where P is the initial investment and A is the amount after t years.

a) Rewrite the formula using a single logarithm.

b) In 1998 a share of Microsoft stock was worth 27 times what it was worth in 1990. What was the annual growth rate for that period?

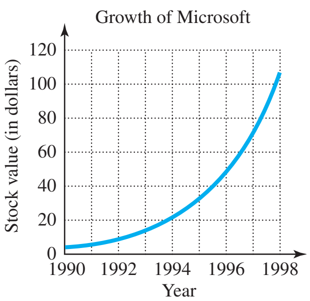


FIGURE FOR EXERCISE 97

98. ***Diversity index.*** The U.S.G.S. measures the quality of a water sample by using the diversity index d, given by

$$d = -[p_1 \cdot \log_2(p_1) + p_2 \cdot \log_2(p_2) + \cdots + p_n \cdot \log_2(p_n)],$$

where n is the number of different taxons (biological classifications) represented in the sample and p_1 through p_n are the percentages of organisms in each of the n taxons. The value of d ranges from 0 when all organisms in the water sample are the same to some positive number when all organisms in the sample are different. If two-thirds of the organisms in a water sample are in one taxon and one-third of the organisms are in a second taxon, then $n = 2$ and

$$d = -\left[\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right].$$

Use the properties of logarithms to write the expression on the right-hand side as $\log_2\left(\frac{3\sqrt[3]{2}}{2}\right)$. (In Section 12.4 you will learn how to evaluate a base-2 logarithm using a calculator.)

GETTING MORE INVOLVED

99. ***Discussion.*** Which of the following equations is an identity? Explain.

a) $\ln(3x) = \ln(3) \cdot \ln(x)$ **b)** $\ln(3x) = \ln(3) + \ln(x)$

c) $\ln(3x) = 3 \cdot \ln(x)$ **d)** $\ln(3x) = \ln(x^3)$

100. ***Discussion.*** Which of the following expressions is not equal to $\log(5^{2/3})$? Explain.

a) $\frac{2}{3}\log(5)$ **b)** $\frac{\log(5) + \log(5)}{3}$

c) $(\log(5))^{2/3}$ **d)** $\frac{1}{3}\log(25)$

GRAPHING CALCULATOR EXERCISES

101. Graph the functions $y_1 = \ln(\sqrt{x})$ and $y_2 = 0.5 \cdot \ln(x)$ on the same screen. Explain your results.

102. Graph the functions $y_1 = \log(x)$, $y_2 = \log(10x)$, $y_3 = \log(100x)$, and $y_4 = \log(1000x)$ using the viewing window $-2 \le x \le 5$ and $-2 \le y \le 5$. Why do these curves appear as they do?

103. Graph the function $y = \log(e^x)$. Explain why the graph is a straight line. What is its slope?

In this section

- Logarithmic Equations
- Exponential Equations
- Changing the Base
- Strategy for Solving Equations
- Applications

12.4 SOLVING EQUATIONS AND APPLICATIONS

We solved some equations involving exponents and logarithms in Sections 12.1 and 12.2. In this section we use the properties of exponents and logarithms to solve more complex equations.

Logarithmic Equations

The main tool that we have for solving logarithmic equations is the definition of logarithms: $y = \log_a(x)$ if and only if $a^y = x$. We can use the definition to rewrite any equation that has only one logarithm as an equivalent exponential equation.

EXAMPLE 1 **A logarithmic equation with only one logarithm**

Solve $\log(x + 3) = 2$.

Solution

Write the equivalent exponential equation:

$$\begin{aligned} \log(x + 3) &= 2 && \text{Original equation} \\ 10^2 &= x + 3 && \text{Definition of logarithm} \\ 100 &= x + 3 \\ 97 &= x \end{aligned}$$

Check: $\log(97 + 3) = \log(100) = 2$. The solution set is $\{97\}$. ■

In the next example we use the product rule for logarithms to write a sum of two logarithms as a single logarithm.

EXAMPLE 2 **Using the product rule to solve an equation**

Solve $\log_2(x + 3) + \log_2(x - 3) = 4$.

