12.3 PROPERTIES OF LOGARITHMS

In this section

- Product Rule for Logarithms
- Quotient Rule for Logarithms
- Power Rule for Logarithms
- Inverse Properties
- Using the Properties

calculator close-up

You can illustrate the product rule for logarithms with a graphing calculator.

```
log(7)+log(8)
          1.748188027
log(56)
          1.748188027
```

The properties of logarithms are very similar to the properties of exponents because *logarithms are exponents.* In this section we use the properties of exponents to write some properties of logarithms. The properties will be used in solving logarithmic equations in Section 12.4.

Product Rule for Logarithms

If $M = a^x$ and $N = a^y$, we can use the product rule for exponents to write

$$MN = a^x \cdot a^y = a^{x+y}.$$

The equation $MN = a^{x+y}$ is equivalent to

$$\log_a(MN) = x + y.$$

Because $M = a^x$ and $N = a^y$ are equivalent to $x = \log_a(M)$ and $y = \log_a(N)$, we can replace x and y in $\log_a(MN) = x + y$ to get

$$\log_a(MN) = \log_a(M) + \log_a(N).$$

So *the logarithm of a product is the sum of the logarithms,* provided that all of the logarithms are defined. This rule is called the **product rule for logarithms.**

Product Rule for Logarithms

$$\log_a(MN) = \log_a(M) + \log_a(N)$$

EXAMPLE 1 **Using the product rule for logarithms**

Write each expression as a single logarithm.

a) $\log_2(7) + \log_2(5)$

b) $\ln(\sqrt{2}) + \ln(\sqrt{3})$

Solution

a) $\log_2(7) + \log_2(5) = \log_2(35)$ Product rule for logarithms

b) $\ln(\sqrt{2}) + \ln(\sqrt{3}) = \ln(\sqrt{6})$ Product rule for logarithms

Quotient Rule for Logarithms

If $M = a^x$ and $N = a^y$, we can use the quotient rule for exponents to write

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}.$$

By the definition of logarithm, $\frac{M}{N} = a^{x-y}$ is equivalent to

$$\log_a\left(\frac{M}{N}\right) = x - y.$$

calculator close-up

You can illustrate the quotient rule for logarithms with a graphing calculator.

```
ln(99/2)
         3.90197267
ln(99)-ln(2)
         3.90197267
```

Because $x = \log_a(M)$ and $y = \log_a(N)$, we have

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N).$$

So *the logarithm of a quotient is equal to the difference of the logarithms,* provided that all logarithms are defined. This rule is called the **quotient rule for logarithms.**

Quotient Rule for Logarithms

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$$

EXAMPLE 2

Using the quotient rule for logarithms

Write each expression as a single logarithm.

a) $\log_2(3) - \log_2(7)$ **b)** $\ln(w^8) - \ln(w^2)$

Solution

a) $\log_2(3) - \log_2(7) = \log_2\left(\frac{3}{7}\right)$ Quotient rule for logarithms

b) $\ln(w^8) - \ln(w^2) = \ln\left(\frac{w^8}{w^2}\right)$ Quotient rule for logarithms

$= \ln(w^6)$ Quotient rule for exponents

Power Rule for Logarithms

calculator close-up

You can illustrate the power rule for logarithms with a graphing calculator.

```
log(11^13)
        13.53810491
13*log(11)
        13.53810491
```

If $M = a^x$, we can use the power rule for exponents to write

$$M^N = (a^x)^N = a^{Nx}.$$

By the definition of logarithms, $M^N = a^{Nx}$ is equivalent to

$$\log_a(M^N) = Nx.$$

Because $x = \log_a(M)$, we have

$$\log_a(M^N) = N \cdot \log_a(M).$$

So *the logarithm of a power of a number is the power times the logarithm of the number,* provided that all logarithms are defined. This rule is called the **power rule for logarithms.**

Power Rule for Logarithms

$$\log_a(M^N) = N \cdot \log_a(M)$$

EXAMPLE 3

Using the power rule for logarithms

Rewrite each logarithm in terms of log(2).

a) $\log(2^{10})$ **b)** $\log(\sqrt{2})$ **c)** $\log\left(\frac{1}{2}\right)$

Solution

a) $\log(2^{10}) = 10 \cdot \log(2)$ Power rule for logarithms

b) $\log(\sqrt{2}) = \log(2^{1/2})$ Write $\sqrt{2}$ as a power of 2.

