10.4 MORE ON QUADRATIC EQUATIONS

In this section

- Using the Discriminant in Factoring
- Writing a Quadratic with Given Solutions
- Equations Quadratic in Form
- Applications

In this section we use the ideas and methods of the previous sections to explore additional topics involving quadratic equations.

Using the Discriminant in Factoring

Consider $ax^2 + bx + c$, where a, b, and c are integers with a greatest common factor of 1. If $b^2 - 4ac$ is a perfect square, then $\sqrt{b^2 - 4ac}$ is a whole number, and the solutions to $ax^2 + bx + c = 0$ are rational numbers. If the solutions to a quadratic equation are rational numbers, then they could be found by the factoring method. So if $b^2 - 4ac$ is a perfect square, then $ax^2 + bx + c$ factors. It is also true that if $b^2 - 4ac$ is not a perfect square, then $ax^2 + bx + c$ is prime.

EXAMPLE 1 **Using the discriminant**

Use the discriminant to determine whether each polynomial can be factored.

a) $6x^2 + x - 15$ **b)** $5x^2 - 3x + 2$

Solution

a) Use $a = 6$, $b = 1$, and $c = -15$ to find $b^2 - 4ac$:

$$b^2 - 4ac = 1^2 - 4(6)(-15) = 361$$

Because $\sqrt{361} = 19$, $6x^2 + x - 15$ can be factored. Using the ac method, we get

$$6x^2 + x - 15 = (2x - 3)(3x + 5).$$

b) Use $a = 5$, $b = -3$, and $c = 2$ to find $b^2 - 4ac$:

$$b^2 - 4ac = (-3)^2 - 4(5)(2) = -31$$

Because the discriminant is not a perfect square, $5x^2 - 3x + 2$ is prime. ■

Writing a Quadratic with Given Solutions

Not every quadratic equation can be solved by factoring, but the factoring method can be used (in reverse) to write a quadratic equation with any given solutions. For example, if the solutions to a quadratic equation are 5 and -3, we can reverse the steps in the factoring method as follows:

$$x = 5 \quad \text{or} \quad x = -3$$
$$x - 5 = 0 \quad \text{or} \quad x + 3 = 0$$
$$(x - 5)(x + 3) = 0 \quad \text{Zero factor property}$$
$$x^2 - 2x - 15 = 0 \quad \text{Multiply the factors.}$$

This method will produce the equation even if the solutions are irrational or imaginary.

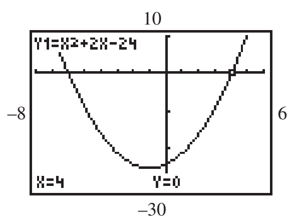
EXAMPLE 2 **Writing a quadratic given the solutions**

Write a quadratic equation that has each given pair of solutions.

a) $4, -6$ **b)** $-\sqrt{2}, \sqrt{2}$ **c)** $-3i, 3i$

calculator close-up

The graph of $y = x^2 + 2x - 24$ supports the conclusion in Example 2(a) because the graph crosses the x-axis at $(4, 0)$ and $(-6, 0)$.



Solution

a) Reverse the factoring method using solutions 4 and -6:

$$x = 4 \quad \text{or} \quad x = -6$$
$$x - 4 = 0 \quad \text{or} \quad x + 6 = 0$$
$$(x - 4)(x + 6) = 0 \quad \text{Zero factor property}$$
$$x^2 + 2x - 24 = 0 \quad \text{Multiply the factors.}$$

b) Reverse the factoring method using solutions $-\sqrt{2}$ and $\sqrt{2}$:

$$x = -\sqrt{2} \quad \text{or} \quad x = \sqrt{2}$$
$$x + \sqrt{2} = 0 \quad \text{or} \quad x - \sqrt{2} = 0$$
$$(x + \sqrt{2})(x - \sqrt{2}) = 0 \quad \text{Zero factor property}$$
$$x^2 - 2 = 0 \quad \text{Multiply the factors.}$$

c) Reverse the factoring method using solutions $-3i$ and $3i$:

$$x = -3i \quad \text{or} \quad x = 3i$$
$$x + 3i = 0 \quad \text{or} \quad x - 3i = 0$$
$$(x + 3i)(x - 3i) = 0 \quad \text{Zero factor property}$$
$$x^2 - 9i^2 = 0 \quad \text{Multiply the factors.}$$
$$x^2 + 9 = 0 \quad \text{Note: } i^2 = -1$$

