10.1 FACTORING AND COMPLETING THE SQUARE

In this section

- Review of Factoring
- Review of the Even-Root Property
- Completing the Square
- Miscellaneous Equations
- Imaginary Solutions

Factoring and the even-root property were used to solve quadratic equations in Chapters 6, 7, and 9. In this section we first review those methods. Then you will learn the method of completing the square, which can be used to solve any quadratic equation.

Review of Factoring

A quadratic equation is a second-degree polynomial equation of the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$. If the second-degree polynomial on the left-hand side can be factored, then we can solve the equation by breaking it into two first-degree polynomial equations (linear equations) using the following strategy.

Strategy for Solving Quadratic Equations by Factoring

1. Write the equation with 0 on the right-hand side.
2. Factor the left-hand side.
3. Use the zero factor property to set each factor equal to zero.
4. Solve the simpler equations.
5. Check the answers in the original equation.

EXAMPLE 1 **Solving a quadratic equation by factoring**

Solve $3x^2 - 4x = 15$ by factoring.

helpful hint

After you have factored the quadratic polynomial, use FOIL to check that you have factored correctly before proceeding to the next step.

Solution

Subtract 15 from each side to get 0 on the right-hand side:

$$3x^2 - 4x - 15 = 0$$

$$(3x + 5)(x - 3) = 0 \quad \text{Factor the left-hand side.}$$

$$3x + 5 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{Zero factor property}$$

$$3x = -5 \quad \text{or} \quad x = 3$$

$$x = -\frac{5}{3}$$

The solution set is $\left\{-\frac{5}{3}, 3\right\}$. Check the solutions in the original equation. ■

Review of the Even-Root Property

In Chapter 9 we solved quadratic equations by using the even-root property.

EXAMPLE 2 **Solving a quadratic equation by the even-root property**

Solve $(a - 1)^2 = 9$.

Solution

By the even-root property $x^2 = k$ is equivalent to $x = \pm\sqrt{k}$.

$$(a - 1)^2 = 9$$

$$a - 1 = \pm\sqrt{9} \quad \text{Even-root property}$$

$$a - 1 = 3 \quad \text{or} \quad a - 1 = -3$$

$$a = 4 \quad \text{or} \quad a = -2$$

Check these solutions in the original equation. The solution set is $\{-2, 4\}$.

Completing the Square

helpful hint

Review the rule for squaring a binomial: square the first term, find twice the product of the two terms, then square the last term. If you are still using FOIL to find the square of a binomial, it is time to learn the proper rule.

We cannot solve every quadratic by factoring because not all quadratic polynomials can be factored. However, we can write any quadratic equation in the form of Example 2 and then apply the even-root property to solve it. This method is called **completing the square.**

The essential part of completing the square is to recognize a perfect square trinomial when given its first two terms. For example, if we are given $x^2 + 6x$, how do we recognize that these are the first two terms of the perfect square trinomial $x^2 + 6x + 9$? To answer this question, recall that $x^2 + 6x + 9$ is a perfect square trinomial because it is the square of the binomial $x + 3$:

$$(x + 3)^2 = x^2 + 2 \cdot 3x + 3^2 = x^2 + 6x + 9$$

Notice that the 6 comes from multiplying 3 by 2 and the 9 comes from squaring the 3. So to find the missing 9 in $x^2 + 6x$, divide 6 by 2 to get 3, then square 3 to get 9. This procedure can be used to find the last term in any perfect square trinomial in which the coefficient of x^2 is 1.

Rule for Finding the Last Term

The last term of a perfect square trinomial is the square of one-half of the coefficient of the middle term. In symbols, the perfect square trinomial whose first two terms are $x^2 + bx$ is $x^2 + bx + \left(\frac{b}{2}\right)^2$.

