100. ***Exploration.*** Solve $x^2 - 4x + k = 0$ for $k = 0, 4, 5,$ and 10.
a) When does the equation have only one solution?
b) For what values of k are the solutions real?
c) For what values of k are the solutions imaginary?

101. ***Cooperative learning.*** Write a quadratic equation of each of the following types, then trade your equations with those of a classmate. Solve the equations and verify that they are of the required types.
a) a single rational solution
b) two rational solutions
c) two irrational solutions
d) two imaginary solutions

102. ***Exploration.*** In the next section we will solve $ax^2 + bx + c = 0$ for x by completing the square. Try it now without looking ahead.

GRAPHING CALCULATOR EXERCISES

For each equation, find approximate solutions rounded to two decimal places.

103. $x^2 - 7.3x + 12.5 = 0$
104. $1.2x^2 - \pi x + \sqrt{2} = 0$
105. $2x - 3 = \sqrt{20 - x}$
106. $x^2 - 1.3x = 22.3 - x^2$

10.2 THE QUADRATIC FORMULA

In this section
- Developing the Formula
- Using the Formula
- Number of Solutions
- Applications

Completing the square from Section 10.1 can be used to solve any quadratic equation. Here we apply this method to the general quadratic equation to get a formula for the solutions to any quadratic equation.

Developing the Formula

Start with the general form of the quadratic equation,

$$ax^2 + bx + c = 0.$$

Assume a is positive for now, and divide each side by a:

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Subtract } \frac{c}{a} \text{ from each side.}$$

One-half of $\frac{b}{a}$ is $\frac{b}{2a}$, and $\frac{b}{2a}$ squared is $\frac{b^2}{4a^2}$:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Factor the left-hand side and get a common denominator for the right-hand side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \quad \frac{c(4a)}{a(4a)} = \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{Even-root property}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Because } a > 0,\ \sqrt{4a^2} = 2a.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We assumed a was positive so that $\sqrt{4a^2} = 2a$ would be correct. If a is negative, then $\sqrt{4a^2} = -2a$, and we get

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{-2a}.$$

However, the negative sign can be omitted in $-2a$ because of the $\pm$ symbol preceding it. For example, the results of $5 \pm (-3)$ and 5 ± 3 are the same. So when a is negative, we get the same formula as when a is positive. It is called the **quadratic formula.**

The Quadratic Formula

The solution to $ax^2 + bx + c = 0$, with $a \neq 0$, is given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Using the Formula

The quadratic formula can be used to solve any quadratic equation.

EXAMPLE 1

Two rational solutions

Solve $x^2 + 2x - 15 = 0$ using the quadratic formula.

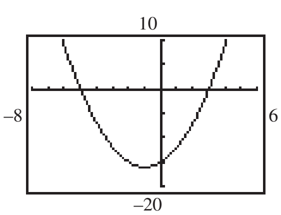
calculator close-up

Note that the two solutions to

$$x^2 + 2x - 15 = 0$$

correspond to the two x-intercepts for the graph of the function

$$y = x^2 + 2x - 15.$$



Solution

To use the formula, we first identify the values of a, b, and c:

$$\underset{a}{1}x^2 + \underset{b}{2}x \underset{c}{-15} = 0$$

The coefficient of x^2 is 1, so $a = 1$. The coefficient of $2x$ is 2, so $b = 2$. The constant term is -15, so $c = -15$. Substitute these values into the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-15)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 + 60}}{2}$$

$$= \frac{-2 \pm \sqrt{64}}{2}$$

$$= \frac{-2 \pm 8}{2}$$

$$x = \frac{-2 + 8}{2} = 3 \quad \text{or} \quad x = \frac{-2 - 8}{2} = -5$$

Check 3 and -5 in the original equation. The solution set is $\{-5, 3\}$. ■

CAUTION To identify a, b, and c for the quadratic formula, the equation must be in the standard form $ax^2 + bx + c = 0$. If it is not in that form, then you must first rewrite the equation.