■

Equations Quadratic in Form

In a quadratic equation we have a variable and its square (x and x^2). An equation that contains an expression and the square of that expression is **quadratic in form** if substituting a single variable for that expression results in a quadratic equation. Equations that are quadratic in form can be solved by using methods for quadratic equations.

EXAMPLE 3 **An equation quadratic in form**

Solve $(x + 15)^2 - 3(x + 15) - 18 = 0$

Solution

Note that $x + 15$ and $(x + 15)^2$ both appear in the equation. Let $a = x + 15$ and substitute a for $x + 15$ in the equation:

$$(x + 15)^2 - 3(x + 15) - 18 = 0$$
$$a^2 - 3a - 18 = 0$$
$$(a - 6)(a + 3) = 0 \quad \text{Factor.}$$
$$a - 6 = 0 \quad \text{or} \quad a + 3 = 0$$
$$a = 6 \quad \text{or} \quad a = -3$$
$$x + 15 = 6 \quad \text{or} \quad x + 15 = -3 \quad \text{Replace } a \text{ by } x + 15.$$
$$x = -9 \quad \text{or} \quad x = -18$$

Check in the original equation. The solution set is $\{-18, -9\}$. ■

In the next example we have a fourth-degree equation that is quadratic in form. Note that the fourth-degree equation has four solutions.

EXAMPLE 4

A fourth-degree equation

Solve $x^4 - 6x^2 + 8 = 0$.

Solution

Note that x^4 is the square of x^2. If we let $w = x^2$, then $w^2 = x^4$. Substitute these expressions into the original equation.

$$
\begin{aligned}
x^4 - 6x^2 + 8 &= 0 \\
w^2 - 6w + 8 &= 0 && \text{Replace } x^4 \text{ by } w^2 \text{ and } x^2 \text{ by } w. \\
(w - 2)(w - 4) &= 0 && \text{Factor.} \\
w - 2 = 0 \quad &\text{or} \quad w - 4 = 0 \\
w = 2 \quad &\text{or} \quad w = 4 \\
x^2 = 2 \quad &\text{or} \quad x^2 = 4 && \text{Substitute } x^2 \text{ for } w. \\
x = \pm\sqrt{2} \quad &\text{or} \quad x = \pm 2 && \text{Even-root property}
\end{aligned}
$$

Check. The solution set is $\{-2, -\sqrt{2}, \sqrt{2}, 2\}$.

helpful hint

The fundamental theorem of algebra says that the number of solutions to a polynomial equation is less than or equal to the degree of the polynomial. This famous theorem was proved by Carl Friedrich Gauss when he was a young man.

CAUTION If you replace x^2 by w, do not quit when you find the values of w. If the variable in the original equation is x, then you must solve for x.

EXAMPLE 5

A quadratic within a quadratic

Solve $(x^2 + 2x)^2 - 11(x^2 + 2x) + 24 = 0$.