Solution

Rewrite the sum of the logarithms as the logarithm of a product:

$$\begin{aligned} \log_2(x + 3) + \log_2(x - 3) &= 4 && \text{Original equation} \\ \log_2[(x + 3)(x - 3)] &= 4 && \text{Product rule} \\ \log_2[x^2 - 9] &= 4 && \text{Multiply the binomials.} \\ x^2 - 9 &= 2^4 && \text{Definition of logarithm} \\ x^2 - 9 &= 16 \\ x^2 &= 25 \\ x &= \pm 5 && \text{Even-root property} \end{aligned}$$

To check, first let $x = -5$ in the original equation:

$$\begin{aligned} \log_2(-5 + 3) + \log_2(-5 - 3) &= 4 \\ \log_2(-2) + \log_2(-8) &= 4 && \text{Incorrect} \end{aligned}$$

Because the domain of any logarithm function is the set of positive real numbers, these logarithms are undefined. Now check $x = 5$ in the original equation:

$$\begin{aligned} \log_2(5 + 3) + \log_2(5 - 3) &= 4 \\ \log_2(8) + \log_2(2) &= 4 \\ 3 + 1 &= 4 && \text{Correct} \end{aligned}$$

The solution set is $\{5\}$. ■

CAUTION Always check that your solutions to a logarithmic equation do not produce undefined logarithms in the original equation.

EXAMPLE 3 **Using the one-to-one property of logarithms**

Solve $\log(x) + \log(x - 1) = \log(8x - 12) - \log(2)$.

Solution

Apply the product rule to the left-hand side and the quotient rule to the right-hand side to get a single logarithm on each side:

$$\log(x) + \log(x - 1) = \log(8x - 12) - \log(2).$$

$$\log[x(x - 1)] = \log\left(\frac{8x - 12}{2}\right) \quad \text{Product rule; quotient rule}$$

$$\log(x^2 - x) = \log(4x - 6) \quad \text{Simplify.}$$

$$x^2 - x = 4x - 6 \quad \text{One-to-one property of logarithms}$$

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 2 \quad \text{or} \quad x = 3$$

Neither $x = 2$ nor $x = 3$ produces undefined terms in the original equation. Use a calculator to check that they both satisfy the original equation. The solution set is $\{2, 3\}$.

calculator close-up

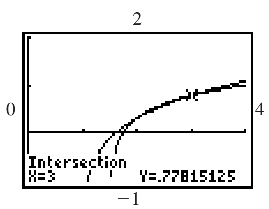
Graph

$y_1 = \log(x) + \log(x - 1)$

and

$y_2 = \log(8x - 12) - \log(2)$

to see the two solutions to the equation in Example 3.



CAUTION The product rule, quotient rule, and power rule do not eliminate logarithms from equations. To do so, we use the definition to change $y = \log_a(x)$ into $a^y = x$ or the one-to-one property to change $\log_a(m) = \log_a(n)$ into $m = n$.

Exponential Equations

If an equation has a single exponential expression, we can write the equivalent logarithmic equation.

EXAMPLE 4 **A single exponential expression**

Find the exact solution to $2^x = 10$.

Solution

The equivalent logarithmic equation is

$$x = \log_2(10).$$

The solution set is $\{\log_2(10)\}$. The number $\log_2(10)$ is the exact solution to the equation. Later in this section you will learn how to use the base-change formula to find an approximate value for an expression of this type.

In Section 12.1 we solved some exponential equations by writing each side as a power of the same base and then applying the one-to-one property of exponential functions. We review that method in the next example.

EXAMPLE 5 **Powers of the same base**

Solve $2^{(x^2)} = 4^{3x-4}$.

Solution

We can write each side as a power of the same base:

$$2^{(x^2)} = (2^2)^{3x-4} \quad \text{Because } 4 = 2^2$$
$$2^{(x^2)} = 2^{6x-8} \quad \text{Power of a power rule}$$
$$x^2 = 6x - 8 \quad \text{One-to-one property of exponential functions}$$
$$x^2 - 6x + 8 = 0$$
$$(x - 4)(x - 2) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 4 \quad \text{or} \quad x = 2$$

Check $x = 2$ and $x = 4$ in the original equation. The solution set is $\{2, 4\}$.

study tip

Success in school depends on effective time management, which is all about goals. Write down your long-term, short-term, and daily goals. Assess them, develop methods for meeting them, and reward yourself when you do.

For some exponential equations we cannot write each side as a power of the same base as we did in Example 5. In this case we take a logarithm of each side and simplify, using the rules for logarithms.

EXAMPLE 6 **Exponential equation with two different bases**

Find the exact and approximate solution to $2^{x-1} = 3^x$.

