In Exercises 81–83, solve each problem.

81. ***Recovering an investment.*** The manager at Cream of the Crop bought a load of watermelons for \$200. She priced the melons so that she would make \$1.50 profit on each melon. When all but 30 had been sold, the manager had recovered her initial investment. How many did she buy originally?

82. ***Sharing cost.*** The members of a flying club plan to share equally the cost of a \$200,000 airplane. The members want to find five more people to join the club so that the cost per person will decrease by \$2000. How many members are currently in the club?

83. ***Kitchen countertop.*** A 30 in. by 40 in. countertop for a work island is to be covered with green ceramic tiles, except for a border of uniform width as shown in the figure. If the area covered by the green tiles is 704 square inches (in.^{2}), then how wide is the border?

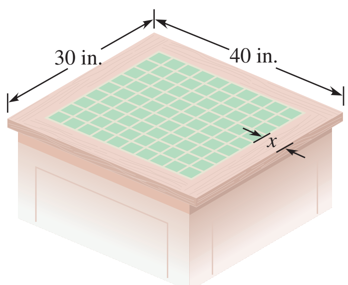


FIGURE FOR EXERCISE 83

GETTING MORE INVOLVED

84. ***Discussion.*** Find the solutions to $6x^2 + 5x - 4 = 0$. Is the sum of your solutions equal to $-\frac{b}{a}$? Explain why the sum of the solutions to any quadratic equation is $-\frac{b}{a}$. (*Hint:* Use the quadratic formula.)

85. ***Discussion.*** Use the result of Exercise 84 to check whether $\left\{\frac{2}{3}, \frac{1}{3}\right\}$ is the solution set to $9x^2 - 3x - 2 = 0$. If this solution set is not correct, then what is the correct solution set?

86. ***Discussion.*** What is the product of the two solutions to $6x^2 + 5x - 4 = 0$? Explain why the product of the solutions to any quadratic equation is $\frac{c}{a}$.

87. ***Discussion.*** Use the result of the previous exercise to check whether $\left\{\frac{9}{2}, -2\right\}$ is the solution set to $2x^2 - 13x + 18 = 0$. If this solution set is not correct, then what is the correct solution set?

88. ***Cooperative learning.*** Work in a group to write a quadratic equation that has each given pair of solutions.

a) -4 and 5 **b)** $2 - \sqrt{3}$ and $2 + \sqrt{3}$

c) $5 + 2i$ and $5 - 2i$

GRAPHING CALCULATOR EXERCISES

Determine the number of real solutions to each equation by examining the calculator graph of the corresponding function. Use the discriminant to check your conclusions.

89. $x^2 - 6.33x + 3.7 = 0$

90. $1.8x^2 + 2.4x - 895 = 0$

91. $4x^2 - 67.1x + 344 = 0$

92. $-2x^2 - 403 = 0$

93. $-x^2 + 30x - 226 = 0$

94. $16x^2 - 648x + 6562 = 0$

10.3 QUADRATIC FUNCTIONS AND THEIR GRAPHS

In this section

- Definition
- Graphing Quadratic Functions
- The Vertex and Intercepts
- Applications

We have seen *quadratic functions* on several occasions in this text, but we have not yet defined the term. In this section we study quadratic functions and their graphs.

Definition

If y is determined from x by a formula involving a quadratic polynomial, then we say that y is a *quadratic function of x.*

Quadratic Function

A **quadratic function** is a function of the form

$$y = ax^2 + bx + c,$$

where a, b, and c are real numbers and $a \neq 0$.

Without the term ax^2, this function would be a linear function. That is why we specify that $a \neq 0$.