Solution

Note that $x^2 + 2x$ and $(x^2 + 2x)^2$ appear in the equation. Let $a = x^2 + 2x$ and substitute.

$$
\begin{aligned}
a^2 - 11a + 24 &= 0 \\
(a - 8)(a - 3) &= 0 && \text{Factor.} \\
a - 8 = 0 \quad &\text{or} \quad a - 3 = 0 \\
a = 8 \quad &\text{or} \quad a = 3 \\
x^2 + 2x = 8 \quad &\text{or} \quad x^2 + 2x = 3 && \text{Replace } a \text{ by } x^2 + 2x. \\
x^2 + 2x - 8 = 0 \quad &\text{or} \quad x^2 + 2x - 3 = 0 \\
(x - 2)(x + 4) = 0 \quad &\text{or} \quad (x + 3)(x - 1) = 0 \\
x - 2 = 0 \quad \text{or} \quad x + 4 = 0 \quad &\text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 1 = 0 \\
x = 2 \quad \text{or} \quad x = -4 \quad &\text{or} \quad x = -3 \quad \text{or} \quad x = 1
\end{aligned}
$$

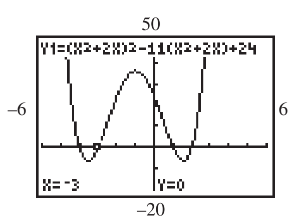
Check. The solution set is $\{-4, -3, 1, 2\}$.

calculator close-up

The four x-intercepts on the graph of

$$y = (x^2 + 2x)^2 - 11(x^2 + 2x) + 24$$

support the conclusion in Example 5.



The next example involves a fractional exponent. To identify this type of equation as quadratic in form, recall how to square an expression with a fractional exponent. For example, $(x^{1/2})^2 = x$, $(x^{1/4})^2 = x^{1/2}$, and $(x^{1/3})^2 = x^{2/3}$.

EXAMPLE 6

A fractional exponent

Solve $x - 9x^{1/2} + 14 = 0$.

Solution

Note that the square of $x^{1/2}$ is x. Let $w = x^{1/2}$; then $w^2 = (x^{1/2})^2 = x$. Now substitute w and w^2 into the original equation:

$$w^2 - 9w + 14 = 0$$
$$(w - 7)(w - 2) = 0$$
$$w - 7 = 0 \quad \text{or} \quad w - 2 = 0$$
$$w = 7 \quad \text{or} \quad w = 2$$
$$x^{1/2} = 7 \quad \text{or} \quad x^{1/2} = 2 \qquad \text{Replace } w \text{ by } x^{1/2}.$$
$$x = 49 \quad \text{or} \quad x = 4 \qquad \text{Square each side.}$$

Because we squared each side, we must check for extraneous roots. First evaluate $x - 9x^{1/2} + 14$ for $x = 49$:

$$49 - 9 \cdot 49^{1/2} + 14 = 49 - 9 \cdot 7 + 14 = 0$$

Now evaluate $x - 9x^{1/2} + 14$ for $x = 4$:

$$4 - 9 \cdot 4^{1/2} + 14 = 4 - 9 \cdot 2 + 14 = 0$$

Because each solution checks, the solution set is $\{4, 49\}$.

CAUTION An equation of quadratic form must have a term that is the square of another. Equations such as $x^4 - 5x^3 + 6 = 0$ or $x^{1/2} - 3x^{1/3} - 18 = 0$ are not quadratic in form and cannot be solved by substitution.

Applications

Applied problems often result in quadratic equations that cannot be factored. For such equations we use the quadratic formula to find exact solutions and a calculator to find decimal approximations for the exact solutions.

EXAMPLE 7

Changing area

Marvin's flower bed is rectangular in shape with a length of 10 feet and a width of 5 feet (ft). He wants to increase the length and width by the same amount to obtain a flower bed with an area of 75 square feet (ft^2). What should the amount of increase be?

