64. If the graph of $y = x^2$ is translated six units to the right, then what is the equation of the curve at that location?

65. If the graph of $y = \sqrt{x}$ is translated five units to the left, then what is the equation of the curve at that location?

66. If the graph of $y = \sqrt{x}$ is translated four units downward, then what is the equation of the curve at that location?

67. If the graph of $y = |x|$ is translated three units to the left and then five units upward, then what is the equation of the curve at that location?

68. If the graph of $y = |x|$ is translated four units downward and then nine units to the right, then what is the equation of the curve at that location?

GRAPHING CALCULATOR EXERCISES

69. Graph $f(x) = |x|$ and $g(x) = |x - 20| + 30$ on the same screen of your calculator. What transformations will transform the graph of f into the graph of g?

70. Graph $f(x) = (x + 3)^2$, $g(x) = x^2 + 3^2$, and $h(x) = x^2 + 6x + 9$ on the same screen of your calculator.

a) Which two of these functions has the same graph? Why are they the same?

b) Is it true that $(x + 3)^2 = x^2 + 9$ for all real numbers x?

c) Describe each graph in terms of a transformation of the graph of $y = x^2$.

11.3 COMBINING FUNCTIONS

In this section

- Basic Operations with Functions
- Composition

In this section you will learn how to combine functions to obtain new functions.

Basic Operations with Functions

An entrepreneur plans to rent a stand at a farmers market for \$25 per day to sell strawberries. If she buys x flats of berries for \$5 per flat and sells them for \$9 per flat, then her daily cost in dollars can be written as a function of x:

$$C(x) = 5x + 25$$

Assuming she sells as many flats as she buys, her revenue in dollars is also a function of x:

$$R(x) = 9x$$

Because profit is revenue minus cost, we can find a function for the profit by subtracting the functions for cost and revenue:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 9x - (5x + 25) \\ &= 4x - 25 \end{aligned}$$

The function $P(x) = 4x - 25$ expresses the daily profit as a function of x. Since $P(6) = -1$ and $P(7) = 3$, the profit is negative if 6 or fewer flats are sold and positive if 7 or more flats are sold.

In the example of the entrepreneur we subtracted two functions to find a new function. In other cases we may use addition, multiplication, or division to combine two functions. For any two given functions we can define the sum, difference, product, and quotient functions as follows.

Sum, Difference, Product, and Quotient Functions

Given two functions f and g, the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$ are defined as follows:

Sum function:	$(f + g)(x) = f(x) + g(x)$	
Difference function:	$(f - g)(x) = f(x) - g(x)$	
Product function:	$(f \cdot g)(x) = f(x) \cdot g(x)$	
Quotient function:	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	provided that $g(x) \neq 0$

The domain of the function $f + g$, $f - g$, $f \cdot g$, or $\frac{f}{g}$ is the intersection of the domain of f and the domain of g. For the function $\frac{f}{g}$ we also rule out any values of x for which $g(x) = 0$.

EXAMPLE 1 **Operations with functions**

Let $f(x) = 4x - 12$ and $g(x) = x - 3$. Find the following.

a) $(f + g)(x)$ **b)** $(f - g)(x)$

c) $(f \cdot g)(x)$ **d)** $\left(\frac{f}{g}\right)(x)$

Solution

a) $$(f + g)(x) = f(x) + g(x) = 4x - 12 + x - 3 = 5x - 15$$

b) $$(f - g)(x) = f(x) - g(x) = 4x - 12 - (x - 3) = 3x - 9$$

c) $$(f \cdot g)(x) = f(x) \cdot g(x) = (4x - 12)(x - 3) = 4x^2 - 24x + 36$$

d) $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4x - 12}{x - 3} = \frac{4(x - 3)}{x - 3} = 4 \qquad \text{for } x \neq 3.$$

helpful hint

Note that we use $f + g$, $f - g$, $f \cdot g$, and f/g to name these functions only because there is no application in mind here. We generally use a single letter to name functions after they are combined as we did when using P for the profit function rather than $R - C$.

EXAMPLE 2 **Evaluating a sum function**

Let $f(x) = 4x - 12$ and $g(x) = x - 3$. Find $(f + g)(2)$.