EXAMPLE 2

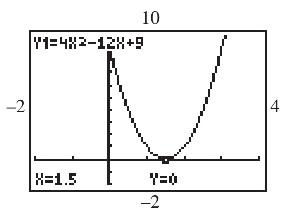
calculator close-up

Note that the single solution to

$$4x^2 - 12x + 9 = 0$$

corresponds to the single x-intercept for the graph of the function

$$y = 4x^2 - 12x + 9.$$



One rational solution

Solve $4x^2 = 12x - 9$ by using the quadratic formula.

Solution

Rewrite the equation in the form $ax^2 + bx + c = 0$ before identifying a, b, and c:

$$4x^2 - 12x + 9 = 0$$

In this form we get $a = 4$, $b = -12$, and $c = 9$.

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} \quad \text{Because } b = -12,\ -b = 12.$$

$$= \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$= \frac{12 \pm 0}{8} = \frac{12}{8} = \frac{3}{2}$$

Check $\frac{3}{2}$ in the original equation. The solution set is $\left\{\frac{3}{2}\right\}$. ■

Because the solutions to the equations in Examples 1 and 2 were rational numbers, these equations could have been solved by factoring. In the next example the solutions are irrational.

EXAMPLE 3

Two irrational solutions

Solve $2x^2 + 6x + 3 = 0$.

Solution

Let $a = 2$, $b = 6$, and $c = 3$ in the quadratic formula:

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(2)(3)}}{2(2)}$$

$$= \frac{-6 \pm \sqrt{36 - 24}}{4} = \frac{-6 \pm \sqrt{12}}{4}$$

$$= \frac{-6 \pm 2\sqrt{3}}{4} = \frac{2(-3 \pm \sqrt{3})}{2 \cdot 2}$$

$$= \frac{-3 \pm \sqrt{3}}{2}$$

Check these values in the original equation. The solution set is $\left\{\frac{-3 \pm \sqrt{3}}{2}\right\}$. ■

EXAMPLE 4

Two imaginary solutions, no real solutions

Find the complex solutions to $x^2 + x + 5 = 0$.

Solution

Let $a = 1$, $b = 1$, and $c = 5$ in the quadratic formula:

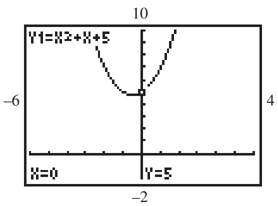
$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(5)}}{2(1)} = \frac{-1 \pm \sqrt{-19}}{2} = \frac{-1 \pm i\sqrt{19}}{2}$$

calculator close-up

Because $x^2 + x + 5 = 0$ has no real solutions, the graph of

$$y = x^2 + x + 5$$

has no x-intercepts.



Check these values in the original equation. The solution set is $\left\{\frac{-1 \pm i\sqrt{19}}{2}\right\}$. There are no real solutions to the equation. ■

You have learned to solve quadratic equations by four different methods: the even-root property, factoring, completing the square, and the quadratic formula. The even-root property and factoring are limited to certain special equations, but you should use those methods when possible. Any quadratic equation can be solved by completing the square or using the quadratic formula. Because the quadratic formula is usually faster, it is used more often than completing the square. However, completing the square is an important skill to learn. It will be used in the study of conic sections in Chapter 13.

Methods for Solving $ax^2 + bx + c = 0$

Method	Comments	Examples
Even-root property	Use when $b = 0$.	$(x - 2)^2 = 8$ $x - 2 = \pm\sqrt{8}$
Factoring	Use when the polynomial can be factored.	$x^2 + 5x + 6 = 0$ $(x + 2)(x + 3) = 0$
Quadratic formula	Solves any quadratic equation	$x^2 + 5x + 3 = 0$ $x = \frac{-5 \pm \sqrt{25 - 4(3)}}{2}$
Completing the square	Solves any quadratic equation, but quadratic formula is faster	$x^2 - 6x + 7 = 0$ $x^2 - 6x + 9 = -7 + 9$ $(x - 3)^2 = 2$

MATH AT WORK

$x^2 + (x + 1)^2 = 5^2$

KITCHEN DESIGNER

Remodeling a kitchen can be an expensive undertaking. Choosing the correct style of cabinets, doors, and floor covering is just a small part of the process. Joe Prendergast, designer for Lee Kimball Kitchens, is involved in every step of creating a new kitchen for a client.