EXAMPLE 1 **Finding ordered pairs of a quadratic function**

Complete each ordered pair so that it satisfies the given equation.

a) $y = x^2 - x - 6$; (2,), (, 0)

b) $s = -16t^2 + 48t + 84$; (0,), (, 20)

Solution

a) If $x = 2$, then $y = 2^2 - 2 - 6 = -4$. So the ordered pair is $(2, -4)$. To find x when $y = 0$, replace y by 0 and solve the resulting quadratic equation:

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 3 \quad \text{or} \quad x = -2$$

The ordered pairs are $(-2, 0)$ and $(3, 0)$.

b) If $t = 0$, then $s = -16 \cdot 0^2 + 48 \cdot 0 + 84 = 84$. The ordered pair is (0, 84). To find t when $s = 20$, replace s by 20 and solve the equation for t:

$$-16t^2 + 48t + 84 = 20$$
$$-16t^2 + 48t + 64 = 0$$
$$t^2 - 3t - 4 = 0$$
$$(t - 4)(t + 1) = 0$$
$$t - 4 = 0 \quad \text{or} \quad t + 1 = 0$$
$$t = 4 \quad \text{or} \quad t = -1$$

The ordered pairs are $(-1, 20)$ and $(4, 20)$. ■

study tip

Although you should avoid cramming, there are times when you have no other choice. In this case concentrate on what is in your class notes and the homework assignments. Try to work one or two problems of each type. Instructors often ask some relatively easy questions on a test to see if you have understood the major ideas.

CAUTION When variables other than x and y are used, the independent variable is the first coordinate of an ordered pair, and the dependent variable is the second coordinate. In Example 1(b), t is the independent variable and first coordinate because s depends on t by the formula $s = -16t^2 + 48t + 84$.

Graphing Quadratic Functions

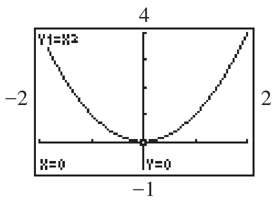
Any real number may be used for x in the formula $y = ax^2 + bx + c$. So the domain (the set of x-coordinates) for any quadratic function is the set of all real numbers, $(-\infty, \infty)$. The range (the set of y-coordinates) can be determined from the graph. All quadratic functions have graphs that are similar in shape. The graph of any quadratic function is called a **parabola.**

EXAMPLE 2 **Graphing the simplest quadratic function**

Graph the function $y = x^2$, and state the domain and range.

calculator close-up

This close-up view of $y = x^2$ shows how rounded the curve is at the bottom. When drawing a parabola by hand, be sure to draw it smoothly.



Solution

Make a table of values for x and y:

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4

See Fig. 10.2 for the graph. The domain is the set of all real numbers, $(-\infty, \infty)$, because we can use any real number for x. From the graph we see that the smallest y-coordinate of the function is 0. So the range is the set of real numbers that are greater than or equal to 0, $[0, \infty)$.

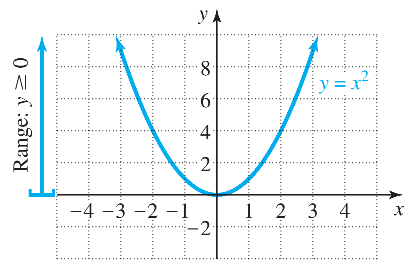


FIGURE 10.2

The parabola in Fig. 10.2 is said to **open upward.** In the next example we see a parabola that **opens downward.** If $a > 0$ in the equation $y = ax^2 + bx + c$, then the parabola opens upward. If $a < 0$, then the parabola opens downward.

EXAMPLE 3

A quadratic function

Graph the function $y = 4 - x^2$, and state the domain and range.

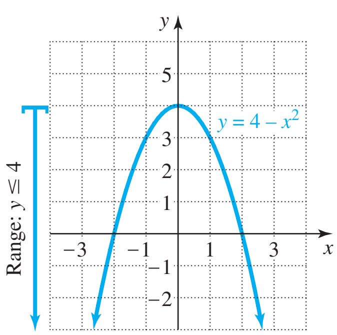


FIGURE 10.3

Solution

We plot enough points to get the correct shape of the graph:

x	-2	-1	0	1	2
$y = 4 - x^2$	0	3	4	3	0

See Fig. 10.3 for the graph. The domain is the set of all real numbers, $(-\infty, \infty)$. From the graph we see that the largest y-coordinate is 4. So the range is $(-\infty, 4]$.