Solution

In Example 1(a) we found a general formula for the function $f + g$, namely, $(f + g)(x) = 5x - 15$. If we replace x by 2, we get

$$(f + g)(2) = 5(2) - 15 = -5.$$

We can also find $(f + g)(2)$ by evaluating each function separately and then adding the results. Because $f(2) = -4$ and $g(2) = -1$, we get

$$\begin{aligned}(f + g)(2) &= f(2) + g(2)\\ &= -4 + (-1)\\ &= -5.\end{aligned}$$

Composition

helpful hint

The difference between the first four operations with functions and composition is like the difference between parallel and series in electrical connections. Components connected in parallel operate simultaneously and separately. If components are connected in series, then electricity must pass through the first component to get to the second component.

A salesperson's monthly salary is a function of the number of cars he sells: \$1000 plus \$50 for each car sold. If we let S be his salary and n be the number of cars sold, then S in dollars is a function of n:

$$S = 1000 + 50n$$

Each month the dealer contributes \$100 plus 5% of his salary to a profit-sharing plan. If P represents the amount put into profit sharing, then P (in dollars) is a function of S:

$$P = 100 + 0.05S$$

Now P is a function of S, and S is a function of n. Is P a function of n? The value of n certainly determines the value of P. In fact, we can write a formula for P in terms of n by substituting one formula into the other:

$$\begin{aligned}P &= 100 + 0.05S\\ &= 100 + 0.05(1000 + 50n) && \text{Substitute } S = 1000 + 50n.\\ &= 100 + 50 + 2.5n && \text{Distributive property}\\ &= 150 + 2.5n\end{aligned}$$

Now P is written as a function of n, bypassing S. We call this idea **composition of functions.**

EXAMPLE 3 **The composition of two functions**

Given that $y = x^2 - 2x + 3$ and $z = 2y - 5$, write z as a function of x.

Solution

Replace y in $z = 2y - 5$ by $x^2 - 2x + 3$:

$$\begin{aligned}z &= 2y - 5\\ &= 2(x^2 - 2x + 3) - 5 && \text{Replace } y \text{ by } x^2 - 2x + 3.\\ &= 2x^2 - 4x + 1\end{aligned}$$

The equation $z = 2x^2 - 4x + 1$ expresses z as a function of x.

The composition of two functions using f-notation is defined as follows.

Composition of Functions

The **composition** of f and g is denoted $f \circ g$ and is defined by the equation

$$(f \circ g)(x) = f(g(x)),$$

provided that $g(x)$ is in the domain of f.

The notation $f \circ g$ is read as "the composition of f and g" or "f compose g." The diagram in Fig. 11.22 shows a function g pairing numbers in its domain with numbers in its range. If the range of g is contained in or equal to the domain of f, then f pairs the second coordinates of g with numbers in the range of f. The composition function $f \circ g$ is a rule for pairing numbers in the domain of g directly with numbers in the range of f, bypassing the middle set. The domain of the function $f \circ g$ is the domain of g (or a subset of it) and the range of $f \circ g$ is the range of f (or a subset of it).

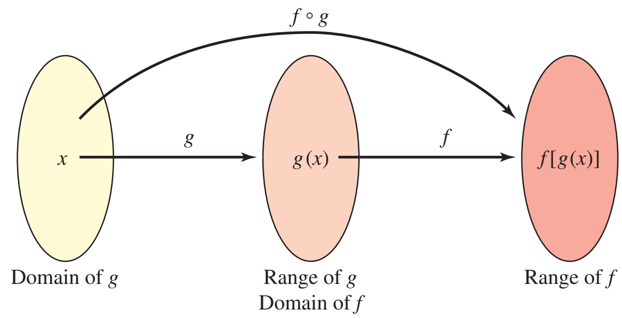


FIGURE 11.22

CAUTION The order in which functions are written is important in composition. For the function $f \circ g$ the function f is applied to $g(x)$. For the function $g \circ f$ the function g is applied to $f(x)$. The function closest to the variable x is applied first.

EXAMPLE 4

Composition of functions

Let $f(x) = 3x - 2$ and $g(x) = x^2 + 2x$. Find the following.

a) $(g \circ f)(2)$

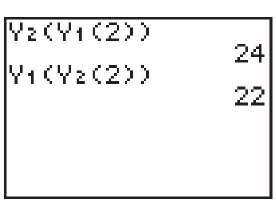
b) $(f \circ g)(2)$

c) $(g \circ f)(x)$

d) $(f \circ g)(x)$

calculator close-up

Set $y_1 = 3x - 2$ and $y_2 = x^2 + 2x$. You can find the composition for Examples 4(a) and 4(b) by evaluating $y_2(y_1(2))$ and $y_1(y_2(2))$. Note that the order in which you evaluate the functions is critical.