$= \frac{1}{2}\log(2)$ Power rule for logarithms

c) $\log\left(\frac{1}{2}\right) = \log(2^{-1})$ Write $\frac{1}{2}$ as a power of 2.

$= -1 \cdot \log(2)$ Power rule for logarithms

$= -\log(2)$

Inverse Properties

An exponential function and logarithmic function with the same base are inverses of each other. For example, the logarithm of 32 base 2 is 5 and the fifth power of 2 is 32. In symbols, we have

$$2^{\log_2(32)} = 2^5 = 32.$$

If we raise 3 to the fourth power, we get 81; and if we find the base-3 logarithm of 81, we get 4. In symbols, we have

$$\log_3(3^4) = \log_3(81) = 4.$$

We can state the inverse relationship between exponential and logarithm functions in general with the following inverse properties.

Inverse Properties

1. $\log_a(a^M) = M$ **2.** $a^{\log_a(M)} = M$

EXAMPLE 4 **Using the inverse properties**

Simplify each expression.

a) $\ln(e^5)$ **b)** $2^{\log_2(8)}$

Solution

a) Using the first inverse property, we get $\ln(e^5) = 5$.

b) Using the second inverse property, we get $2^{\log_2(8)} = 8$.

Note that there is more than one way to simplify the expressions in Example 4. Using the power rule for logarithms and the fact that $\ln(e) = 1$, we have $\ln(e^5) = 5 \cdot \ln(e) = 5$. Using $\log_2(8) = 3$, we have $2^{\log_2(8)} = 2^3 = 8$.

Using the Properties

We have already seen many properties of logarithms. There are three properties that we have not yet formally stated. Because $a^1 = a$ and $a^0 = 1$, we have $\log_a(a) = 1$ and $\log_a(1) = 0$ for any positive number a. If we apply the quotient rule to $\log_a(1/N)$, we get

$$\log_a\left(\frac{1}{N}\right) = \log_a(1) - \log_a(N) = 0 - \log_a(N) = -\log_a(N).$$

study tip

Keep track of your time for one entire week. Account for how you spend every half hour. Add up your totals for sleep, study, work, and recreation. You should be sleeping 50 to 60 hours per week and studying 1 to 2 hours for every hour you spend in the classroom.

So $\log_a\left(\frac{1}{N}\right) = -\log_a(N)$. These three new properties along with all of the other properties of logarithms are summarized as follows.

Properties of Logarithms

If M, N, and a are positive numbers, $a \neq 1$, then

1. $\log_a(a) = 1$ **2.** $\log_a(1) = 0$

3. $\log_a(a^M) = M$ Inverse properties **4.** $a^{\log_a(M)} = M$

5. $\log_a(MN) = \log_a(M) + \log_a(N)$ Product rule

6. $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$ Quotient rule

7. $\log_a\left(\frac{1}{N}\right) = -\log_a(N)$ **8.** $\log_a(M^N) = N \cdot \log_a(M)$ Power rule

We have already seen several ways in which to use the properties of logarithms. In the next three examples we see more uses of the properties. First we use the rules of logarithms to write the logarithm of a complicated expression in terms of logarithms of simpler expressions.

EXAMPLE 5 **Using the properties of logarithms**

Rewrite each expression in terms of log(2) and/or log(3).

a) $\log(6)$ **b)** $\log(16)$

c) $\log\left(\frac{9}{2}\right)$ **d)** $\log\left(\frac{1}{3}\right)$

Solution

a) $\log(6) = \log(2 \cdot 3)$

$= \log(2) + \log(3)$ Product rule

b) $\log(16) = \log(2^4)$

$= 4 \cdot \log(2)$ Power rule

c) $\log\left(\frac{9}{2}\right) = \log(9) - \log(2)$ Quotient rule

$= \log(3^2) - \log(2)$

$= 2 \cdot \log(3) - \log(2)$ Power rule

d) $\log\left(\frac{1}{3}\right) = -\log(3)$ Property 7 ■

calculator close-up

Examine the values of log(9/2), log(9) − log(2), and log(9)/log(2).

```
log(9/2)
       .6532125138
log(9)-log(2)
       .6532125138
log(9)/log(2)
       3.169925001
```

CAUTION Do not confuse $\frac{\log(9)}{\log(2)}$ with $\log\left(\frac{9}{2}\right)$. We can use the quotient rule to write $\log\left(\frac{9}{2}\right) = \log(9) - \log(2)$, but $\frac{\log(9)}{\log(2)} \neq \log(9) - \log(2)$. The expression $\frac{\log(9)}{\log(2)}$ means $\log(9) \div \log(2)$. Use your calculator to verify these two statements.