EXAMPLE 3 **Finding the last term**

Find the perfect square trinomial whose first two terms are given.

a) $x^2 + 8x$ **b)** $x^2 - 5x$

c) $x^2 + \frac{4}{7}x$ **d)** $x^2 - \frac{3}{2}x$

Solution

a) One-half of 8 is 4, and 4 squared is 16. So the perfect square trinomial is

$$x^2 + 8x + 16.$$

b) One-half of -5 is $-\frac{5}{2}$, and $-\frac{5}{2}$ squared is $\frac{25}{4}$. So the perfect square trinomial is $x^2 - 5x + \frac{25}{4}$.

c) Since $\frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$ and $\frac{2}{7}$ squared is $\frac{4}{49}$, the perfect square trinomial is

$$x^2 + \frac{4}{7}x + \frac{4}{49}.$$

d) Since $\frac{1}{2}\left(-\frac{3}{2}\right) = -\frac{3}{4}$ and $\left(-\frac{3}{4}\right)^2 = \frac{9}{16}$, the perfect square trinomial is

$$x^2 - \frac{3}{2}x + \frac{9}{16}.$$

Another essential step in completing the square is to write the perfect square trinomial as the square of a binomial. Recall that

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2.$$

EXAMPLE 4 **Factoring perfect square trinomials**

Factor each trinomial.

a) $x^2 + 12x + 36$

b) $y^2 - 7y + \frac{49}{4}$

c) $z^2 - \frac{4}{3}z + \frac{4}{9}$

Solution

a) The trinomial $x^2 + 12x + 36$ is of the form $a^2 + 2ab + b^2$ with $a = x$ and $b = 6$. So

$$x^2 + 12x + 36 = (x + 6)^2.$$

Check by squaring $x + 6$.

b) The trinomial $y^2 - 7y + \frac{49}{4}$ is of the form $a^2 - 2ab + b^2$ with $a = y$ and $b = \frac{7}{2}$. So

$$y^2 - 7y + \frac{49}{4} = \left(y - \frac{7}{2}\right)^2.$$

Check by squaring $y - \frac{7}{2}$.

c) The trinomial $z^2 - \frac{4}{3}z + \frac{4}{9}$ is of the form $a^2 - 2ab + b^2$ with $a = z$ and $b = -\frac{2}{3}$. So

$$z^2 - \frac{4}{3}z + \frac{4}{9} = \left(z - \frac{2}{3}\right)^2.$$

study tip

Most instructors believe that what they do in class is important. If you miss class, then you miss what is important to your instructor and what is most likely to appear on the test.

In the next example we use the skills that we practiced in Examples 2, 3, and 4 to solve the quadratic equation $ax^2 + bx + c = 0$ with $a = 1$ by the method of completing the square.

EXAMPLE 5 **Completing the square with $a = 1$**

Solve $x^2 + 6x + 5 = 0$ by completing the square.

Solution

The perfect square trinomial whose first two terms are $x^2 + 6x$ is

$$x^2 + 6x + 9.$$

So we move 5 to the right-hand side of the equation, then add 9 to each side:

$$x^2 + 6x = -5 \qquad \text{Subtract 5 from each side.}$$

$$x^2 + 6x + 9 = -5 + 9 \qquad \text{Add 9 to each side to get a perfect square trinomial.}$$

$$(x + 3)^2 = 4 \qquad \text{Factor the left-hand side.}$$

$$x + 3 = \pm\sqrt{4} \qquad \text{Even-root property}$$

$$x + 3 = 2 \quad \text{or} \quad x + 3 = -2$$

$$x = -1 \quad \text{or} \quad x = -5$$

Check in the original equation:

$$(-1)^2 + 6(-1) + 5 = 0$$

and

$$(-5)^2 + 6(-5) + 5 = 0$$

The solution set is $\{-1, -5\}$. ■

calculator close-up

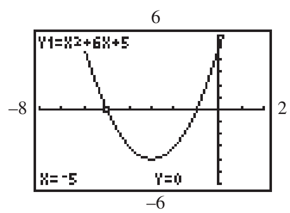
The solutions to

$$x^2 + 6x + 5 = 0$$

correspond to the x-intercepts for the graph of the function

$$y = x^2 + 6x + 5.$$

So we can check our solutions by graphing the function and using the TRACE feature as shown here.



