which apartment it was done. Find the function $C^{-1}(x)$ that gives the length of a living room as a function of the total cost of the carpeting job x.

GETTING MORE INVOLVED

77. Discussion. Let $f(x) = x^n$ for n a positive integer. For which values of n is f an invertible function? Explain.

78. Discussion. Suppose f is a function with range $(-\infty, \infty)$ and g is a function with domain $(0, \infty)$. Is it possible that g and f are inverse functions? Explain.

GRAPHING CALCULATOR EXERCISES

79. Most graphing calculators can form compositions of functions. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. To graph the composition $g \circ f$, let $y_1 = x^2$ and $y_2 = \sqrt{y_1}$. The graph of y_2 is the graph of $g \circ f$. Use the graph of y_2 to determine whether f and g are inverse functions.

80. Let $y_1 = x^3 - 4$, $y_2 = \sqrt[3]{x + 4}$, and $y_3 = \sqrt[3]{y_1 + 4}$. The function y_3 is the composition of the first two functions. Graph all three functions on the same screen. What do the graphs indicate about the relationship between y_1 and y_2?

11.5 VARIATION

In this section

- Direct Variation
- Finding the Proportionality Constant
- Inverse Variation
- Joint Variation
- More Variation

If $y = 3x$, then as x varies so does y. Certain functions are customarily expressed in terms of variation. In this section you will learn to write formulas for those functions from verbal descriptions of the functions.

Direct Variation

In a community with an 8% sales tax rate, the amount of tax, t (in dollars), is a function of the amount of the purchase, a (in dollars). This function is expressed by the formula

$$t = 0.08a.$$

If the amount increases, then the tax increases. If a decreases, then t decreases. In this situation we say that *t varies directly with a, or t is directly proportional to a.* The constant tax rate, 0.08, is called the **variation constant** or **proportionality constant.** Notice that t is just a simple linear function of a. We are merely introducing some new terms to express an old idea.
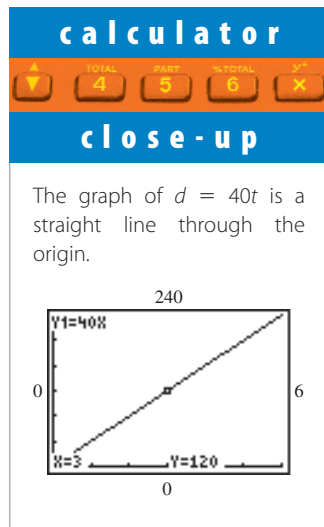
> **Direct Variation**
>
> The statement **y varies directly as x,** or **y is directly proportional to x,** means that
>
> $$y = kx$$
>
> for some constant, k. The constant, k, is a fixed nonzero real number.

Finding the Proportionality Constant

If y varies directly as x and we know corresponding values for x and y, then we can find the proportionality constant.

E X A M P L E   1

240

Y1=40X

0

6

X=3   Y=120

0

Finding the proportionality constant

Joyce is traveling by car, and the distance she travels, d, varies directly with the amount of time, t, that she drives. In 3 hours she drove 120 miles. Find the proportionality constant and write d as a function of t.

Solution

Because d varies directly as t, we must have a constant k such that

$$d = kt.$$
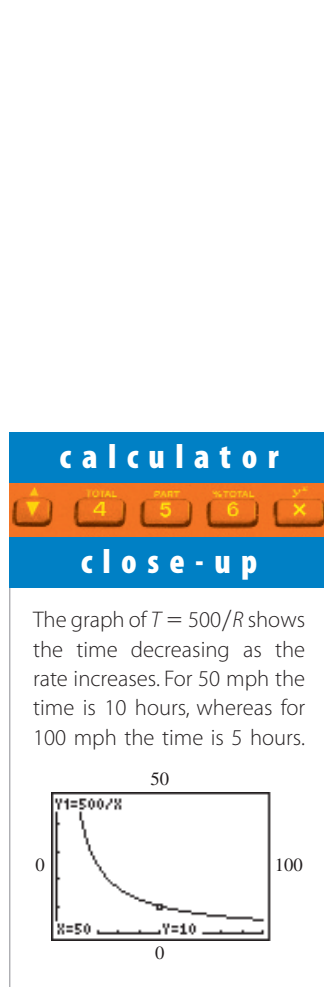
Because $d = 120$ when $t = 3$, we can write

$$120 = k \cdot 3,$$

or

$$40 = k.$$

So the proportionality constant is 40 mph, and $d = 40t$. ■

E X A M P L E   2

Direct variation

In a downtown office building the monthly rent for an office is directly proportional to the size of the office. If a 420-square-foot office rents for \$1260 per month, then what is the rent for a 900-square-foot office?