The properties of logarithms can be used to combine several logarithms into a single logarithm (as in Examples 1 and 2) or to write a logarithm of a complicated expression in terms of logarithms of simpler expressions.

EXAMPLE 6 **Using the properties of logarithms**

Rewrite each expression as a sum or difference of multiples of logarithms.

a) $\log\left(\frac{xz}{y}\right)$ **b)** $\log_3\left(\frac{(x-3)^{2/3}}{\sqrt{x}}\right)$

Solution

a) $\log\left(\frac{xz}{y}\right) = \log(xz) - \log(y)$ Quotient rule

$= \log(x) + \log(z) - \log(y)$ Product rule

b) $\log_3\left(\frac{(x-3)^{2/3}}{\sqrt{x}}\right) = \log_3((x-3)^{2/3}) - \log_3(x^{1/2})$ Quotient rule

$= \frac{2}{3}\log_3(x-3) - \frac{1}{2}\log_3(x)$ Power rule ■

In the next example we use the properties of logarithms to convert expressions involving several logarithms into a single logarithm. The skills we are learning here will be used to solve logarithmic equations in Section 12.4.

EXAMPLE 7 **Combining logarithms**

Rewrite each expression as a single logarithm.

a) $\frac{1}{2}\log(x) - 2 \cdot \log(x+1)$

b) $3 \cdot \log(y) + \frac{1}{2}\log(z) - \log(x)$

Solution

a) $\frac{1}{2}\log(x) - 2 \cdot \log(x+1) = \log(x^{1/2}) - \log((x+1)^2)$ Power rule

$= \log\left(\frac{\sqrt{x}}{(x+1)^2}\right)$ Quotient rule

b) $3 \cdot \log(y) + \frac{1}{2}\log(z) - \log(x) = \log(y^3) + \log(\sqrt{z}) - \log(x)$ Power rule

$= \log(y^3 \cdot \sqrt{z}) - \log(x)$ Product rule

$= \log\left(\frac{y^3 \cdot \sqrt{z}}{x}\right)$ Quotient rule ■

WARM-UPS

True or false? Explain your answer.

1. $\log_2\left(\frac{x^2}{8}\right) = \log_2(x^2) - 3$

2. $\frac{\log(100)}{\log(10)} = \log(100) - \log(10)$

WARM-UPS

(continued)

3. $\ln(\sqrt{2}) = \dfrac{\ln(2)}{2}$

4. $3^{\log_3(17)} = 17$

5. $\log_2\left(\dfrac{1}{8}\right) = \dfrac{1}{\log_2(8)}$

6. $\ln(8) = 3 \cdot \ln(2)$

7. $\ln(1) = e$

8. $\dfrac{\log(100)}{10} = \log(10)$

9. $\dfrac{\log_2(8)}{\log_2(2)} = \log_2(4)$

10. $\ln(2) + \ln(3) - \ln(7) = \ln\left(\dfrac{6}{7}\right)$

12.3 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the product rule for logarithms?

2. What is the quotient rule for logarithms?

3. What is the power rule for logarithms?

4. Why is it true that $\log_a(a^M) = M$?

5. Why is it true that $a^{\log_a(M)} = M$?

6. Why is it true that $\log_a(1) = 0$ for $a > 0$ and $a \neq 1$?

Assume all variables involved in logarithms represent numbers for which the logarithms are defined.

Write each expression as a single logarithm and simplify. See Example 1.

7. $\log(3) + \log(7)$

8. $\ln(5) + \ln(4)$

9. $\log_3(\sqrt{5}) + \log_3(\sqrt{x})$

10. $\ln(\sqrt{x}) + \ln(\sqrt{y})$

11. $\log(x^2) + \log(x^3)$

12. $\ln(a^3) + \ln(a^5)$

13. $\ln(2) + \ln(3) + \ln(5)$

14. $\log_2(x) + \log_2(y) + \log_2(z)$

15. $\log(x) + \log(x + 3)$

16. $\ln(x - 1) + \ln(x + 1)$

17. $\log_2(x - 3) + \log_2(x + 2)$

18. $\log_3(x - 5) + \log_3(x - 4)$

Write each expression as a single logarithm. See Example 2.