The process begins with customers visiting the store to see what products are available. Once the client has decided on material and style, he or she fills out an extensive questionnaire so that Prendergast can get a sense of the client's lifestyle. Design plans are drawn using a scale of $\frac{1}{2}$ inch to 1 foot. Consideration is given to traffic patterns, doorways, and especially work areas where the cook would want to work unencumbered. Storage areas are a big consideration, as are lighting and color.

A new kitchen can take anywhere from 5 weeks to a few months or more, depending on how complicated the design and construction is. In Exercise 83 of this section we will find the dimensions of a border around a countertop.

Number of Solutions

The quadratic equations in Examples 1 and 3 had two real solutions each. In each of those examples the value of $b^2 - 4ac$ was positive. In Example 2 the quadratic equation had only one solution because the value of $b^2 - 4ac$ was zero. In Example 4 the quadratic equation had no real solutions because $b^2 - 4ac$ was negative. Because $b^2 - 4ac$ determines the kind and number of solutions to a quadratic equation, it is called the **discriminant.**

Number of Solutions to a Quadratic Equation

The quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$ has
two real solutions if $b^2 - 4ac > 0$,
one real solution if $b^2 - 4ac = 0$, and
no real solutions (two imaginary solutions) if $b^2 - 4ac < 0$.

EXAMPLE 5 **Using the discriminant**

Use the discriminant to determine the number of real solutions to each quadratic equation.

a) $x^2 - 3x - 5 = 0$ **b)** $x^2 = 3x - 9$ **c)** $4x^2 - 12x + 9 = 0$

Solution

a) For $x^2 - 3x - 5 = 0$, use $a = 1$, $b = -3$, and $c = -5$ in $b^2 - 4ac$:

$$b^2 - 4ac = (-3)^2 - 4(1)(-5) = 9 + 20 = 29$$

Because the discriminant is positive, there are two real solutions to this quadratic equation.

b) For $x^2 - 3x + 9 = 0$, use $a = 1$, $b = -3$, and $c = 9$ in $b^2 - 4ac$:

$$b^2 - 4ac = (-3)^2 - 4(1)(9) = 9 - 36 = -27$$

Because the discriminant is negative, the equation has no real solutions. It has two imaginary solutions.

c) For $4x^2 - 12x + 9 = 0$, use $a = 4$, $b = -12$, and $c = 9$ in $b^2 - 4ac$:

$$b^2 - 4ac = (-12)^2 - 4(4)(9) = 144 - 144 = 0$$

Because the discriminant is zero, there is only one real solution to this quadratic equation. ■

Applications

With the quadratic formula we can easily solve problems whose solutions are irrational numbers. When the solutions are irrational numbers, we usually use a calculator to find rational approximations and to check.

EXAMPLE 6 **Area of a tabletop**

The area of a rectangular tabletop is 6 square feet. If the width is 2 feet shorter than the length, then what are the dimensions?

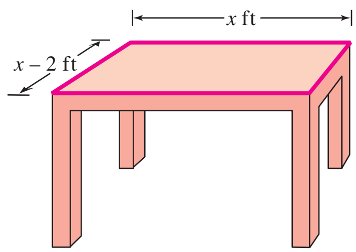


FIGURE 10.1

Solution

Let x be the length and $x - 2$ be the width, as shown in Fig. 10.1. Because the area is 6 square feet and $A = LW$, we can write the equation

$$x(x - 2) = 6$$

or

$$x^2 - 2x - 6 = 0.$$

Because this equation cannot be factored, we use the quadratic formula with $a = 1$, $b = -2$, and $c = -6$:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$$

Because $1 - \sqrt{7}$ is a negative number, it cannot be the length of a tabletop. If $x = 1 + \sqrt{7}$, then $x - 2 = 1 + \sqrt{7} - 2 = \sqrt{7} - 1$. Checking the product of $\sqrt{7} + 1$ and $\sqrt{7} - 1$, we get

$$(\sqrt{7} + 1)(\sqrt{7} - 1) = 7 - 1 = 6.$$

The exact length is $\sqrt{7} + 1$ feet, and the width is $\sqrt{7} - 1$ feet. Using a calculator, we find that the approximate length is 3.65 feet and the approximate width is 1.65 feet.