Solution

Because the rent, R, varies directly with the area of the office, A, we have

$$R = kA.$$

Because a 420-square-foot office rents for \$1260, we can substitute to find k:

$$1260 = k \cdot 420$$
$$3 = k$$

Now that we know the value of k, we can write

$$R = 3A.$$

To get the rent for a 900-square-foot office, insert 900 into this formula:

$$R = 3 \cdot 900$$
$$= 2700$$

So a 900-square-foot office rents for \$2700 per month. ■

50

Y1=500/X

0

100

X=50   Y=10

0

Inverse Variation

In making a 500-mile trip by car, the time it takes is a function of the speed of the car. The greater the speed, the less time it will take. If you decrease the speed, the time increases. We say that the time is *inversely proportional* to the speed. Using the formula $D = RT$ or $T = \dfrac{D}{R}$, we can write

$$T = \frac{500}{R}.$$

In general, we make the following definition.

> ### Inverse Variation
>
> The statement **y varies inversely as x,** or **y is inversely proportional to x,** means that
>
> $$y = \frac{k}{x}$$
>
> for some nonzero constant, k.

C A U T I O N   Be sure to understand the difference between direct and inverse variation. If y varies directly as x (with $k > 0$), then as x increases, y increases. If y varies inversely as x (with $k > 0$), then as x increases, y decreases.

E X A M P L E   3

Inverse variation

Suppose a is inversely proportional to b, and when $b = 5$, $a = \frac{1}{2}$. Find a when $b = 12$.

Solution

Because a is inversely proportional to b, we have

$$a = \frac{k}{b}$$

for some constant, k. Because $a = \frac{1}{2}$ when $b = 5$, we can find k by substituting these values into the formula:

$$\frac{1}{2} = \frac{k}{5}$$

$$\frac{5}{2} = k \qquad \text{Multiply each side by 5.}$$

Now to find a when $b = 12$, we use the formula with k replaced by $\frac{5}{2}$:

$$a = \frac{\frac{5}{2}}{b}$$

$$a = \frac{\frac{5}{2}}{12} = \frac{5}{2} \cdot \frac{1}{12} = \frac{5}{24}$$ ■

Joint Variation

On a deposit of $5000 in a savings account, the interest earned, I, depends on the rate, r, and the time, t. Assuming the interest is simple interest, we can use the formula $I = Prt$ to write

$$I = 5000rt.$$

The variable I is a function of two independent variables, r and t. In this case we say that I *varies jointly* as r and t.

> ### Joint Variation
>
> The statement *y* **varies jointly as** *x* **and** *z,* or *y* **is jointly proportional to** *x* **and** *z,* means that
>
> $$y = kxz$$
>
> for some nonzero constant, k.

E X A M P L E 4
Joint variation
Suppose y varies jointly with x and z, and $y = 12$ when $x = 5$ and $z = 2$. Find y when $x = 10$ and $z = -3$.

Solution
Because y varies jointly with x and z, we can write

$$y = kxz$$

for some constant, k. Now substitute $y = 12$, $x = 5$, and $z = 2$, and solve for k:

$$12 = k \cdot 5 \cdot 2$$
$$12 = 10k$$
$$\frac{6}{5} = k$$

Now that we know the value of k, we can rewrite the equation as

$$y = \frac{6}{5}xz.$$

To find y when $x = 10$ and $z = -3$, substitute into the equation:

$$y = \frac{6}{5}(10)(-3)$$
$$y = -36$$ ■

More Variation

We frequently combine the ideas of direct, inverse, and joint variation with powers and roots. A combination of direct and inverse variation is referred to as **combined variation.** Study the examples that follow.

h e l p f u l h i n t

The language of variation is popular in science. Instead of saying $V = kT/P$, a chemist would say that the volume of a gas varies directly with the temperature and inversely with the pressure.

