(-2 - 3i) - (1 - i)$

Solution

a) $(2 + 3i) + (6 + i) = 8 + 4i$

b) $(-2 + 3i) + (-2 - 5i) = -4 - 2i$

c) $(3 + 5i) - (1 + 2i) = 2 + 3i$

d) $(-2 - 3i) - (1 - i) = -3 - 2i$

We can give a symbolic definition of addition and subtraction as follows.

Addition and Subtraction of Complex Numbers

The sum and difference of $a + bi$ and $c + di$ are defined as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Complex numbers are multiplied as if they were algebraic expressions. Whenever i^2 appears, we replace it by -1.

EXAMPLE 2 **Products of complex numbers**

Find each product.

a) $2i(1 + i)$ **b)** $(2 + 3i)(4 + 5i)$

c) $(3 + i)(3 - i)$

Solution

a) $2i(1 + i) = 2i + 2i^2$ Distributive property

$= 2i + 2(-1)$ $i^2 = -1$

$= -2 + 2i$

b) Use the FOIL method to find the product:

$$(2 + 3i)(4 + 5i) = 8 + 10i + 12i + 15i^2$$
$$= 8 + 22i + 15(-1) \quad \text{Replace } i^2 \text{ by } -1.$$
$$= 8 + 22i - 15$$
$$= -7 + 22i$$

c) This product is the product of a sum and a difference.

$$(3 + i)(3 - i) = 9 - 3i + 3i - i^2$$
$$= 9 - (-1) \quad i^2 = -1$$
$$= 10$$

calculator close-up

Many graphing calculators can perform operations with complex numbers.

```
2i(1+i)
                -2+2i
(2+3i)(4+5i)
               -7+22i
(3+i)(3-i)
                   10
```

We can find powers of i using the fact that $i^2 = -1$. For example,

$$i^3 = i^2 \cdot i = -1 \cdot i = -i.$$

The value of i^4 is found from the value of i^3:

$$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = 1.$$

In the next example we find more powers of imaginary numbers.

EXAMPLE 3 **Powers of imaginary numbers**

Write each expression in the form $a + bi$.

a) $(2i)^2$ **b)** $(-2i)^2$ **c)** i^6

Solution

a) $(2i)^2 = 2^2 \cdot i^2 = 4(-1) = -4$

b) $(-2i)^2 = (-2)^2 \cdot i^2 = 4i^2 = 4(-1) = -4$

c) $i^6 = i^2 \cdot i^4 = -1 \cdot 1 = -1$

For completeness we give the following symbolic definition of multiplication of complex numbers. However, it is simpler to find products as we did in Examples 2 and 3 than to use this definition.

Multiplication of Complex Numbers

The complex numbers $a + bi$ and $c + di$ are multiplied as follows:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Division of Complex Numbers

To divide a complex number by a real number, divide each term by the real number, just as we would divide a binomial by a number. For example,

$$\frac{4 + 6i}{2} = \frac{2(2 + 3i)}{2}$$
$$= 2 + 3i.$$

helpful hint

Here is that word "conjugate" again. It is generally used to refer to two things that go together in some way.

To understand division by a complex number, we first look at imaginary numbers that have a real product. The product of the two imaginary numbers in Example 2(c) is a real number:

$$(3 + i)(3 - i) = 10$$

We say that $3 + i$ and $3 - i$ are complex conjugates of each other.

Complex Conjugates

The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates** of one another. Their product is the real number $a^2 + b^2$.

EXAMPLE 4 **Products of conjugates**

Find the product of the given complex number and its conjugate.

a) $2 + 3i$ **b)** $5 - 4i$

Solution

a) The conjugate of $2 + 3i$ is $2 - 3i$.

$$\begin{aligned}(2 + 3i)(2 - 3i) &= 4 - 9i^2 \\ &= 4 + 9 \\ &= 13\end{aligned}$$

b) The conjugate of $5 - 4i$ is $5 + 4i$.

$$\begin{aligned}(5 - 4i)(5 + 4i) &= 25 + 16 \\ &= 41\end{aligned}$$

We use the idea of complex conjugates to divide complex numbers. The process is similar to rationalizing the denominator. Multiply the numerator and denominator of the quotient by the complex conjugate of the denominator.