WARM-UPS

True or false? Explain.

1. Completing the square is used to develop the quadratic formula.
2. For the equation $3x^2 = 4x - 7$, we have $a = 3$, $b = 4$, and $c = -7$.
3. If $dx^2 + ex + f = 0$ and $d \neq 0$, then $x = \frac{-e \pm \sqrt{e^2 - 4df}}{2d}$.
4. The quadratic formula will not work on the equation $x^2 - 3 = 0$.
5. If $a = 2$, $b = -3$, and $c = -4$, then $b^2 - 4ac = 41$.
6. If the discriminant is zero, then there are no imaginary solutions.
7. If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has two real solutions.
8. To solve $2x - x^2 = 0$ by the quadratic formula, let $a = -1$, $b = 2$, and $c = 0$.
9. Two numbers that have a sum of 6 can be represented by x and $x + 6$.
10. Some quadratic equations have one real and one imaginary solution.

10.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the quadratic formula used for?
2. When do you use the even-root property to solve a quadratic equation?
3. When do you use factoring to solve a quadratic equation?
4. When do you use the quadratic formula to solve a quadratic equation?
5. What is the discriminant?
6. How many solutions are there to any quadratic equation in the complex number system?

Solve each equation by using the quadratic formula. See Example 1.

7. $x^2 + 5x + 6 = 0$
8. $x^2 - 7x + 12 = 0$
9. $y^2 + y = 6$
10. $m^2 + 2m = 8$
11. $6z^2 - 7z - 3 = 0$
12. $8q^2 + 2q - 1 = 0$

Solve each equation by using the quadratic formula. See Example 2.

13. $4x^2 - 4x + 1 = 0$
14. $4x^2 - 12x + 9 = 0$
15. $9x^2 - 6x + 1 = 0$
16. $9x^2 - 24x + 16 = 0$
17. $9 + 24x + 16x^2 = 0$
18. $4 + 20x = -25x^2$

Solve each equation by using the quadratic formula. See Example 3.

19. $v^2 + 8v + 6 = 0$
20. $p^2 + 6p + 4 = 0$
21. $-x^2 - 5x + 1 = 0$
22. $-x^2 - 3x + 5 = 0$
23. $2t^2 - 6t + 1 = 0$
24. $3z^2 - 8z + 2 = 0$

Solve each equation by using the quadratic formula. See Example 4.

25. $2t^2 - 6t + 5 = 0$
26. $2y^2 + 1 = 2y$
27. $-2x^2 + 3x = 6$
28. $-3x^2 - 2x - 5 = 0$
29. $\frac{1}{2}x^2 + 13 = 5x$
30. $\frac{1}{4}x^2 + \frac{17}{4} = 2x$

Find $b^2 - 4ac$ and the number of real solutions to each equation. See Example 5.

31. $x^2 - 6x + 2 = 0$
32. $x^2 + 6x + 9 = 0$
33. $2x^2 - 5x + 6 = 0$
34. $-x^2 + 3x - 4 = 0$
35. $4m^2 + 25 = 20m$
36. $v^2 = 3v + 5$
37. $y^2 - \frac{1}{2}y + \frac{1}{4} = 0$
38. $\frac{1}{2}w^2 - \frac{1}{3}w + \frac{1}{4} = 0$
39. $-3t^2 + 5t + 6 = 0$
40. $9m^2 + 16 = 24m$
41. $9 - 24z + 16z^2 = 0$
42. $12 - 7x + x^2 = 0$
43. $5x^2 - 7 = 0$
44. $-6x^2 - 5 = 0$
45. $x^2 = x$
46. $-3x^2 + 7x = 0$

Solve each equation by the method of your choice.