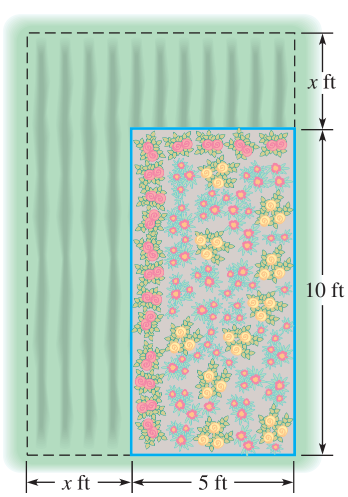


FIGURE 10.8

Solution

Let x be the amount of increase. The length and width of the new flower bed are $x + 10$ ft and $x + 5$ ft, as shown in Fig. 10.8. Because the area is to be 75 ft^2, we have

$$(x + 10)(x + 5) = 75.$$

Write this equation in the form $ax^2 + bx + c = 0$:

$$x^2 + 15x + 50 = 75$$
$$x^2 + 15x - 25 = 0 \qquad \text{Get 0 on the right.}$$
$$x = \frac{-15 \pm \sqrt{225 - 4(1)(-25)}}{2(1)}$$
$$= \frac{-15 \pm \sqrt{325}}{2} = \frac{-15 \pm 5\sqrt{13}}{2}$$

Because the value of x must be positive, the exact increase is

$$\frac{-15 + 5\sqrt{13}}{2} \text{ ft.}$$

Using a calculator, we can find that x is approximately 1.51 ft. If $x = 1.51$ ft, then the new length is 11.51 ft, and the new width is 6.51 ft. The area of a rectangle with these dimensions is 74.93 ft^2. Of course, the approximate dimensions do not give exactly 75 ft^2. ■

EXAMPLE 8 **Mowing the lawn**

It takes Carla 1 hour longer to mow the lawn than it takes Sharon to mow the lawn. If they can mow the lawn in 5 hours working together, then how long would it take each girl by herself?

Solution

If Sharon can mow the lawn by herself in x hours, then she works at the rate of $\frac{1}{x}$ of the lawn per hour. If Carla can mow the lawn by herself in $x + 1$ hours, then she works at the rate of $\frac{1}{x+1}$ of the lawn per hour. We can use a table to list all of the important quantities.

	Rate	Time	Work
Sharon	$\frac{1}{x}\frac{\text{lawn}}{\text{hr}}$	5 hr	$\frac{5}{x}$ lawn
Carla	$\frac{1}{x+1}\frac{\text{lawn}}{\text{hr}}$	5 hr	$\frac{5}{x+1}$ lawn

Because they complete the lawn in 5 hours, the portion of the lawn done by Sharon and the portion done by Carla have a sum of 1:

$$\frac{5}{x} + \frac{5}{x+1} = 1$$

$$x(x + 1)\frac{5}{x} + x(x + 1)\frac{5}{x+1} = x(x + 1)1 \quad \text{Multiply by the LCD.}$$

$$5x + 5 + 5x = x^2 + x$$

$$10x + 5 = x^2 + x$$

$$-x^2 + 9x + 5 = 0$$

$$x^2 - 9x - 5 = 0$$

$$x = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{9 \pm \sqrt{101}}{2}$$

Using a calculator, we find that $\frac{9 - \sqrt{101}}{2}$ is negative. So Sharon's time alone is

$$\frac{9 + \sqrt{101}}{2} \text{ hours.}$$

helpful hint

Note that the equation concerns the portion of the job done by each girl. We could have written an equation about the rates at which the two girls work. Because they can finish the lawn together in 5 hours, they are mowing together at the rate of $\frac{1}{5}$ lawn per hour. So

$$\frac{1}{x} + \frac{1}{x+1} = \frac{1}{5}.$$

To find Carla's time alone, we add one hour to Sharon's time. So Carla's time alone is

$$\frac{9+\sqrt{101}}{2}+1=\frac{9+\sqrt{101}}{2}+\frac{2}{2}=\frac{11+\sqrt{101}}{2}\text{ hours.}$$

Sharon's time alone is approximately 9.525 hours, and Carla's time alone is approximately 10.525 hours. ■

WARM-UPS

True or false? Explain your answer.