EXAMPLE 5 **Dividing complex numbers**

Find each quotient. Write the answer in the form $a + bi$.

a) $\dfrac{5}{3 - 4i}$ **b)** $\dfrac{3 - i}{2 + i}$ **c)** $\dfrac{3 + 2i}{i}$

Solution

a) Multiply the numerator and denominator by $3 + 4i$, the conjugate of $3 - 4i$:

$$\begin{aligned}\frac{5}{3 - 4i} &= \frac{5(3 + 4i)}{(3 - 4i)(3 + 4i)} \\ &= \frac{15 + 20i}{9 - 16i^2} \\ &= \frac{15 + 20i}{25} \qquad 9 - 16i^2 = 9 - 16(-1) = 25 \\ &= \frac{15}{25} + \frac{20}{25}i \\ &= \frac{3}{5} + \frac{4}{5}i\end{aligned}$$

b) Multiply the numerator and denominator by $2 - i$, the conjugate of $2 + i$:

$$\begin{aligned}\frac{3 - i}{2 + i} &= \frac{(3 - i)(2 - i)}{(2 + i)(2 - i)} \\ &= \frac{6 - 5i + i^2}{4 - i^2} \\ &= \frac{6 - 5i - 1}{4 - (-1)} \\ &= \frac{5 - 5i}{5} \\ &= 1 - i\end{aligned}$$

c) Multiply the numerator and denominator by $-i$, the conjugate of i:

$$\begin{aligned}\frac{3 + 2i}{i} &= \frac{(3 + 2i)(-i)}{i(-i)} \\ &= \frac{-3i - 2i^2}{-i^2} \\ &= \frac{-3i + 2}{1} \\ &= 2 - 3i\end{aligned}$$

The symbolic definition of division of complex numbers follows.

Division of Complex Numbers

We divide the complex number $a + bi$ by the complex number $c + di$ as follows:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$$

Square Roots of Negative Numbers

In Examples 3(a) and 3(b) we saw that both

$$(2i)^2 = -4 \quad \text{and} \quad (-2i)^2 = -4.$$

Because the square of each of these complex numbers is -4, both $2i$ and $-2i$ are square roots of -4. We write $\sqrt{-4} = 2i$. In the complex number system the square root of any negative number is an imaginary number.

Square Root of a Negative Number

For any positive real number b,

$$\sqrt{-b} = i\sqrt{b}.$$

For example, $\sqrt{-9} = i\sqrt{9} = 3i$ and $\sqrt{-7} = i\sqrt{7}$. Note that the expression $\sqrt{7}i$ could easily be mistaken for the expression $\sqrt{7i}$, where i is under the radical. For this reason, when the coefficient of i is a radical, we write i preceding the radical.

EXAMPLE 6 **Square roots of negative numbers**

Write each expression in the form $a + bi$, where a and b are real numbers.

a) $3 + \sqrt{-9}$ **b)** $\sqrt{-12} + \sqrt{-27}$ **c)** $\dfrac{-1 - \sqrt{-18}}{3}$

Solution

a) $3 + \sqrt{-9} = 3 + i\sqrt{9}$
$= 3 + 3i$

b) $\sqrt{-12} + \sqrt{-27} = i\sqrt{12} + i\sqrt{27}$
$= 2i\sqrt{3} + 3i\sqrt{3}$ $\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$
$\sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$
$= 5i\sqrt{3}$

c) $$\frac{-1 - \sqrt{-18}}{3} = \frac{-1 - i\sqrt{18}}{3}$$
$$= \frac{-1 - 3i\sqrt{2}}{3}$$
$$= -\frac{1}{3} - i\sqrt{2}$$

Imaginary Solutions to Equations

In the complex number system the even-root property can be restated so that $x^2 = k$ is equivalent to $x = \pm\sqrt{k}$ for any $k \neq 0$. So an equation such as $x^2 = -9$ that has no real solutions has two imaginary solutions in the complex numbers.