47. $\frac{1}{3}x^2 + \frac{1}{2}x = \frac{1}{3}$
48. $\frac{1}{2}x^2 + x = 1$
49. $\frac{w}{w - 2} = \frac{w}{w - 3}$
50. $\frac{y}{3y - 4} = \frac{2}{y + 4}$
51. $\frac{9(3x - 5)^2}{4} = 1$
52. $\frac{25(2x + 1)^2}{9} = 0$

53. $1 + \frac{20}{x^2} = \frac{8}{x}$

54. $\frac{34}{x^2} = \frac{6}{x} - 1$

55. $(x - 8)(x + 4) = -42$

56. $(x - 10)(x - 2) = -20$

57. $y = \frac{3(2y + 5)}{8(y - 1)}$

58. $z = \frac{7z - 4}{12(z - 1)}$

Use the quadratic formula and a calculator to solve each equation. Round answers to three decimal places and check your answers.

59. $x^2 + 3.2x - 5.7 = 0$

60. $x^2 + 7.15x + 3.24 = 0$

61. $x^2 - 7.4x + 13.69 = 0$

62. $1.44x^2 + 5.52x + 5.29 = 0$

63. $1.85x^2 + 6.72x + 3.6 = 0$

64. $3.67x^2 + 4.35x - 2.13 = 0$

65. $3x^2 + 14{,}379x + 243 = 0$

66. $x^2 + 12{,}347x + 6741 = 0$

67. $x^2 + 0.00075x - 0.0062 = 0$

68. $4.3x^2 - 9.86x - 3.75 = 0$

Solve each problem. See Example 6.

69. ***Missing numbers.*** Find two positive real numbers that differ by 1 and have a product of 16.

70. ***Missing numbers.*** Find two positive real numbers that differ by 2 and have a product of 10.

71. ***More missing numbers.*** Find two real numbers that have a sum of 6 and a product of 4.

72. ***More missing numbers.*** Find two real numbers that have a sum of 8 and a product of 2.

73. ***Bulletin board.*** The length of a bulletin board is one foot more than the width. The diagonal has a length of $\sqrt{3}$ feet (ft). Find the length and width of the bulletin board.

74. ***Diagonal brace.*** The width of a rectangular gate is 2 meters (m) larger than its height. The diagonal brace measures $\sqrt{6}$ m. Find the width and height.

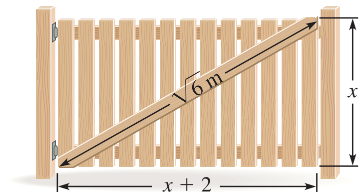


FIGURE FOR EXERCISE 74

75. ***Area of a rectangle.*** The length of a rectangle is 4 ft longer than the width, and its area is 10 square feet (ft^2). Find the length and width.

76. ***Diagonal of a square.*** The diagonal of a square is 2 m longer than a side. Find the length of a side.

If an object is given an initial velocity of v_0 feet per second from a height of s_0 feet, then its height S after t seconds is given by the formula $S = -16t^2 + v_0 t + s_0$.

77. ***Projected pine cone.*** If a pine cone is projected upward at a velocity of 16 ft/sec from the top of a 96-foot pine tree, then how long does it take to reach the earth?

78. ***Falling pine cone.*** If a pine cone falls from the top of a 96-foot pine tree, then how long does it take to reach the earth?

79. ***Penny tossing.*** If a penny is thrown downward at 30 ft/sec from the bridge at Royal Gorge, Colorado, how long does it take to reach the Arkansas River 1000 ft below?

80. ***Foul ball.*** Suppose Charlie O'Brian of the Braves hits a baseball straight upward at 150 ft/sec from a height of 5 ft.

a) Use the formula to determine how long it takes the ball to return to the earth.

b) Use the accompanying graph to estimate the maximum height reached by the ball?