Solution

a) Because $(g \circ f)(2) = g(f(2))$, we first find $f(2)$:

$$f(2) = 3 \cdot 2 - 2 = 4$$

Because $f(2) = 4$, we have

$$(g \circ f)(2) = g(f(2)) = g(4) = 4^2 + 2 \cdot 4 = 24.$$

So $(g \circ f)(2) = 24$.

b) Because $(f \circ g)(2) = f(g(2))$, we first find $g(2)$:

$$g(2) = 2^2 + 2 \cdot 2 = 8$$

Because $g(2) = 8$, we have

$$(f \circ g)(2) = f(g(2)) = f(8) = 3 \cdot 8 - 2 = 22.$$

Thus $(f \circ g)(2) = 22$.

c) $(g \circ f)(x) = g(f(x))$
$$= g(3x - 2)$$
$$= (3x - 2)^2 + 2(3x - 2)$$
$$= 9x^2 - 12x + 4 + 6x - 4 = 9x^2 - 6x$$

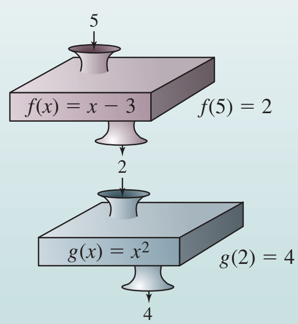
So $(g \circ f)(x) = 9x^2 - 6x$.

d) $(f \circ g)(x) = f(g(x))$
$$= f(x^2 + 2x)$$
$$= 3(x^2 + 2x) - 2 = 3x^2 + 6x - 2$$

So $(f \circ g)(x) = 3x^2 + 6x - 2$. ■

helpful hint

A composition of functions can be viewed as two function machines where the output of the first is the input of the second.



Notice that in Example 4(a) and (b), $(g \circ f)(2) \neq (f \circ g)(2)$. In Example 4(c) and (d) we see that $(g \circ f)(x)$ and $(f \circ g)(x)$ have different formulas defining them. In general, $f \circ g \neq g \circ f$. However, in Section 11.4 we will see some functions for which the composition in either order results in the same function.

It is often useful to view a complicated function as a composition of simpler functions. For example, the function $Q(x) = (x - 3)^2$ consists of two operations, subtracting 3 and squaring. So Q can be described as a composition of the functions $f(x) = x - 3$ and $g(x) = x^2$. To check this, we find $(g \circ f)(x)$:

$$(g \circ f)(x) = g(f(x))$$
$$= g(x - 3)$$
$$= (x - 3)^2$$

We can express the fact that Q is the same as the composition function $g \circ f$ by writing $Q = g \circ f$ or $Q(x) = (g \circ f)(x)$.

EXAMPLE 5 **Expressing a function as a composition of simpler functions**

Let $f(x) = x - 2$, $g(x) = 3x$, and $h(x) = \sqrt{x}$. Write each of the following functions as a composition, using f, g, and h.

a) $F(x) = \sqrt{x - 2}$

b) $H(x) = x - 4$

c) $K(x) = 3x - 6$

Solution

a) The function F consists of first subtracting 2 from x and then taking the square root of that result. So $F = h \circ f$. Check this result by finding $(h \circ f)(x)$:

$$(h \circ f)(x) = h(f(x)) = h(x - 2) = \sqrt{x - 2}$$

b) Subtracting 4 from x can be accomplished by subtracting 2 from x and then subtracting 2 from that result. So $H = f \circ f$. Check by finding $(f \circ f)(x)$:

$$(f \circ f)(x) = f(f(x)) = f(x - 2) = x - 2 - 2 = x - 4$$

c) Notice that $K(x) = 3(x - 2)$. The function K consists of subtracting 2 from x and then multiplying the result by 3. So $K = g \circ f$. Check by finding $(g \circ f)(x)$:

$$(g \circ f)(x) = g(f(x)) = g(x - 2) = 3(x - 2) = 3x - 6$$ ■

study tip

Effective studying involves actively digging into the subject. Be sure that you are making steady progress. At the end of each week take note of the progress that you have made. What do you know on Friday that you did not know on Monday?