CAUTION All of the perfect square trinomials that we have used so far had a leading coefficient of 1. If $a \neq 1$, then we must divide each side of the equation by a to get an equation with a leading coefficient of 1.

The strategy for solving a quadratic equation by completing the square is stated in the following box.

Strategy for Solving Quadratic Equations by Completing the Square

1. The coefficient of x^2 must be 1.
2. Get only the x^2 and the x terms on the left-hand side.
3. Add to each side the square of $\frac{1}{2}$ the coefficient of x.
4. Factor the left-hand side as the square of a binomial.
5. Apply the even-root property.
6. Solve for x.
7. Simplify.

In our procedure for completing the square the coefficient of x^2 must be 1. We can solve $ax^2 + bx + c = 0$ with $a \neq 1$ by completing the square if we first divide each side of the equation by a.

EXAMPLE 6 **Completing the square with $a \neq 1$**

Solve $2x^2 + 3x - 2 = 0$ by completing the square.

Solution

For completing the square, the coefficient of x^2 must be 1. So we first divide each side of the equation by 2:

$$\frac{2x^2 + 3x - 2}{2} = \frac{0}{2} \qquad \text{Divide each side by 2.}$$

$$x^2 + \frac{3}{2}x - 1 = 0 \qquad \text{Simplify.}$$

$$x^2 + \frac{3}{2}x = 1 \qquad \text{Get only } x^2 \text{ and } x \text{ terms on the left-hand side.}$$

$$x^2 + \frac{3}{2}x + \frac{9}{16} = 1 + \frac{9}{16} \qquad \text{One-half of } \tfrac{3}{2} \text{ is } \tfrac{3}{4}, \text{ and } \left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16}.$$

$$\left(x + \frac{3}{4}\right)^2 = \frac{25}{16} \qquad \text{Factor the left-hand side.}$$

$$x + \frac{3}{4} = \pm\sqrt{\frac{25}{16}} \qquad \text{Even-root property}$$

$$x + \frac{3}{4} = \frac{5}{4} \quad \text{or} \quad x + \frac{3}{4} = -\frac{5}{4}$$

$$x = \frac{2}{4} = \frac{1}{2} \quad \text{or} \quad x = -\frac{8}{4} = -2$$

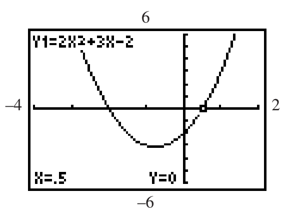
Check these values in the original equation. The solution set is $\left\{-2, \frac{1}{2}\right\}$. ■

calculator close-up

Note that the x-intercepts for the graph of the function:

$$y = 2x^2 + 3x - 2$$

are $(-2, 0)$ and $\left(\frac{1}{2}, 0\right)$:



In Examples 5 and 6 the solutions were rational numbers, and the equations could have been solved by factoring. In the next example the solutions are irrational numbers, and factoring will not work.

EXAMPLE 7 **A quadratic equation with irrational solutions**

Solve $x^2 - 3x - 6 = 0$ by completing the square.

Solution

Because $a = 1$, we first get the x^2 and x terms on the left-hand side:

$$x^2 - 3x - 6 = 0$$

$$x^2 - 3x = 6 \qquad \text{Add 6 to each side.}$$

$$x^2 - 3x + \frac{9}{4} = 6 + \frac{9}{4} \qquad \text{One-half of } -3 \text{ is } -\tfrac{3}{2}, \text{ and } \left(-\tfrac{3}{2}\right)^2 = \tfrac{9}{4}.$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{33}{4} \qquad 6 + \tfrac{9}{4} = \tfrac{24}{4} + \tfrac{9}{4} = \tfrac{33}{4}$$

$$x - \frac{3}{2} = \pm\sqrt{\frac{33}{4}} \qquad \text{Even-root property}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{33}}{2} \qquad \text{Add } \tfrac{3}{2} \text{ to each side.}$$

$$x = \frac{3 \pm \sqrt{33}}{2}$$

The solution set is $\left\{\frac{3 + \sqrt{33}}{2}, \frac{3 - \sqrt{33}}{2}\right\}$. ■

Miscellaneous Equations

The next two examples show equations that are not originally in the form of quadratic equations. However, after simplifying these equations, we get quadratic equations. Even though completing the square can be used on any quadratic equation, factoring and the square root property are usually easier and we can use them when applicable. In the next examples we will use the most appropriate method.