1. To solve $x^4 - 5x^2 + 6 = 0$ by substitution, we can let $w = x^2$.
2. We can solve $x^5 - 3x^3 - 10 = 0$ by substitution if we let $w = x^3$.
3. We always use the quadratic formula on equations of quadratic form.
4. If $w = x^{1/6}$, then $w^2 = x^{1/3}$.
5. To solve $x - 7\sqrt{x} + 10 = 0$ by substitution, we let $\sqrt{w} = x$.
6. If $y = 2^{1/2}$, then $y^2 = 2^{1/4}$.
7. If John paints a 100-foot fence in x hours, then his rate is $\frac{100}{x}$ of the fence per hour.
8. If Elvia drives 300 miles in x hours, then her rate is $\frac{300}{x}$ miles per hour (mph).
9. If Ann's boat goes 10 mph in still water, then against a 5-mph current, it will go 2 mph.
10. If squares with sides of length x inches are cut from the corners of an 11-inch by 14-inch rectangular piece of sheet metal and the sides are folded up to form a box, then the dimensions of the bottom will be $11 - x$ by $14 - x$.

10.4 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. How can you use the discriminant to determine if a quadratic polynomial can be factored?
2. What is the relationship between solutions to a quadratic equation and factors of a quadratic polynomial?
3. How do we write a quadratic equation with given solutions?
4. What is an equation quadratic in form?

Use the discriminant to determine whether each quadratic polynomial can be factored, then factor the ones that are not prime. See Example 1.

5. $2x^2 - x + 4$
6. $2x^2 + 3x - 5$
7. $2x^2 + 6x - 5$
8. $3x^2 + 5x - 1$
9. $6x^2 + 19x - 36$
10. $8x^2 + 6x - 27$
11. $4x^2 - 5x - 12$
12. $4x^2 - 27x + 45$

13. $8x^2 - 18x - 45$

14. $6x^2 + 9x - 16$

Write a quadratic equation that has each given pair of solutions. See Example 2.

15. $3, -7$

16. $-8, 2$

17. $4, 1$

18. $3, 2$

19. $\sqrt{5}, -\sqrt{5}$

20. $-\sqrt{7}, \sqrt{7}$

21. $4i, -4i$

22. $-3i, 3i$

23. $i\sqrt{2}, -i\sqrt{2}$

24. $3i\sqrt{2}, -3i\sqrt{2}$

25. $\frac{1}{2}, \frac{1}{3}$

26. $-\frac{1}{5}, -\frac{1}{2}$

Find all real solutions to each equation. See Example 3.

27. $(2a - 1)^2 + 2(2a - 1) - 8 = 0$

28. $(3a + 2)^2 - 3(3a + 2) = 10$

29. $(w - 1)^2 + 5(w - 1) + 5 = 0$

30. $(2x - 1)^2 - 4(2x - 1) + 2 = 0$

Find all real solutions to each equation. See Example 4.

31. $x^4 - 14x^2 + 45 = 0$

32. $x^4 + 2x^2 = 15$

33. $x^6 + 7x^3 = 8$

34. $a^6 + 6a^3 = 16$

Find all real solutions to each equation. See Example 5.

35. $(x^2 + 2x)^2 - 7(x^2 + 2x) + 12 = 0$

36. $(x^2 + 3x)^2 + (x^2 + 3x) - 20 = 0$

37. $(y^2 + y)^2 - 8(y^2 + y) + 12 = 0$

38. $(w^2 - 2w)^2 + 24 = 11(w^2 - 2w)$

Find all real solutions to each equation. See Example 6.

39. $x^{1/2} - 5x^{1/4} + 6 = 0$

40. $2x - 5\sqrt{x} + 2 = 0$

41. $2x - 5x^{1/2} - 3 = 0$

42. $x^{1/4} + 2 = x^{1/2}$

Find all real solutions to each equation.