CAUTION In Example 5(a) we have $F = h \circ f$ because in F we subtract 2 before taking the square root. If we had the function $G(x) = \sqrt{x} - 2$, we would take the square root before subtracting 2. So $G = f \circ h$. Notice how important the order of operations is here.

In the next example we see functions for which the composition is the identity function. Each function undoes what the other function does. We will study functions of this type further in Section 11.4.

EXAMPLE 6 **Composition of functions**

Show that $(f \circ g)(x) = x$ for each pair of functions.

a) $f(x) = 2x - 1$ and $g(x) = \dfrac{x + 1}{2}$

b) $f(x) = x^3 + 5$ and $g(x) = (x - 5)^{1/3}$

Solution

a)
$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f\left(\frac{x+1}{2}\right)\\ &= 2\left(\frac{x+1}{2}\right) - 1\\ &= x + 1 - 1\\ &= x\end{aligned}$$

b)
$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f((x - 5)^{1/3})\\ &= ((x - 5)^{1/3})^3 + 5\\ &= x - 5 + 5\\ &= x\end{aligned}$$

WARM-UPS

True or false? Explain your answer.

1. If $f(x) = x - 2$ and $g(x) = x + 3$, then $(f - g)(x) = -5$.
2. If $f(x) = x + 4$ and $g(x) = 3x$, then $\left(\frac{f}{g}\right)(2) = 1$.
3. The functions $f \circ g$ and $g \circ f$ are always the same.
4. If $f(x) = x^2$ and $g(x) = x + 2$, then $(f \circ g)(x) = x^2 + 2$.
5. The functions $f \circ g$ and $f \cdot g$ are always the same.
6. If $f(x) = \sqrt{x}$ and $g(x) = x - 9$, then $g(f(x)) = f(g(x))$ for every x.
7. If $f(x) = 3x$ and $g(x) = \frac{x}{3}$, then $(f \circ g)(x) = x$.
8. If $a = 3b^2 - 7b$, and $c = a^2 + 3a$, then c is a function of b.
9. The function $F(x) = \sqrt{x - 5}$ is a composition of two functions.
10. If $F(x) = (x - 1)^2$, $h(x) = x - 1$, and $g(x) = x^2$, then $F = g \circ h$.

11.3 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What are the basic operations with functions?
2. How do we perform the basic operations with functions?
3. What is the composition of two functions?
4. How is the order of operations related to composition of functions?

Let $f(x) = 4x - 3$, *and* $g(x) = x^2 - 2x$. *Find the following. See Examples 1 and 2.*

5. $(f + g)(x)$
6. $(f - g)(x)$
7. $(f \cdot g)(x)$
8. $\left(\frac{f}{g}\right)(x)$
9. $(f + g)(3)$
10. $(f + g)(2)$
11. $(f - g)(-3)$
12. $(f - g)(-2)$
13. $(f \cdot g)(-1)$
14. $(f \cdot g)(-2)$
15. $\left(\frac{f}{g}\right)(4)$
16. $\left(\frac{f}{g}\right)(-2)$

For Exercises 17–24, use the two functions to write y as a function of x. See Example 3.

17. $y = 3a - 2$, $a = 2x - 6$
18. $y = 2c + 3$, $c = -3x + 4$
19. $y = 2d + 1$, $d = \dfrac{x + 1}{2}$
20. $y = -3d + 2$, $d = \dfrac{2 - x}{3}$
21. $y = m^2 - 1$, $m = x + 1$
22. $y = n^2 - 3n + 1$, $n = x + 2$
23. $y = \dfrac{a - 3}{a + 2}$, $a = \dfrac{2x + 3}{1 - x}$
24. $y = \dfrac{w + 2}{w - 5}$, $w = \dfrac{5x + 2}{x - 1}$

Let $f(x) = 2x - 3$, $g(x) = x^2 + 3x$, *and* $h(x) = \frac{x + 3}{2}$. *Find the following. See Example 4.*