EXAMPLE 8

An equation containing a radical

Solve $x + 3 = \sqrt{153 - x}$.

Solution

Square both sides of the equation to eliminate the radical:

$$\begin{aligned} x + 3 &= \sqrt{153 - x} && \text{The original equation} \\ (x + 3)^2 &= (\sqrt{153 - x})^2 && \text{Square each side.} \\ x^2 + 6x + 9 &= 153 - x && \text{Simplify.} \\ x^2 + 7x - 144 &= 0 && \\ (x - 9)(x + 16) &= 0 && \text{Factor.} \end{aligned}$$

$$x - 9 = 0 \quad \text{or} \quad x + 16 = 0 \qquad \text{Zero factor property}$$

$$x = 9 \quad \text{or} \quad x = -16$$

Because we squared each side of the original equation, we must check for extraneous roots. Let $x = 9$ in the original equation:

$$\begin{aligned} 9 + 3 &= \sqrt{153 - 9} \\ 12 &= \sqrt{144} && \text{Correct} \end{aligned}$$

Let $x = -16$ in the original equation:

$$\begin{aligned} -16 + 3 &= \sqrt{153 - (-16)} \\ -13 &= \sqrt{169} && \text{Incorrect because } \sqrt{169} = 13 \end{aligned}$$

Because -16 is an extraneous root, the solution set is $\{9\}$. ■

calculator close-up

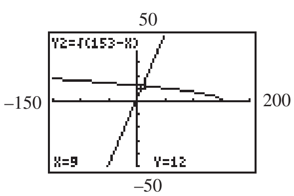
You can provide graphical support for the solution to Example 8 by graphing

$$y_1 = x + 3$$

and

$$y_2 = \sqrt{53 - x}.$$

It appears that the only point of intersection occurs when $x = 9$.



EXAMPLE 9

An equation containing rational expressions

Solve $\frac{1}{x} + \frac{3}{x - 2} = \frac{5}{8}$.

Solution

The least common denominator (LCD) for x, $x - 2$, and 8 is $8x(x - 2)$.

$$\frac{1}{x} + \frac{3}{x - 2} = \frac{5}{8}$$

$$8x(x - 2)\frac{1}{x} + 8x(x - 2)\frac{3}{x - 2} = 8x(x - 2)\frac{5}{8} \qquad \text{Multiply each side by the LCD.}$$

$$\begin{aligned} 8x - 16 + 24x &= 5x^2 - 10x \\ 32x - 16 &= 5x^2 - 10x \\ -5x^2 + 42x - 16 &= 0 \\ 5x^2 - 42x + 16 &= 0 && \text{Multiply each side by } -1 \text{ for easier factoring.} \\ (5x - 2)(x - 8) &= 0 && \text{Factor.} \end{aligned}$$

$$5x - 2 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = \frac{2}{5} \quad \text{or} \quad x = 8$$

Check these values in the original equation. The solution set is $\left\{\frac{2}{5}, 8\right\}$. ■

Imaginary Solutions

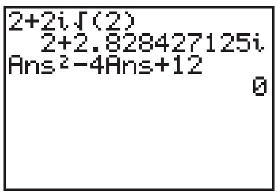
In Chapter 9 we found imaginary solutions to quadratic equations using the even-root property. We can get imaginary solutions also by completing the square.

EXAMPLE 10 **An equation with imaginary solutions**

Find the complex solutions to $x^2 - 4x + 12 = 0$.

calculator close-up

The answer key (ANS) can be used to check imaginary answers as shown here.