43. $x^{-2} + x^{-1} - 6 = 0$

44. $x^{-2} - 2x^{-1} = 8$

45. $x^{1/6} - x^{1/3} + 2 = 0$

46. $x^{2/3} - x^{1/3} - 20 = 0$

47. $\left(\frac{1}{y - 1}\right)^2 + \left(\frac{1}{y - 1}\right) = 6$

48. $\left(\frac{1}{w + 1}\right)^2 - 2\left(\frac{1}{w + 1}\right) - 24 = 0$

49. $2x^2 - 3 - 6\sqrt{2x^2 - 3} + 8 = 0$

50. $x^2 + x + \sqrt{x^2 + x} - 2 = 0$

51. $x^{-2} - 2x^{-1} - 1 = 0$

52. $x^{-2} - 6x^{-1} + 6 = 0$

Find the exact solution to each problem. If the exact solution is an irrational number, then also find an approximate decimal solution. See Examples 7 and 8.

53. ***Country singers.*** Harry and Gary are traveling to Nashville to make their fortunes. Harry leaves on the train at 8:00 A.M. and Gary travels by car, starting at 9:00 A.M. To complete the 300-mile trip and arrive at the same time as Harry, Gary travels 10 miles per hour (mph) faster than the train. At what time will they both arrive in Nashville?

54. ***Gone fishing.*** Debbie traveled by boat 5 miles upstream to fish in her favorite spot. Because of the 4-mph current, it took her 20 minutes longer to get there than to return. How fast will her boat go in still water?

55. ***Cross-country cycling.*** Erin was traveling across the desert on her bicycle. Before lunch she traveled 60 miles (mi); after lunch she traveled 46 mi. She put in one hour more after lunch than before lunch, but her speed was 4 mph slower than before. What was her speed before lunch and after lunch?

FIGURE FOR EXERCISE 55

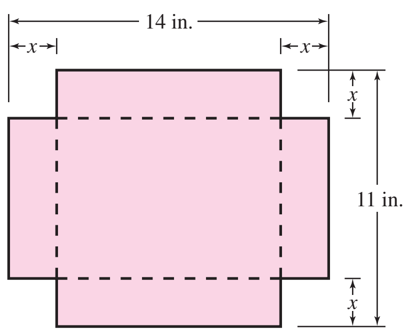
56. ***Extreme hardship.*** Kim starts to walk 3 mi to school at 7:30 A.M. with a temperature of 0°F. Her brother Bryan starts at 7:45 A.M. on his bicycle, traveling 10 mph faster than Kim. If they get to school at the same time, then how fast is each one traveling?

57. ***American pie.*** John takes 3 hours longer than Andrew to peel 500 pounds (lb) of apples. If together they can peel 500 lb of apples in 8 hours, then how long would it take each one working alone?

58. ***On the half shell.*** It takes Brent one hour longer than Calvin to shuck a sack of oysters. If together they shuck a sack of oysters in 45 minutes, then how long would it take each one working alone?

59. ***The growing garden.*** Eric's garden is 20 ft by 30 ft. He wants to increase the length and width by the same amount to have a 1000-ft^2 garden. What should be the new dimensions of the garden?

60. ***Open-top box.*** Thomas is going to make an open-top box by cutting equal squares from the four corners of an 11 inch by 14 inch sheet of cardboard and folding up the sides. If the area of the base is to be 80 square inches, then what size square should be cut from each corner?



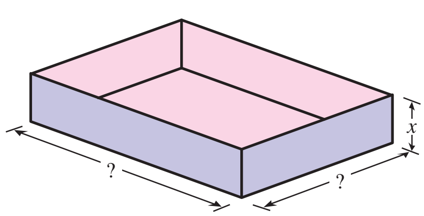


FIGURE FOR EXERCISE 60

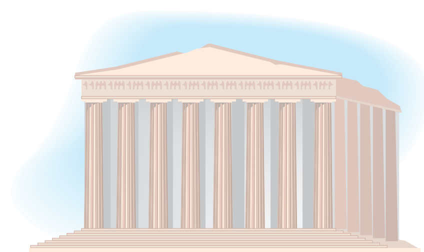
61. ***Pumping the pool.*** It takes pump A 2 hours less time than pump B to empty a certain swimming pool. Pump A is started at 8:00 A.M., and pump B is started at 11:00 A.M. If the pool is still half full at 5:00 P.M., then how long would it take pump A working alone?