25. $(g \circ f)(1)$
26. $(f \circ g)(-2)$
27. $(f \circ g)(1)$
28. $(g \circ f)(-2)$
29. $(f \circ f)(4)$
30. $(h \circ h)(3)$
31. $(h \circ f)(5)$
32. $(f \circ h)(0)$
33. $(f \circ h)(5)$
34. $(h \circ f)(0)$
35. $(g \circ h)(-1)$
36. $(h \circ g)(-1)$
37. $(f \circ g)(2.36)$
38. $(h \circ f)(23.761)$
39. $(g \circ f)(x)$
40. $(g \circ h)(x)$
41. $(f \circ g)(x)$
42. $(h \circ g)(x)$
43. $(h \circ f)(x)$
44. $(f \circ h)(x)$
45. $(f \circ f)(x)$
46. $(g \circ g)(x)$
47. $(h \circ h)(x)$
48. $(f \circ f \circ f)(x)$

Let $f(x) = \sqrt{x}$, $g(x) = x^2$, *and* $h(x) = x - 3$. *Write each of the following functions as a composition using f, g, or h. See Example 5.*

49. $F(x) = \sqrt{x - 3}$
50. $N(x) = \sqrt{x} - 3$
51. $G(x) = x^2 - 6x + 9$
52. $P(x) = x$ for $x \geq 0$
53. $H(x) = x^2 - 3$
54. $M(x) = x^{1/4}$
55. $J(x) = x - 6$
56. $R(x) = \sqrt{x^2 - 3}$
57. $K(x) = x^4$
58. $Q(x) = \sqrt{x^2 - 6x + 9}$

Show that $(f \circ g)(x) = x$ *and* $(g \circ f)(x) = x$ *for each given pair of functions. See Example 6.*

59. $f(x) = 3x + 5$, $g(x) = \dfrac{x - 5}{3}$
60. $f(x) = 3x - 7$, $g(x) = \dfrac{x + 7}{3}$
61. $f(x) = x^3 - 9$, $g(x) = \sqrt[3]{x + 9}$
62. $f(x) = x^3 + 1$, $g(x) = \sqrt[3]{x - 1}$

63. $f(x) = \frac{x - 1}{x + 1}$, $g(x) = \frac{x + 1}{1 - x}$

64. $f(x) = \frac{x + 1}{x - 3}$, $g(x) = \frac{3x + 1}{x - 1}$

65. $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x}$

66. $f(x) = 2x^3$, $g(x) = \left(\frac{x}{2}\right)^{1/3}$

Solve each problem.

67. ***Color monitor.*** The CTX CMS-1561 color monitor has a square viewing area that has a diagonal measure of 15 inches (Midwest Micro Catalog). Find the area of the viewing area in square inches (in.^{2}). Write a formula for the area of a square as a function of the length of its diagonal.

68. ***Perimeter.*** Write a formula for the perimeter of a square as a function of its area.

69. ***Profit function.*** A plastic bag manufacturer has determined that the company can sell as many bags as it can produce each month. If it produces x thousand bags in a month, the revenue is $R(x) = x^2 - 10x + 30$ dollars, and the cost is $C(x) = 2x^2 - 30x + 200$ dollars. Use the fact that profit is revenue minus cost to write the profit as a function of x.

70. ***Area of a sign.*** A sign is in the shape of a square with a semicircle of radius x adjoining one side and a semicircle of diameter x removed from the opposite side. If the sides of the square are length $2x$, then write the area of the sign as a function of x.

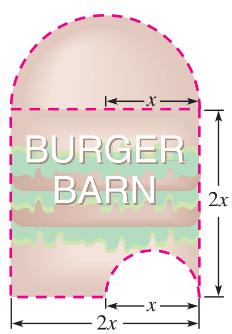


FIGURE FOR EXERCISE 70

71. ***Junk food expenditures.*** Suppose the average family spends 25% of its income on food, $F = 0.25I$, and 10% of each food dollar on junk food, $J = 0.10F$. Write J as a function of I.

72. ***Area of an inscribed circle.*** A pipe of radius r must pass through a square hole of area M as shown in the figure. Write the cross-sectional area of the pipe A as a function of M.

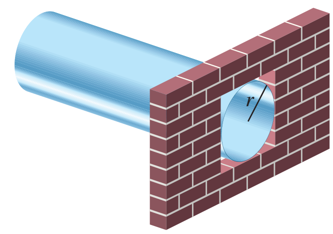


FIGURE FOR EXERCISE 72

73. ***Displacement-length ratio.*** To find the displacement-length ratio D for a sailboat, first find x, where $x = (L/100)^3$ and L is the length at the water line in feet (*Sail,* September 1997). Next find D, where $D = (d/2240)/x$ and d is the displacement in pounds.

a) For the Pacific Seacraft 40, $L = 30$ ft 3 in. and $d = 24{,}665$ pounds. Find D.

b) For a boat with a displacement of 25,000 pounds, write D as a function of L.

c) The graph for the function in part (b) is shown in the accompanying figure. For a fixed displacement, does the displacement-length ratio increase or decrease as the length increases?