The Vertex and Intercepts

The lowest point on a parabola that opens upward or the highest point on a parabola that opens downward is called the **vertex.** The y-coordinate of the vertex is the **minimum value** of the function if the parabola opens upward, and it is the **maximum value** of the function if the parabola opens downward. For $y = x^2$ the vertex is $(0, 0)$, and 0 is the minimum value of the function. For $g(x) = 4 - x^2$ the vertex is $(0, 4)$, and 4 is the maximum value of the function.

Because the vertex is either the highest or lowest point on a parabola, it is an important point to find before drawing the graph. The vertex can be found by using the following fact.

helpful hint

To draw a parabola or any curve by hand, use your hand like a compass. The two halves of a parabola should be drawn in two steps. Position your paper so that your hand is approximately at the "center" of the arc you are trying to draw.

Vertex of a Parabola

The x-coordinate of the vertex of $y = ax^2 + bx + c$ is $\frac{-b}{2a}$, provided that $a \neq 0$.

You can remember $\frac{-b}{2a}$ by observing that it is part of the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When you graph a parabola, you should always locate the vertex because it is the point at which the graph "turns around." With the vertex and several nearby points you can see the correct shape of the parabola.

EXAMPLE 4 **Using the vertex in graphing a quadratic function**

Graph $y = -x^2 - x + 2$, and state the domain and range.

Solution

First find the x-coordinate of the vertex:

$$x = \frac{-b}{2a} = \frac{-(-1)}{2(-1)} = \frac{1}{-2} = -\frac{1}{2}$$

Now find y for $x = -\frac{1}{2}$:

$$y = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 2 = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{9}{4}$$

The vertex is $\left(-\frac{1}{2}, \frac{9}{4}\right)$. Now find a few points on either side of the vertex:

x	-2	-1	$-\frac{1}{2}$	0	1
$y = -x^2 - x + 2$	0	2	$\frac{9}{4}$	2	0

Sketch a parabola through these points as in Fig. 10.4. The domain is $(-\infty, \infty)$. Because the graph goes no higher than $\frac{9}{4}$, the range is $\left(-\infty, \frac{9}{4}\right]$.

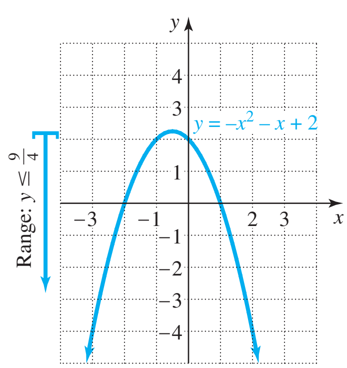


FIGURE 10.4

The y-intercept of a parabola is the point that has 0 as the first coordinate. The x-intercepts are the points that have 0 as their second coordinates.

EXAMPLE 5 **Using the intercepts in graphing a quadratic function**

Find the vertex and intercepts, and sketch the graph of each function.

a) $y = x^2 - 2x - 8$ **b)** $s = -16t^2 + 64t$

Solution

a) Use $x = \frac{-b}{2a}$ to get $x = 1$ as the x-coordinate of the vertex. If $x = 1$, then

$$\begin{aligned} y &= 1^2 - 2 \cdot 1 - 8 \\ &= -9. \end{aligned}$$

So the vertex is $(1, -9)$. If $x = 0$, then

$$\begin{aligned} y &= 0^2 - 2 \cdot 0 - 8 \\ &= -8. \end{aligned}$$

The y-intercept is $(0, -8)$. To find the x-intercepts, replace y by 0:

$$\begin{aligned} x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x + 2 = 0 \\ x = 4 \quad &\text{or} \quad x = -2 \end{aligned}$$

The x-intercepts are $(-2, 0)$ and $(4, 0)$. The graph is shown in Fig. 10.5.