EXAMPLE 7 **Complex solutions to equations**

Find the complex solutions to each equation.

a) $x^2 = -9$ **b)** $3x^2 + 2 = 0$

Solution

a) First apply the even-root property:
$$x^2 = -9$$
$$x = \pm\sqrt{-9} \quad \text{Even-root property}$$
$$= \pm i\sqrt{9}$$
$$= \pm 3i$$

Check these solutions in the original equation:
$$(3i)^2 = 9i^2 = 9(-1) = -9$$
$$(-3i)^2 = 9i^2 = -9$$

The solution set is $\{\pm 3i\}$.

b) First solve the equation for x^2:
$$3x^2 + 2 = 0$$
$$x^2 = -\frac{2}{3}$$
$$x = \pm\sqrt{-\frac{2}{3}} = \pm i\sqrt{\frac{2}{3}} = \pm i\frac{\sqrt{6}}{3}$$

Check these solutions in the original equation. The solution set is $\left\{\pm i\frac{\sqrt{6}}{3}\right\}$.

The basic facts about complex numbers are listed in the following box.

Complex Numbers

1. Definition of i: $i = \sqrt{-1}$, and $i^2 = -1$.
2. A complex number has the form $a + bi$, where a and b are real numbers.
3. The complex number $a + 0i$ is the real number a.
4. If b is a positive real number, then $\sqrt{-b} = i\sqrt{b}$.
5. The numbers $a + bi$ and $a - bi$ are called complex conjugates of each other. Their product is the real number $a^2 + b^2$.
6. Add, subtract, and multiply complex numbers as if they were algebraic expressions with i being the variable, and replace i^2 by -1.
7. Divide complex numbers by multiplying the numerator and denominator by the conjugate of the denominator.
8. In the complex number system $x^2 = k$ for any real number k is equivalent to $x = \pm\sqrt{k}$.

WARM-UPS

True or false? Explain your answer.

1. The set of real numbers is a subset of the set of complex numbers.
2. $2 - \sqrt{-6} = 2 - 6i$
3. $\sqrt{-9} = \pm 3i$
4. The solution set to the equation $x^2 = -9$ is $\{\pm 3i\}$.
5. $2 - 3i - (4 - 2i) = -2 - i$
6. $i^4 = 1$
7. $(2 - i)(2 + i) = 5$
8. $i^3 = i$
9. $i^{48} = 1$
10. The equation $x^2 = k$ has two complex solutions for any real number k.

9.6 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What are complex numbers?
2. What is an imaginary number?
3. What is the relationship among the real numbers, the imaginary numbers, and the complex numbers?
4. How do we add, subtract, and multiply complex numbers?
5. What is the conjugate of a complex number?
6. How do we divide complex numbers?

Find the indicated sums and differences of complex numbers. See Example 1.

7. $(2 + 3i) + (-4 + 5i)$
8. $(-1 + 6i) + (5 - 4i)$
9. $(2 - 3i) - (6 - 7i)$
10. $(2 - 3i) - (6 - 2i)$
11. $(-1 + i) + (-1 - i)$
12. $(-5 + i) + (-5 - i)$
13. $(-2 - 3i) - (6 - i)$
14. $(-6 + 4i) - (2 - i)$

Find the indicated products of complex numbers. See Example 2.

15. $3(2 + 5i)$ **16.** $4(1 - 3i)$ **17.** $2i(i - 5)$

18. $3i(2 - 6i)$ **19.** $-4i(3 - i)$ **20.** $-5i(2 + 3i)$

21. $(2 + 3i)(4 + 6i)$ **22.** $(2 + i)(3 + 4i)$

23. $(-1 + i)(2 - i)$ **24.** $(3 - 2i)(2 - 5i)$

25. $(-1 - 2i)(2 + i)$ **26.** $(1 - 3i)(1 + 3i)$

27. $(5 - 2i)(5 + 2i)$ **28.** $(4 + 3i)(4 + 3i)$

29. $(1 - i)(1 + i)$ **30.** $(2 + 6i)(2 - 6i)$

31. $(4 + 2i)(4 - 2i)$ **32.** $(4 - i)(4 + i)$

Find the indicated powers of complex numbers. See Example 3.