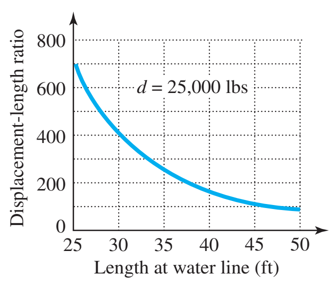


FIGURE FOR EXERCISE 73

74. ***Sail area-displacement ratio.*** To find the sail area-displacement ratio S, first find y, where $y = (d/64)^{2/3}$ and d is the displacement in pounds. Next find S, where $S = A/y$ and A is the sail area in square feet.

a) For the Pacific Seacraft 40, $A = 846$ square feet (ft^2) and $d = 24{,}665$ pounds. Find S.

b) For a boat with a sail area of 900 ft^2, write S as a function of d.

c) For a fixed sail area, does S increase or decrease as the displacement increases?

GETTING MORE INVOLVED

75. ***Discussion.*** Let $f(x) = \sqrt{x} - 4$ and $g(x) = \sqrt{x}$. Find the domains of f, g, and $g \circ f$.

76. ***Discussion.*** Let $f(x) = \sqrt{x - 4}$ and $g(x) = \sqrt{x - 8}$. Find the domains of f, g, and $f + g$.

GRAPHING CALCULATOR EXERCISES

77. Graph $y_1 = x$, $y_2 = \sqrt{x}$, and $y_3 = x + \sqrt{x}$ in the same screen. Find the domain and range of $y_3 = x + \sqrt{x}$ by examining its graph. (On some graphing calculators you can enter y_3 as $y_3 = y_1 + y_2$.)

78. Graph $y_1 = |x|$, $y_2 = |x - 3|$, and $y_3 = |x| + |x - 3|$. Find the domain and range of $y_3 = |x| + |x - 3|$ by examining its graph.

11.4 INVERSE FUNCTIONS

In this section

- Inverse of a Function
- Identifying Inverse Functions
- Switch-and-Solve Strategy
- Even Roots or Even Powers
- Graphs of f and f^{-1}

In Section 11.3 we introduced the idea of a pair of functions such that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. Each function reverses what the other function does. In this section we explore that idea further.

Inverse of a Function

You can buy a 6-, 7-, or 8-foot conference table in the K-LOG Catalog for \$299, \$329, or \$349, respectively. The set

$$f = \{(6, 299), (7, 329), (8, 349)\}$$

gives the price as a function of the length. We use the letter f as a name for this set or function, just as we use the letter f as a name for a function in the function notation. In the function f, lengths in the domain $\{6, 7, 8\}$ are paired with prices in the range $\{299, 329, 349\}$. The **inverse** of the function f, denoted f^{-1}, is a function whose ordered pairs are obtained from f by interchanging the x- and y-coordinates:

$$f^{-1} = \{(299, 6), (329, 7), (349, 8)\}$$

We read f^{-1} as "f inverse." The domain of f^{-1} is $\{299, 329, 349\}$, and the range of f^{-1} is $\{6, 7, 8\}$. The inverse function reverses what the function does: it pairs prices in the range of f with lengths in the domain of f. For example, to find the cost of a 7-foot table, we use the function f to get $f(7) = 329$. To find the length of a table, that costs \$349, we use the function f^{-1} to get $f^{-1}(349) = 8$. Of course, we could find the length of a \$349 table by looking at the function f, but f^{-1} is a function whose input is price and whose output is length. In general, *the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.* See Fig. 11.23.

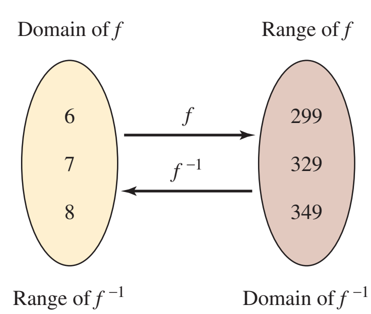


FIGURE 11.23