Solution

Because the quadratic polynomial cannot be factored, we solve the equation by completing the square.

$$x^2 - 4x + 12 = 0 \quad \text{The original equation}$$
$$x^2 - 4x = -12 \quad \text{Subtract 12 from each side.}$$
$$x^2 - 4x + 4 = -12 + 4 \quad \text{One-half of } -4 \text{ is } -2 \text{, and } (-2)^2 = 4.$$
$$(x - 2)^2 = -8$$
$$x - 2 = \pm\sqrt{-8} \quad \text{Even-root property}$$
$$x = 2 \pm i\sqrt{8}$$
$$= 2 \pm 2i\sqrt{2}$$

Check these values in the original equation. The solution set is $\{2 \pm 2i\sqrt{2}\}$. ■

WARM-UPS

True or false? Explain your answer.

1. Completing the square means drawing the fourth side.
2. The equation $(x - 3)^2 = 12$ is equivalent to $x - 3 = 2\sqrt{3}$.
3. Every quadratic equation can be solved by factoring.
4. The trinomial $x^2 + \frac{4}{3}x + \frac{16}{9}$ is a perfect square trinomial.
5. Every quadratic equation can be solved by completing the square.
6. To complete the square for $2x^2 + 6x = 4$, add 9 to each side.
7. $(2x - 3)(3x + 5) = 0$ is equivalent to $x = \frac{3}{2}$ or $x = \frac{5}{3}$.
8. In completing the square for $x^2 - 3x = 4$, add $\frac{9}{4}$ to each side.
9. The equation $x^2 = -8$ is equivalent to $x = \pm 2\sqrt{2}$.
10. All quadratic equations have two distinct complex solutions.

10.1 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What are the three methods discussed in this section for solving a quadratic equation?
2. Which quadratic equations can be solved by the even-root property?
3. How do you find the last term for a perfect square trinomial when completing the square?

4. How do you complete the square when the leading coefficient is not 1?

Solve by factoring. See Example 1.

5. $x^2 - x - 6 = 0$

6. $x^2 + 6x + 8 = 0$

7. $a^2 + 2a = 15$

8. $w^2 - 2w = 15$

9. $2x^2 - x - 3 = 0$

10. $6x^2 - x - 15 = 0$

11. $y^2 + 14y + 49 = 0$

12. $a^2 - 6a + 9 = 0$

13. $a^2 - 16 = 0$

14. $4w^2 - 25 = 0$

Use the even-root property to solve each equation. See Example 2.

15. $x^2 = 81$

16. $x^2 = \frac{9}{4}$

17. $x^2 = \frac{16}{9}$

18. $a^2 = 32$

19. $(x - 3)^2 = 16$

20. $(x + 5)^2 = 4$

21. $(z + 1)^2 = 5$

22. $(a - 2)^2 = 8$

23. $\left(w - \frac{3}{2}\right)^2 = \frac{7}{4}$

24. $\left(w + \frac{2}{3}\right)^2 = \frac{5}{9}$

Find the perfect square trinomial whose first two terms are given. See Example 3.

25. $x^2 + 2x$

26. $m^2 + 14m$

27. $x^2 - 3x$

28. $w^2 - 5w$

29. $y^2 + \frac{1}{4}y$

30. $z^2 + \frac{3}{2}z$

31. $x^2 + \frac{2}{3}x$

32. $p^2 + \frac{6}{5}p$

Factor each perfect square trinomial. See Example 4.

33. $x^2 + 8x + 16$

34. $x^2 - 10x + 25$

35. $y^2 - 5y + \frac{25}{4}$

36. $w^2 + w + \frac{1}{4}$

37. $z^2 - \frac{4}{7}z + \frac{4}{49}$

38. $m^2 - \frac{6}{5}m + \frac{9}{25}$

39. $t^2 + \frac{3}{5}t + \frac{9}{100}$

40. $h^2 + \frac{3}{2}h + \frac{9}{16}$

Solve by completing the square. See Examples 5–7. Use your calculator to check.