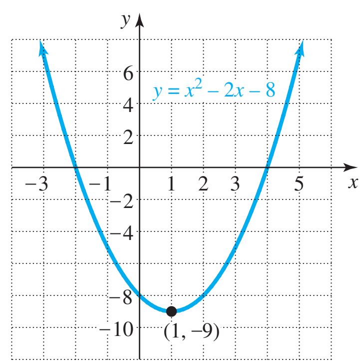


FIGURE 10.5

b) Because s is expressed as a function of t, the first coordinate is t. Use $t = \frac{-b}{2a}$ to get

$$t = \frac{-64}{2(-16)} = 2.$$

If $t = 2$, then

$$\begin{aligned} s &= -16 \cdot 2^2 + 64 \cdot 2 \\ &= 64. \end{aligned}$$

So the vertex is $(2, 64)$. If $t = 0$, then

$$\begin{aligned} s &= -16 \cdot 0^2 + 64 \cdot 0 \\ &= 0. \end{aligned}$$

So the *s*-intercept is (0, 0). To find the *t*-intercepts, replace *s* by 0:

$$-16t^2 + 64t = 0$$
$$-16t(t - 4) = 0$$
$$-16t = 0 \quad \text{or} \quad t - 4 = 0$$
$$t = 0 \quad \text{or} \quad t = 4$$

The *t*-intercepts are (0, 0) and (4, 0). The graph is shown in Fig. 10.6.

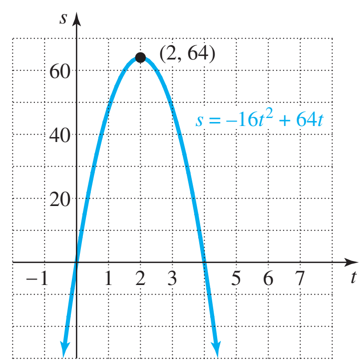
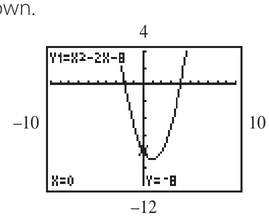


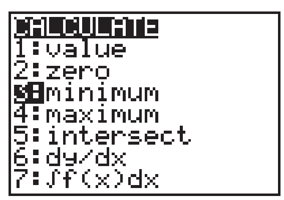
FIGURE 10.6

calculator close-up

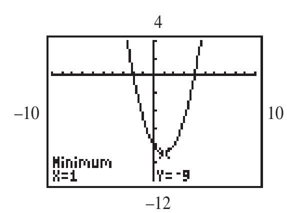
You can find the vertex of a parabola with a calculator by using either the maximum or minimum feature. First graph the parabola as shown.



Because this parabola opens upward, the *y*-coordinate of the vertex is the minimum *y*-coordinate on the graph. Press CALC and choose minimum.



The calculator will ask for a left bound, a right bound, and a guess. For the left bound choose a point to the left of the vertex by moving the cursor to the point and pressing ENTER. For the right bound choose a point to the right of the vertex. For the guess choose a point close to the vertex.



Applications

In applications we are often interested in finding the maximum or minimum value of a variable. If the graph of a quadratic function opens downward, then the maximum value of the second coordinate is the second coordinate of the vertex. If the parabola opens upward, then the minimum value of the second coordinate is the second coordinate of the vertex.

EXAMPLE 6 **Finding the maximum height**

If a projectile is launched with an initial velocity of v_0 feet per second from an initial height of s_0 feet, then its height $s(t)$ in feet is determined by the quadratic function $s(t) = -16t^2 + v_0t + s_0$, where *t* is the time in seconds. If a ball is tossed upward with velocity 64 feet per second from a height of 5 feet, then what is the maximum height reached by the ball?

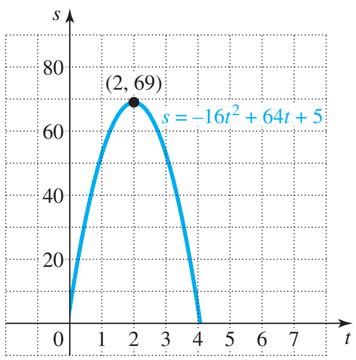


FIGURE 10.7

Solution

The height $s(t)$ of the ball for any time t is given by $s(t) = -16t^2 + 64t + 5$. Because the maximum height occurs at the vertex of the parabola, we use $t = \frac{-b}{2a}$ to find the vertex:

$$t = \frac{-64}{2(-16)} = 2$$

Now use $t = 2$ to find the second coordinate of the vertex:

$$s(2) = -16(2)^2 + 64(2) + 5 = 69$$

The maximum height reached by the ball is 69 feet. See Fig. 10.7.