19. $\log(8) - \log(2)$

20. $\ln(3) - \ln(6)$

21. $\log_2(x^6) - \log_2(x^2)$

22. $\ln(w^9) - \ln(w^3)$

23. $\log(\sqrt{10}) - \log(\sqrt{2})$

24. $\log_3(\sqrt{6}) - \log_3(\sqrt{3})$

25. $\ln(4h - 8) - \ln(4)$

26. $\log(3x - 6) - \log(3)$

27. $\log_2(w^2 - 4) - \log_2(w + 2)$

28. $\log_3(k^2 - 9) - \log_3(k - 3)$

29. $\ln(x^2 + x - 6) - \ln(x + 3)$

30. $\ln(t^2 - t - 12) - \ln(t - 4)$

Write each expression in terms of $\log(3)$*. See Example 3.*

31. $\log(27)$

32. $\log\left(\dfrac{1}{9}\right)$

33. $\log(\sqrt{3})$ **34.** $\log(\sqrt[4]{3})$

35. $\log(3^x)$ **36.** $\log(3^{-99})$

Simplify each expression. See Example 4.

37. $\log_2(2^{10})$ **38.** $\ln(e^9)$

39. $5^{\log_5(19)}$ **40.** $10^{\log(2.3)}$

41. $\log(10^8)$ **42.** $\log_4(4^5)$

43. $e^{\ln(4.3)}$ **44.** $3^{\log_3(5.5)}$

Rewrite each expression in terms of $\log(3)$ *and/or* $\log(5)$. *See Example 5.*

45. $\log(15)$ **46.** $\log(9)$

47. $\log\left(\frac{5}{3}\right)$ **48.** $\log\left(\frac{3}{5}\right)$

49. $\log(25)$ **50.** $\log\left(\frac{1}{27}\right)$

51. $\log(75)$ **52.** $\log(0.6)$

53. $\log\left(\frac{1}{3}\right)$ **54.** $\log(45)$

55. $\log(0.2)$ **56.** $\log\left(\frac{9}{25}\right)$

Rewrite each expression as a sum or a difference of multiples of logarithms. See Example 6.

57. $\log(xyz)$

58. $\log(3y)$

59. $\log_2(8x)$

60. $\log_2(16y)$

61. $\ln\left(\frac{x}{y}\right)$

62. $\ln\left(\frac{z}{3}\right)$

63. $\log(10x^2)$

64. $\log(100\sqrt{x})$

65. $\log_5\left(\frac{(x-3)^2}{\sqrt{w}}\right)$

66. $\log_3\left(\frac{(y+6)^3}{y-5}\right)$

67. $\ln\left(\frac{yz\sqrt{x}}{w}\right)$

68. $\ln\left(\frac{(x-1)\sqrt{w}}{x^3}\right)$

Rewrite each expression as a single logarithm. See Example 7.

69. $\log(x) + \log(x - 1)$

70. $\log_2(x - 2) + \log_2(5)$

71. $\ln(3x - 6) - \ln(x - 2)$

72. $\log_3(x^2 - 1) - \log_3(x - 1)$

73. $\ln(x) - \ln(w) + \ln(z)$

74. $\ln(x) - \ln(3) - \ln(7)$

75. $3 \cdot \ln(y) + 2 \cdot \ln(x) - \ln(w)$

76. $5 \cdot \ln(r) + 3 \cdot \ln(t) - 4 \cdot \ln(s)$

77. $\frac{1}{2}\log(x - 3) - \frac{2}{3}\log(x + 1)$

78. $\frac{1}{2}\log(y - 4) + \frac{1}{2}\log(y + 4)$

79. $\frac{2}{3}\log_2(x - 1) - \frac{1}{4}\log_2(x + 2)$

80. $\frac{1}{2}\log_3(y + 3) + 6 \cdot \log_3(y)$

Determine whether each equation is true or false.