41. $x^2 - 2x - 15 = 0$

42. $x^2 - 6x - 7 = 0$

43. $x^2 + 8x = 20$

44. $x^2 + 10x = -9$

45. $2x^2 - 4x = 70$

46. $3x^2 - 6x = 24$

47. $w^2 - w - 20 = 0$

48. $y^2 - 3y - 10 = 0$

49. $q^2 + 5q = 14$

50. $z^2 + z = 2$

51. $2h^2 - h - 3 = 0$

52. $2m^2 - m - 15 = 0$

53. $x^2 + 4x = 6$

54. $x^2 + 6x - 8 = 0$

55. $x^2 + 8x - 4 = 0$

56. $x^2 + 10x - 3 = 0$

57. $2x^2 + 3x - 4 = 0$

58. $2x^2 + 5x - 1 = 0$

Solve each equation by an appropriate method. See Examples 8 and 9.

59. $\sqrt{2x + 1} = x - 1$

60. $\sqrt{2x - 4} = x - 14$

61. $w = \frac{\sqrt{w + 1}}{2}$

62. $y - 1 = \frac{\sqrt{y + 1}}{2}$

63. $\frac{t}{t - 2} = \frac{2t - 3}{t}$

64. $\frac{z}{z + 3} = \frac{3z}{5z - 1}$

65. $\frac{2}{x^2} + \frac{4}{x} + 1 = 0$

66. $\frac{1}{x^2} + \frac{3}{x} + 1 = 0$

Find the complex solutions to each equation. See Example 10.

67. $x^2 + 2x + 5 = 0$

68. $x^2 + 4x + 5 = 0$

69. $x^2 + 12 = 0$

70. $-3x^2 - 21 = 0$

71. $5z^2 - 4z + 1 = 0$

72. $2w^2 - 3w + 2 = 0$

Find all real or imaginary solutions to each equation. Use the method of your choice.

73. $4x^2 + 25 = 0$

74. $5w^2 - 3 = 0$

75. $\left(p + \frac{1}{2}\right)^2 = \frac{9}{4}$

76. $\left(y - \frac{2}{3}\right)^2 = \frac{4}{9}$

77. $5t^2 + 4t - 3 = 0$

78. $3v^2 + 4v - 1 = 0$

79. $m^2 + 2m - 24 = 0$

80. $q^2 + 6q - 7 = 0$

81. $\left(a + \frac{2}{3}\right)^2 = -\frac{32}{9}$

82. $\left(w + \frac{1}{2}\right)^2 = -6$

83. $-x^2 + x + 6 = 0$

84. $-x^2 + x + 12 = 0$

85. $x^2 - 6x + 10 = 0$

86. $x^2 - 8x + 17 = 0$

87. $2x - 5 = \sqrt{7x + 7}$

88. $\sqrt{7x + 29} = x + 3$

89. $\frac{1}{x} + \frac{1}{x - 1} = \frac{1}{4}$

90. $\frac{1}{x} - \frac{2}{1 - x} = \frac{1}{2}$

If the solution to an equation is imaginary or irrational, it takes a bit more effort to check. Replace x by each given number to verify each statement.

91. Both $2 + \sqrt{3}$ and $2 - \sqrt{3}$ satisfy $x^2 - 4x + 1 = 0$.

92. Both $1 + \sqrt{2}$ and $1 - \sqrt{2}$ satisfy $x^2 - 2x - 1 = 0$.

93. Both $1 + i$ and $1 - i$ satisfy $x^2 - 2x + 2 = 0$.

94. Both $2 + 3i$ and $2 - 3i$ satisfy $x^2 - 4x + 13 = 0$.

Solve each problem.

95. ***Approach speed.*** The formula $1211.1L = CA^2S$ is used to determine the approach speed for landing an aircraft, where L is the gross weight of the aircraft in pounds, C is the coefficient of lift, S is the surface area of the wings in square feet (ft^2), and A is approach speed in feet per second. Find A for the Piper Cheyenne, which has a gross weight of 8700 lbs, a coefficient of lift of 2.81, and wing surface area of 200 ft^2.