More Variation Examples

Statement	Formula
y varies directly as the square root of x.	$y = k\sqrt{x}$
y is directly proportional to the cube of x.	$y = kx^3$
y is inversely proportional to x^2.	$y = \dfrac{k}{x^2}$
y varies inversely as the square root of x.	$y = \dfrac{k}{\sqrt{x}}$
y varies jointly as x and the square of z.	$y = kxz^2$
y varies directly with x and inversely with the square root of z (combined variation).	$y = \dfrac{kx}{\sqrt{z}}$

C A U T I O N   The variation terms never signify addition or subtraction. We always use multiplication unless we see the word "inversely." In that case we divide.

E X A M P L E   5

Newton's law of gravity

According to Newton's law of gravity, the gravitational attraction F between two objects with masses m_1 and m_2 is directly proportional to the product of their masses and inversely proportional to the square of the distance r between their centers. Write a formula for Newton's law of gravity.

Solution

Letting k be the constant of proportionality, we have

$$F = \frac{km_1 m_2}{r^2}.$$

■

E X A M P L E   6

House framing

The time t that it takes to frame a house varies directly with the size of the house s in square feet and inversely with the number of framers n working on the job. If three framers can complete a 2500-square-foot house in 6 days, then how long will it take six framers to complete a 4500-square-foot house?

Solution

Because t varies directly with s and inversely with n, we have

$$t = \frac{ks}{n}.$$

Substitute $t = 6$, $s = 2500$, and $n = 3$ into this equation to find k:

$$6 = \frac{k \cdot 2500}{3}$$

$$18 = 2500k$$

$$0.0072 = k$$

Now use $k = 0.0072$, $s = 4500$, and $n = 6$ to find t:

$$t = \frac{0.0072 \cdot 4500}{6}$$

$$t = 5.4$$

So six framers can frame a 4500-square-foot house in 5.4 days.   ■

WARM-UPS

True or false? Explain your answer.

1. If a varies directly as b, then $a = kb$.
2. If a is inversely proportional to b, then $a = bk$.
3. If a is jointly proportional to b and c, then $a = bc$.
4. If a is directly proportional to the square root of c, then $a = k\sqrt{c}$.
5. If b is directly proportional to a, then $b = ka^2$.

(continued)

6. If a varies directly as b and inversely as c, then $a = \frac{kb}{c}$.

7. If a is jointly proportional to c and the square of b, then $a = \frac{kc}{b^2}$.

8. If a varies directly as c and inversely as the square root of b, then $a = \frac{kc}{b}$.

9. If b varies directly as a and inversely as the square of c, then $b = ka\sqrt{c}$.

10. If b varies inversely with the square of c, then $b = \frac{k}{c^2}$.

11.5   EXERCISES

Reading and Writing   After reading this section, write out the answers to these questions. Use complete sentences.

1. What does it mean that y varies directly as x?

2. What is the constant of proportionality in a direct variation?

3. What does it mean that y is inversely proportional to x?

4. What is the difference between direct and inverse variation?

5. What does it mean that y is jointly proportional to x and z?

6. What is the difference between varies directly and directly proportional?

Write a formula that expresses the relationship described by each statement. Use k as a constant of variation. See Examples 1–6.

7. a varies directly as m.

8. w varies directly with P.

9. d varies inversely with e.

10. y varies inversely as x.

11. I varies jointly as r and t.

12. q varies jointly as w and v.

13. m is directly proportional to the square of p.

14. g is directly proportional to the cube of r.

15. B is directly proportional to the cube root of w.

16. F is directly proportional to the square of m.

17. t is inversely proportional to the square of x.

18. y is inversely proportional to the square root of z.

19. v varies directly as m and inversely as n.

20. b varies directly as the square of n and inversely as the square root of v.

Find the proportionality constant and write a formula that expresses the indicated variation. See Example 1.

21. y varies directly as x, and $y = 6$ when $x = 4$.

22. m varies directly as w, and $m = \frac{1}{3}$ when $w = \frac{1}{4}$.

23. A varies inversely as B, and $A = 10$ when $B = 3$.

24. c varies inversely as d, and $c = 0.31$ when $d = 2$.

25. m varies inversely as the square root of p, and $m = 12$ when $p = 9$.

26. s varies inversely as the square root of v, and $s = 6$ when $v = \frac{3}{2}$.