33. $(3i)^2$ **34.** $(5i)^2$ **35.** $(-5i)^2$

36. $(-9i)^2$ **37.** $(2i)^4$ **38.** $(-2i)^3$

39. i^9 **40.** i^{12}

Find the product of the given complex number and its conjugate. See Example 4.

41. $3 + 5i$ **42.** $3 + i$ **43.** $1 - 2i$

44. $4 - 6i$ **45.** $-2 + i$ **46.** $-3 - 2i$

47. $2 - i\sqrt{3}$ **48.** $\sqrt{5} - 4i$

Find each quotient. See Example 5.

49. $\dfrac{3}{4 + i}$ **50.** $\dfrac{6}{7 - 2i}$

51. $\dfrac{2 + i}{3 - 2i}$ **52.** $\dfrac{3 + 5i}{2 - i}$

53. $\dfrac{4 + 3i}{i}$ **54.** $\dfrac{5 - 6i}{3i}$

55. $\dfrac{2 + 6i}{2}$ **56.** $\dfrac{9 - 3i}{-6}$

Write each expression in the form a + bi, where a and b are real numbers. See Example 6.

57. $2 + \sqrt{-4}$ **58.** $3 + \sqrt{-9}$ **59.** $2\sqrt{-9} + 5$

60. $3\sqrt{-16} + 2$ **61.** $7 - \sqrt{-6}$ **62.** $\sqrt{-5} + 3$

63. $\sqrt{-8} + \sqrt{-18}$ **64.** $2\sqrt{-20} - \sqrt{-45}$

65. $\dfrac{2 + \sqrt{-12}}{2}$ **66.** $\dfrac{-6 - \sqrt{-18}}{3}$

67. $\dfrac{-4 - \sqrt{-24}}{4}$ **68.** $\dfrac{8 + \sqrt{-20}}{-4}$

Find the complex solutions to each equation. See Example 7.

69. $x^2 = -36$ **70.** $x^2 + 4 = 0$

71. $x^2 = -12$ **72.** $x^2 = -25$

73. $2x^2 + 5 = 0$ **74.** $3x^2 + 4 = 0$

75. $3x^2 + 6 = 0$ **76.** $x^2 + 1 = 0$

Write each expression in the form a + bi, where a and b are real numbers.

77. $(2 - 3i)(3 + 4i)$ **78.** $(2 - 3i)(2 + 3i)$

79. $(2 - 3i) + (3 + 4i)$ **80.** $(3 - 5i) - (2 - 7i)$

81. $\dfrac{2 - 3i}{3 + 4i}$ **82.** $\dfrac{-3i}{3 - 6i}$

83. $i(2 - 3i)$ **84.** $-3i(4i - 1)$

85. $(-3i)^2$ **86.** $(-2i)^6$

87. $\sqrt{-12} + \sqrt{-3}$ **88.** $\sqrt{-49} - \sqrt{-25}$

89. $(2 - 3i)^2$ **90.** $(5 + 3i)^2$

91. $\dfrac{-4 + \sqrt{-32}}{2}$ **92.** $\dfrac{-2 - \sqrt{-27}}{-6}$

GETTING MORE INVOLVED

93. ***Writing.*** Explain why $2 - i$ is a solution to

$$x^2 - 4x + 5 = 0.$$

94. ***Cooperative learning.*** Work with a group to verify that $-1 + i\sqrt{3}$ and $-1 - i\sqrt{3}$ satisfy the equation

$$x^3 - 8 = 0.$$

In the complex number system there are three cube roots of 8. What are they?

95. ***Discussion.*** What is wrong with using the product rule for radicals to get

$$\sqrt{-4} \cdot \sqrt{-4} = \sqrt{(-4)(-4)} = \sqrt{16} = 4?$$

What is the correct product?