96. ***Time to swing.*** The period T (time in seconds for one complete cycle) of a simple pendulum is related to the length L (in feet) of the pendulum by the formula $8T^2 = \pi^2L$. If a child is on a swing with a 10-foot chain, then how long does it take to complete one cycle of the swing?

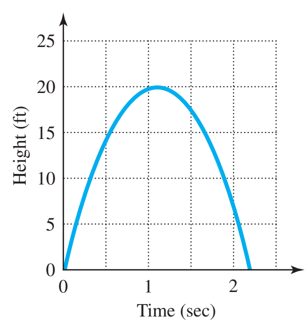
97. ***Time for a swim.*** Tropical Pools figures that its monthly revenue in dollars on the sale of x above-ground pools is given by $R = 1500x - 3x^2$, where x is less than 25. What number of pools sold would provide a revenue of $17,568?

98. ***Pole vaulting.*** In 1981 Vladimir Poliakov (USSR) set a world record of 19 ft $\frac{3}{4}$ in. for the pole vault (Doubleday Almanac). To reach that height, Poliakov obtained a speed of approximately 36 feet per second on the runway. The function $h = -16t^2 + 36t$ gives his height t seconds after leaving the ground.

a) Use the formula to find the exact values of t for which his height was 18 feet.

b) Use the accompanying graph to estimate the value of t for which he was at his maximum height.

c) Approximately how long was he in the air?



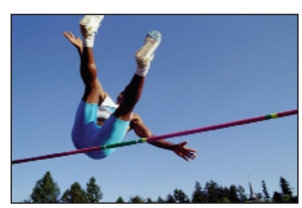


FIGURE FOR EXERCISE 98

GETTING MORE INVOLVED

99. ***Discussion.*** Which of the following equations is not a quadratic equation?

a) $\pi x^2 - \sqrt{5}x - 1 = 0$ **b)** $3x^2 - 1 = 0$

c) $4x + 5 = 0$ **d)** $0.009x^2 = 0$

100. ***Exploration.*** Solve $x^2 - 4x + k = 0$ for $k = 0, 4, 5,$ and 10.
a) When does the equation have only one solution?
b) For what values of k are the solutions real?
c) For what values of k are the solutions imaginary?

101. ***Cooperative learning.*** Write a quadratic equation of each of the following types, then trade your equations with those of a classmate. Solve the equations and verify that they are of the required types.
a) a single rational solution
b) two rational solutions
c) two irrational solutions
d) two imaginary solutions

102. ***Exploration.*** In the next section we will solve $ax^2 + bx + c = 0$ for x by completing the square. Try it now without looking ahead.

GRAPHING CALCULATOR EXERCISES

For each equation, find approximate solutions rounded to two decimal places.

103. $x^2 - 7.3x + 12.5 = 0$
104. $1.2x^2 - \pi x + \sqrt{2} = 0$
105. $2x - 3 = \sqrt{20 - x}$
106. $x^2 - 1.3x = 22.3 - x^2$

10.2 THE QUADRATIC FORMULA

In this section
- Developing the Formula
- Using the Formula
- Number of Solutions
- Applications

Completing the square from Section 10.1 can be used to solve any quadratic equation. Here we apply this method to the general quadratic equation to get a formula for the solutions to any quadratic equation.

Developing the Formula

Start with the general form of the quadratic equation,

$$ax^2 + bx + c = 0.$$

Assume a is positive for now, and divide each side by a:

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Subtract } \frac{c}{a} \text{ from each side.}$$

One-half of $\frac{b}{a}$ is $\frac{b}{2a}$, and $\frac{b}{2a}$ squared is $\frac{b^2}{4a^2}$:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Factor the left-hand side and get a common denominator for the right-hand side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \quad \frac{c(4a)}{a(4a)} = \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{Even-root property}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Because } a > 0, \sqrt{4a^2} = 2a.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$