WARM-UPS

True or false? Explain your answer.

1. The ordered pair $(-2, -1)$ satisfies $y = x^2 - 5$.
2. The y-intercept for $y = x^2 - 3x + 9$ is $(9, 0)$.
3. The x-intercepts for $y = x^2 - 5$ are $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$.
4. The graph of $y = x^2 - 12$ opens upward.
5. The graph of $y = 4 + x^2$ opens downward.
6. The vertex of $y = x^2 + 2x$ is $(-1, -1)$.
7. The parabola $y = x^2 + 1$ has no x-intercepts.
8. The y-intercept for $y = ax^2 + bx + c$ is $(0, c)$.
9. If $w = -2v^2 + 9$, then the maximum value of w is 9.
10. If $y = 3x^2 - 7x + 9$, then the maximum value of y occurs when $x = \frac{7}{6}$.

10.3 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a quadratic function?
2. What is a parabola?
3. When does a parabola open upward and when does a parabola open downward?
4. What is the domain of any quadratic function?
5. What is the vertex of a parabola?
6. How can you find the vertex of a parabola?

Complete each ordered pair so that it satisfies the given equation. See Example 1.

7. $y = x^2 - x - 12$ (3,), (, 0)
8. $y = -\frac{1}{2}x^2 - x + 1$ (0,), (, −3)

9. $s = -16t^2 + 32t$ $(4, \quad), (\quad, 0)$

10. $a = b^2 + 4b + 5$ $(-2, \quad), (\quad, 2)$

Graph each quadratic function, and state its domain and range. See Examples 2 and 3.

11. $y = x^2 + 2$

12. $y = x^2 - 4$

13. $y = \frac{1}{2}x^2 - 4$

14. $y = \frac{1}{3}x^2 - 6$

15. $y = -2x^2 + 5$

16. $y = -x^2 - 1$

17. $y = -\frac{1}{3}x^2 + 5$

18. $y = -\frac{1}{2}x^2 + 3$

19. $y = (x - 2)^2$

20. $y = (x + 3)^2$

Find the vertex and intercepts for each quadratic function. Sketch the graph, and state the domain and range. See Examples 4 and 5.

21. $y = x^2 - x - 2$

22. $y = x^2 + 2x - 3$

23. $y = x^2 + 2x - 8$

24. $y = x^2 + x - 6$

25. $y = -x^2 - 4x - 3$

26. $y = -x^2 - 5x - 4$

27. $y = -x^2 + 3x + 4$

28. $y = -x^2 - 2x + 8$

29. $a = b^2 - 6b - 16$

30. $v = -u^2 - 8u + 9$

Find the maximum or minimum value of y for each function.

31. $y = x^2 - 8$

32. $y = 33 - x^2$

33. $y = -3x^2 + 14$

34. $y = 6 + 5x^2$

35. $y = x^2 + 2x + 3$

36. $y = x^2 - 2x + 5$

37. $y = -2x^2 - 4x$

38. $y = -3x^2 + 24x$

Solve each problem. See Example 6.

39. ***Maximum height.*** If a baseball is projected upward from ground level with an initial velocity of 64 feet per second, then its height is a function of time, given by $s(t) = -16t^2 + 64t$. Graph this function for $0 \le t \le 4$. What is the maximum height reached by the ball?

40. ***Maximum height.*** If a soccer ball is kicked straight up with an initial velocity of 32 feet per second, then its height above the earth is a function of time given by $s(t) = -16t^2 + 32t$. Graph this function for $0 \le t \le 2$. What is the maximum height reached by this ball?

41. ***Minimum cost.*** It costs Acme Manufacturing C dollars per hour to operate its golf ball division. An analyst has determined that C is related to the number of golf balls produced per hour, x, by the equation $C = 0.009x^2 - 1.8x + 100$. What number of balls per hour should Acme produce to minimize the cost per hour of manufacturing these golf balls?

42. ***Maximum profit.*** A chain store manager has been told by the main office that daily profit, P, is related to the number of clerks working that day, x, according to the equation $P = -25x^2 + 300x$. What number of clerks will maximize the profit, and what is the maximum possible profit?