27. A varies jointly as t and u, and $A = 6$ when $t = 5$ and $u = 3$.

28. N varies jointly as the square of p and the cube of q, and $N = 72$ when $p = 3$ and $q = 2$.

29. y varies directly as x and inversely as z, and $y = 2.37$ when $x = \pi$ and $z = \sqrt{2}$.

30. a varies directly as the square root of m and inversely as the square of n, and $a = 5.47$ when $m = 3$ and $n = 1.625$.

Solve each variation problem. See Examples 2–6.

31. If y varies directly as x, and $y = 7$ when $x = 5$, find y when $x = -3$.

32. If n varies directly as p, and $n = 0.6$ when $p = 0.2$, find n when $p = \sqrt{2}$.

33. If w varies inversely as z, and $w = 6$ when $z = 2$, find w when $z = -8$.

34. If p varies inversely as q, and $p = 5$ when $q = \sqrt{3}$, find p when $q = 5$.

35. If A varies jointly as F and T, and $A = 6$ when $F = 3\sqrt{2}$ and $T = 4$, find A when $F = 2\sqrt{2}$ and $T = \frac{1}{2}$.

36. If j varies jointly as the square of r and the cube of v, and $j = -3$ when $r = 2\sqrt{3}$ and $v = \frac{1}{2}$, find j when $r = 3\sqrt{5}$ and $v = 2$.

37. If D varies directly with t and inversely with the square of s, and $D = 12.35$ when $t = 2.8$ and $s = 2.48$, find D when $t = 5.63$ and $s = 6.81$.

38. If M varies jointly with x and the square of v, and $M = 39.5$ when $x = \sqrt{10}$ and $v = 3.87$, find M when $x = \sqrt{30}$ and $v = 7.21$.

Determine whether each equation represents direct, inverse, joint, or combined variation.

39. $y = \dfrac{78}{x}$

40. $y = \dfrac{\pi}{x}$

41. $y = \dfrac{1}{2}x$

42. $y = \dfrac{x}{4}$

43. $y = \dfrac{3x}{w}$

44. $y = \dfrac{4t^2}{\sqrt{x}}$

45. $y = \dfrac{1}{3}xz$

46. $y = 99qv$

In Exercises 47–61, solve each problem.

47. *Lawn maintenance.* At Larry's Lawn Service the cost of lawn maintenance varies directly with the size of the lawn. If the monthly maintenance on a 4000-square-foot lawn is $280, then what is the maintenance fee for a 6000-square-foot lawn?

48. *Weight of the iguana.* The weight of an iguana is directly proportional to its length. If a 4-foot iguana weighs 30 pounds, then how much should a 5-foot iguana weigh?

49. *Gas laws.* The volume of a gas in a cylinder at a fixed temperature is inversely proportional to the weight on the piston. If the gas has a volume of 6 cubic centimeters (cm^3) for a weight of 30 kilograms (kg), then what would the volume be for a weight of 20 kg?

50. *Selling software.* A software vendor sells a software package at a price that is inversely proportional to the number of packages sold per month. When they are selling 900 packages per month, the price is $80 each. If they sell 1000 packages per month, then what should the new price be?

51. *Costly culvert.* The price of an aluminum culvert is jointly proportional to its radius and length. If a 12-foot culvert with a 6-inch radius costs $324, then what is the price of a 10-foot culvert with an 8-inch radius?

52. *Pricing plastic.* The cost of a piece of PVC water pipe varies jointly as its diameter and length. If a 20-foot pipe with a diameter of 1 inch costs $6.80, then what will be the cost of a 10-foot pipe with a $\frac{3}{4}$-inch diameter?

53. *Reinforcing rods.* The price of a steel rod varies jointly as the length and the square of the diameter. If an 18-foot rod with a 2-inch diameter costs $12.60, then what is the cost of a 12-foot rod with a 3-inch diameter?

54. *Pea soup.* The weight of a cylindrical can of pea soup varies jointly with the height and the square of the radius. If a 4-inch-high can with a 1.5-inch radius weighs 16 ounces, then what is the weight of a 5-inch-high can with a radius of 3 inches?