81. $\log(56) = \log(7) \cdot \log(8)$ **82.** $\log\left(\frac{5}{9}\right) = \frac{\log(5)}{\log(9)}$

83. $\log_2(4^2) = (\log_2(4))^2$ **84.** $\ln(4^2) = (\ln(4))^2$

85. $\ln(25) = 2 \cdot \ln(5)$ **86.** $\ln(3e) = 1 + \ln(3)$

87. $\frac{\log_2(64)}{\log_2(8)} = \log_2(8)$ **88.** $\frac{\log_2(16)}{\log_2(4)} = \log_2(4)$

89. $\log\left(\frac{1}{3}\right) = -\log(3)$ **90.** $\log_2(8 \cdot 2^{59}) = 62$

91. $\log_2(16^5) = 20$ **92.** $\log_2\left(\frac{5}{2}\right) = \log_2(5) - 1$

93. $\log(10^3) = 3$ **94.** $\log_3(3^7) = 7$

95. $\log(100 + 3) = 2 + \log(3)$ **96.** $\frac{\log_7(32)}{\log_7(8)} = \frac{5}{3}$

Solve each problem.

97. ***Growth rate.*** The annual growth rate for continuous growth is given by

$$r = \frac{\ln(A) - \ln(P)}{t},$$

where P is the initial investment and A is the amount after t years.

a) Rewrite the formula using a single logarithm.

b) In 1998 a share of Microsoft stock was worth 27 times what it was worth in 1990. What was the annual growth rate for that period?

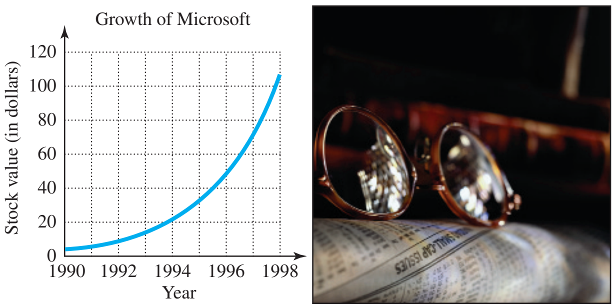

FIGURE FOR EXERCISE 97

98. ***Diversity index.*** The U.S.G.S. measures the quality of a water sample by using the diversity index d, given by

$$d = -[p_1 \cdot \log_2(p_1) + p_2 \cdot \log_2(p_2) + \cdots + p_n \cdot \log_2(p_n)],$$

where n is the number of different taxons (biological classifications) represented in the sample and p_1 through p_n are the percentages of organisms in each of the n taxons. The value of d ranges from 0 when all organisms in the water sample are the same to some positive number when all organisms in the sample are different. If two-thirds of the organisms in a water sample are in one taxon and one-third of the organisms are in a second taxon, then $n = 2$ and

$$d = -\left[\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right].$$

Use the properties of logarithms to write the expression on the right-hand side as $\log_2\left(\frac{3\sqrt[3]{2}}{2}\right)$. (In Section 12.4 you will learn how to evaluate a base-2 logarithm using a calculator.)

GETTING MORE INVOLVED

99. ***Discussion.*** Which of the following equations is an identity? Explain.

a) $\ln(3x) = \ln(3) \cdot \ln(x)$ **b)** $\ln(3x) = \ln(3) + \ln(x)$

c) $\ln(3x) = 3 \cdot \ln(x)$ **d)** $\ln(3x) = \ln(x^3)$

100. ***Discussion.*** Which of the following expressions is not equal to $\log(5^{2/3})$? Explain.

a) $\frac{2}{3}\log(5)$ **b)** $\frac{\log(5) + \log(5)}{3}$

c) $(\log(5))^{2/3}$ **d)** $\frac{1}{3}\log(25)$

GRAPHING CALCULATOR EXERCISES

101. Graph the functions $y_1 = \ln(\sqrt{x})$ and $y_2 = 0.5 \cdot \ln(x)$ on the same screen. Explain your results.

102. Graph the functions $y_1 = \log(x)$, $y_2 = \log(10x)$, $y_3 = \log(100x)$, and $y_4 = \log(1000x)$ using the viewing window $-2 \le x \le 5$ and $-2 \le y \le 5$. Why do these curves appear as they do?

103. Graph the function $y = \log(e^x)$. Explain why the graph is a straight line. What is its slope?

12.4 SOLVING EQUATIONS AND APPLICATIONS

In this section

- Logarithmic Equations
- Exponential Equations
- Changing the Base
- Strategy for Solving Equations
- Applications

We solved some equations involving exponents and logarithms in Sections 12.1 and 12.2. In this section we use the properties of exponents and logarithms to solve more complex equations.

Logarithmic Equations

The main tool that we have for solving logarithmic equations is the definition of logarithms: $y = \log_a(x)$ if and only if $a^y = x$. We can use the definition to rewrite any equation that has only one logarithm as an equivalent exponential equation.