43. ***Maximum area.*** Jason plans to fence a rectangular area with 100 meters of fencing. He has written the formula $A = w(50 - w)$ to express the area in terms of the width w. What is the maximum possible area that he can enclose with his fencing?

FIGURE FOR EXERCISE 43

44. ***Minimizing cost.*** A company uses the function $C(x) = 0.02x^2 - 3.4x + 150$ to model the unit cost in dollars for producing x stabilizer bars. For what number of bars is the unit cost at its minimum? What is the unit cost at that level of production?

45. ***Air pollution.*** The amount of nitrogen dioxide A in parts per million (ppm) that was present in the air in the city of Homer on a certain day in June is modeled by the function

$$A(t) = -2t^2 + 32t + 12,$$

where t is the number of hours after 6:00 A.M. Use this function to find the time at which the nitrogen dioxide level was at its maximum.

46. ***Stabilization ratio.*** The stabilization ratio (births/deaths) for South and Central America can be modeled by the function

$$y = -0.0012x^2 + 0.074x + 2.69$$

where y is the number of births divided by the number of deaths in the year $1950 + x$ (World Resources Institute, www.wri.org).

a) Use the graph to estimate the year in which the stabilization ratio was at its maximum.

b) Use the function to find the year in which the stabilization ratio was at its maximum.

c) What was the maximum stabilization ratio from part (b)?

d) What is the significance of a stabilization ratio of 1?

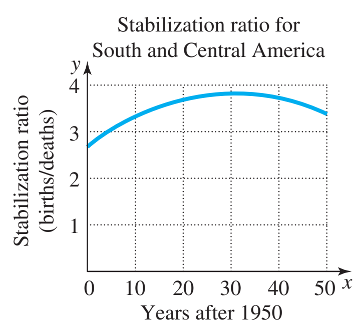


FIGURE FOR EXERCISE 46

47. ***Suspension bridge.*** The cable of the suspension bridge shown in the accompanying figure hangs in the shape of a parabola with equation $y = 0.0375x^2$, where x and y are in meters. What is the height of each tower above the roadway? What is the length z for the cable bracing the tower?

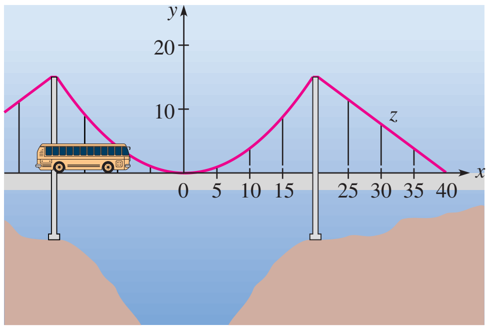


FIGURE FOR EXERCISE 47

GETTING MORE INVOLVED

48. ***Exploration.***

a) Write the function $y = 3(x - 2)^2 + 6$ in the form $y = ax^2 + bx + c$, and find the vertex of the parabola using the formula $x = \frac{-b}{2a}$.

b) Repeat part (a) with the functions $y = -4(x - 5)^2 - 9$ and $y = 3(x + 2)^2 - 6$.

c) What is the vertex for a parabola that is written in the form $y = a(x - h)^2 + k$? Explain your answer.

GRAPHING CALCULATOR EXERCISES

49. Graph $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$ on the same coordinate system. What can you say about the graph of $y = kx^2$?

50. Graph $y = x^2$, $y = (x - 3)^2$, and $y = (x + 3)^2$ on the same coordinate system. How does the graph of $y = (x - k)^2$ compare to the graph of $y = x^2$?

51. The equation $x = y^2$ is equivalent to $y = \pm\sqrt{x}$. Graph both $y = \sqrt{x}$ and $y = -\sqrt{x}$ on a graphing calculator. How does the graph of $x = y^2$ compare to the graph of $y = x^2$?

52. Graph each of the following equations by solving for y.

a) $x = y^2 - 1$

b) $x = -y^2$

c) $x^2 + y^2 = 4$

53. Determine the approximate vertex, domain, range, and x-intercepts for each quadratic function.

a) $y = 3.2x^2 - 5.4x + 1.6$

b) $y = -1.09x^2 + 13x + 7.5$