55. **Falling objects.** The distance an object falls in a vacuum varies directly with the square of the time it is falling. In the first 0.1 second after an object is dropped, it falls 0.16 feet.

  a) Find the formula that expresses the distance d an object falls as a function of the time it is falling t.
  b) How far does an object fall in the first 0.5 second after it is dropped?
  c) How long does it take for a watermelon to reach the ground when dropped from a height of 100 feet?
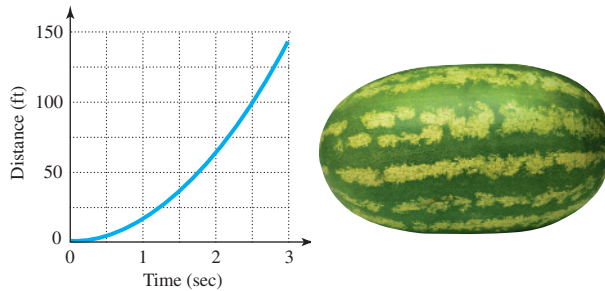

FIGURE FOR EXERCISE 55

56. **Making Frisbees.** The cost of material used in making a Frisbee varies directly with the square of the diameter. If it costs the manufacturer $0.45 for the material in a Frisbee with a 9-inch diameter, then what is the cost for the material in a 12-inch-diameter Frisbee?

57. **Using leverage.** The basic law of leverage is that the force required to lift an object is inversely proportional to the length of the lever. If a force of 2000 pounds applied 2 feet from the pivot point would lift a car, then what force would be required at 10 feet to lift the car?

58. **Resistance.** The resistance of a wire varies directly with the length and inversely as the square of the diameter. If a wire of length 20 feet and diameter 0.1 inch has a resistance of 2 ohms, then what is the resistance of a 30-foot wire with a diameter of 0.2 inch?

59. **Computer programming.** The time t required to complete a programming job varies directly with the complexity of the job and inversely with the number n of programmers working on the job. The complexity c is an arbitrarily assigned number between 1 and 10, with 10 being the most complex. It takes 8 days for a team of three programmers to complete a job with complexity 6. How long will it take five programmers to complete a job with complexity 9?

60. **Shock absorbers.** The volume of gas in a gas shock absorber varies directly with the temperature and inversely

with the pressure. The volume is 10 cubic centimeters (cm^3) when the temperature is 20°C and the pressure is 40 kg. What is the volume when the temperature is 30°C and the pressure is 25 kg?

61. **Bicycle gear ratio.** A bicycle's gear ratio G varies jointly with the number of teeth on the chain ring N (by the pedals) and the diameter of the wheel d, and inversely with the number of teeth on the cog c (on the rear wheel). A bicycle with 27-inch-diameter wheels, 26 teeth on the cog, and 52 teeth on the chain ring has a gear ratio of 54.

  a) Find a formula that expresses the gear ratio as a function of N, d, and c.
  b) What is the gear ratio for a bicycle with 26-inch-diameter wheels, 42 teeth on the chain ring, and 13 teeth on the cog?
  c) A five-speed bicycle with 27-inch-diameter wheels and 44 teeth on the chain ring has gear ratios of 52, 59, 70, 79, and 91. Find the number of teeth on the cog for each gear ratio.
  d) For a fixed wheel size and chain ring, does the gear ratio increase or decrease as the number of teeth on the cog increases?
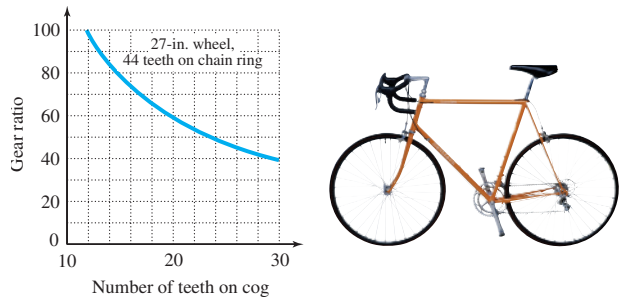

FIGURE FOR EXERCISE 61

 GRAPHING CALCULATOR EXERCISES

62. To see the difference between direct and inverse variation, graph $y_1 = 2x$ and $y_2 = \frac{2}{x}$ using $0 \le x \le 5$ and $0 \le y \le 10$. Which of these functions is increasing and which is decreasing?

63. Graph $y_1 = 2\sqrt{x}$ and $y_2 = \frac{2}{\sqrt{x}}$ by using $0 \le x \le 5$ and $0 \le y \le 10$. At what point in the first quadrant do the curves cross? Which function is increasing and which is decreasing? Which represents direct variation and which represents inverse variation?