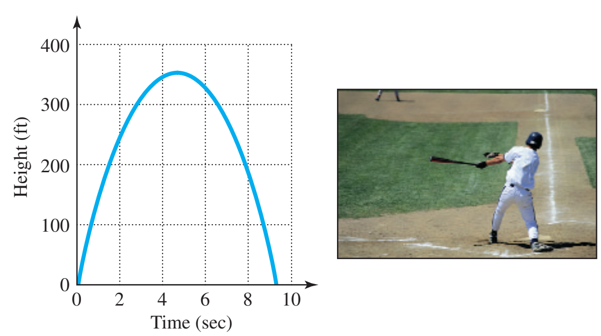


FIGURE FOR EXERCISE 80

In Exercises 81–83, solve each problem.

81. ***Recovering an investment.*** The manager at Cream of the Crop bought a load of watermelons for \$200. She priced the melons so that she would make \$1.50 profit on each melon. When all but 30 had been sold, the manager had recovered her initial investment. How many did she buy originally?

82. ***Sharing cost.*** The members of a flying club plan to share equally the cost of a \$200,000 airplane. The members want to find five more people to join the club so that the cost per person will decrease by \$2000. How many members are currently in the club?

83. ***Kitchen countertop.*** A 30 in. by 40 in. countertop for a work island is to be covered with green ceramic tiles, except for a border of uniform width as shown in the figure. If the area covered by the green tiles is 704 square inches (in.^{2}), then how wide is the border?

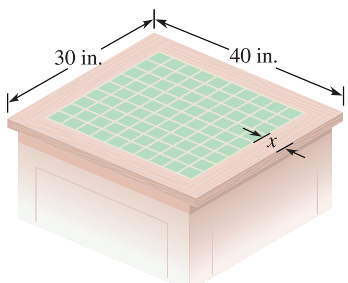


FIGURE FOR EXERCISE 83

GETTING MORE INVOLVED

84. ***Discussion.*** Find the solutions to $6x^2 + 5x - 4 = 0$. Is the sum of your solutions equal to $-\frac{b}{a}$? Explain why the sum of the solutions to any quadratic equation is $-\frac{b}{a}$. (*Hint:* Use the quadratic formula.)

85. ***Discussion.*** Use the result of Exercise 84 to check whether $\left\{\frac{2}{3}, \frac{1}{3}\right\}$ is the solution set to $9x^2 - 3x - 2 = 0$. If this solution set is not correct, then what is the correct solution set?

86. ***Discussion.*** What is the product of the two solutions to $6x^2 + 5x - 4 = 0$? Explain why the product of the solutions to any quadratic equation is $\frac{c}{a}$.

87. ***Discussion.*** Use the result of the previous exercise to check whether $\left\{\frac{9}{2}, -2\right\}$ is the solution set to $2x^2 - 13x + 18 = 0$. If this solution set is not correct, then what is the correct solution set?

88. ***Cooperative learning.*** Work in a group to write a quadratic equation that has each given pair of solutions.

a) -4 and 5 **b)** $2 - \sqrt{3}$ and $2 + \sqrt{3}$

c) $5 + 2i$ and $5 - 2i$

GRAPHING CALCULATOR EXERCISES

Determine the number of real solutions to each equation by examining the calculator graph of the corresponding function. Use the discriminant to check your conclusions.

89. $x^2 - 6.33x + 3.7 = 0$

90. $1.8x^2 + 2.4x - 895 = 0$

91. $4x^2 - 67.1x + 344 = 0$

92. $-2x^2 - 403 = 0$

93. $-x^2 + 30x - 226 = 0$

94. $16x^2 - 648x + 6562 = 0$

10.3 QUADRATIC FUNCTIONS AND THEIR GRAPHS

In this section

- Definition
- Graphing Quadratic Functions
- The Vertex and Intercepts
- Applications

We have seen *quadratic functions* on several occasions in this text, but we have not yet defined the term. In this section we study quadratic functions and their graphs.

Definition

If y is determined from x by a formula involving a quadratic polynomial, then we say that y is a *quadratic function of x*.

Quadratic Function

A **quadratic function** is a function of the form

$$y = ax^2 + bx + c,$$

where a, b, and c are real numbers and $a \neq 0$.