62. ***Time off for lunch.*** It usually takes Eva 3 hours longer to do the monthly payroll than it takes Cicely. They start working on it together at 9:00 A.M. and at 5:00 P.M. they have 90% of it done. If Eva took a 2-hour lunch break while Cicely had none, then how much longer will it take for them to finish the payroll working together?

63. ***Golden Rectangle.*** One principle used by the ancient Greeks to get shapes that are pleasing to the eye in art and architecture was the Golden Rectangle. If a square is removed from one end of a Golden Rectangle, as shown in the figure, the sides of the remaining rectangle are proportional to the original rectangle. So the length and width of the original rectangle satisfy

$$\frac{L}{W} = \frac{W}{L - W}.$$

If the length of a Golden Rectangle is 10 meters, then what is its width?



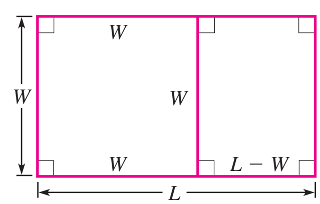


FIGURE FOR EXERCISE 63

GETTING MORE INVOLVED

64. ***Exploration.***

a) Given that $P(x) = x^4 + 6x^2 - 27$, find $P(3i)$, $P(-3i)$, $P(\sqrt{3})$, and $P(-\sqrt{3})$.

b) What can you conclude about the values $3i$, $-3i$, $\sqrt{3}$, and $-\sqrt{3}$ and their relationship to each other?

65. ***Cooperative learning.*** Work with a group to write a quadratic equation that has each given pair of solutions.

a) $3 + \sqrt{5}, 3 - \sqrt{5}$ **b)** $4 - 2i, 4 + 2i$

c) $\frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}$

GRAPHING CALCULATOR EXERCISES

Solve each equation by locating the x-intercepts on the graph of a corresponding function. Round approximate answers to two decimal places.

66. $(5x - 7)^2 - (5x - 7) - 6 = 0$

67. $x^4 - 116x^2 + 1600 = 0$

68. $(x^2 + 3x)^2 - 7(x^2 + 3x) + 9 = 0$

69. $x^2 - 3x^{1/2} - 12 = 0$

10.5 QUADRATIC AND RATIONAL INEQUALITIES

In this section

- Solving Quadratic Inequalities with a Sign Graph
- Solving Rational Inequalities with a Sign Graph
- Quadratic Inequalities That Cannot Be Factored
- Applications

In this section we solve inequalities involving quadratic polynomials. We use a new technique based on the rules for multiplying real numbers.

Solving Quadratic Inequalities with a Sign Graph

An inequality involving a quadratic polynomial is called a **quadratic** inequality.

Quadratic Inequality

A quadratic inequality is an inequality of the form

$$ax^2 + bx + c > 0,$$

where a, b, and c are real numbers with $a \neq 0$. The inequality symbols $<$, $\leq$, and $\geq$ may also be used.

If we can factor a quadratic inequality, then the inequality can be solved with a **sign graph,** which shows where each factor is positive, negative, or zero.

EXAMPLE 1

Solving a quadratic inequality

Use a sign graph to solve the inequality $x^2 + 3x - 10 > 0$.

Solution

Because the left-hand side can be factored, we can write the inequality as

$$(x + 5)(x - 2) > 0.$$

This inequality says that the product of $x + 5$ and $x - 2$ is positive. If both factors are negative or both are positive, the product is positive. To analyze the signs of each factor, we make a sign graph as follows. First consider the possible values of the factor $x + 5$:

Value	Where	On the number line
$x + 5 = 0$	if $x = -5$	Put a 0 above -5.
$x + 5 > 0$	if $x > -5$	Put + signs to the right of -5.
$x + 5 < 0$	if $x < -5$	Put − signs to the left of -5.