Solution

We first take the base-10 logarithm of each side:

$$2^{x-1} = 3^x \quad \text{Original equation}$$
$$\log(2^{x-1}) = \log(3^x) \quad \text{Take log of each side.}$$
$$(x - 1)\log(2) = x \cdot \log(3) \quad \text{Power rule}$$
$$x \cdot \log(2) - \log(2) = x \cdot \log(3) \quad \text{Distributive property}$$
$$x \cdot \log(2) - x \cdot \log(3) = \log(2) \quad \text{Get all } x\text{-terms on one side.}$$
$$x[\log(2) - \log(3)] = \log(2) \quad \text{Factor out } x.$$
$$x = \frac{\log(2)}{\log(2) - \log(3)} \quad \text{Exact solution}$$
$$x \approx -1.7095 \quad \text{Approximate solution}$$

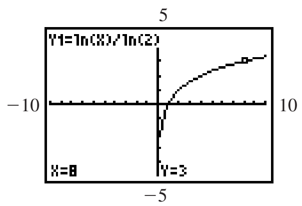
You can use a calculator to check -1.7095 in the original equation. As the first step of the solution, we could have taken the logarithm of each side using any base. We chose base 10 so that we could use a calculator to find an approximate solution from the exact solution.

Changing the Base

Scientific calculators have an x^y key for computing any power of any base, in addition to the function keys for computing 10^x and e^x. For logarithms we have the keys ln and log, but there are no function keys for logarithms using other bases. To solve this problem, we develop a formula for expressing a base-a logarithm in terms of base-b logarithms.

calculator close-up

The base-change formula allows you to graph logarithmic functions with bases other than e and 10. For example, to graph $y = \log_2(x)$, graph $y = \ln(x)/\ln(2)$.



If $y = \log_a(M)$, then $a^y = M$. Now we solve $a^y = M$ for y, using base-b logarithms:

$$\begin{aligned} a^y &= M \\ \log_b(a^y) &= \log_b(M) && \text{Take the base-}b\text{ logarithm of each side.} \\ y \cdot \log_b(a) &= \log_b(M) && \text{Power rule} \\ y &= \frac{\log_b(M)}{\log_b(a)} && \text{Divide each side by } \log_b(a). \end{aligned}$$

Because $y = \log_a(M)$, we can write $\log_a(M)$ in terms of base-b logarithms.

Base-Change Formula

If a and b are positive numbers not equal to 1 and M is positive, then

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}.$$

In words, we take the logarithm with the new base and divide by the logarithm of the old base. The most important use of the base-change formula is to find base-a logarithms using a calculator. If the new base is 10 or e, then

$$\log_a(M) = \frac{\log(M)}{\log(a)} = \frac{\ln(M)}{\ln(a)}.$$

EXAMPLE 7 **Using the base-change formula**

Find $\log_7(99)$ to four decimal places.

Solution

Use the base-change formula with $a = 7$ and $b = 10$:

$$\log_7(99) = \frac{\log(99)}{\log(7)} \approx 2.3614$$

Check by finding $7^{2.3614}$ with your calculator. Note that we also have

$$\log_7(99) = \frac{\ln(99)}{\ln(7)} \approx 2.3614.$$

■

Strategy for Solving Equations

There is no formula that will solve every equation in this section. However, we have a strategy for solving exponential and logarithmic equations. The following list summarizes the ideas that we need for solving these equations.

Solving Exponential and Logarithmic Equations

1. If the equation has a single logarithm or a single exponential expression, rewrite the equation using the definition $y = \log_a(x)$ if and only if $a^y = x$.
2. Use the properties of logarithms to combine logarithms as much as possible.
3. Use the one-to-one properties:
   a) If $\log_a(m) = \log_a(n)$, then $m = n$.
   b) If $a^m = a^n$, then $m = n$.
4. To get an approximate solution of an exponential equation, take the common or natural logarithm of each side of the equation.

Applications

In compound interest problems, logarithms are used to find the time it takes for money to grow to a specified amount.

EXAMPLE 8 **Finding the time**

If \$500 is deposited into an account paying 8% compounded quarterly, then in how many quarters will the account have \$1000 in it?

Solution

We use the compound interest formula $A = P(1 + i)^n$ with a principal of \$500, an amount of \$1000, and an interest rate of 2% each quarter:

$$\begin{aligned} A &= P(1 + i)^n \\ 1000 &= 500(1.02)^n && \text{Substitute.} \\ 2 &= (1.02)^n && \text{Divide each side by 500.} \\ n &= \log_{1.02}(2) && \text{Definition of logarithm} \\ &= \frac{\ln(2)}{\ln(1.02)} && \text{Base-change formula} \\ &\approx 35.0028 && \text{Use a calculator.} \end{aligned}$$

It takes approximately 35 quarters, or 8 years and 9 months, for the initial investment to be worth \$1000. Note that we could also solve $2 = (1.02)^n$ by taking the common or natural logarithm of each side. Try it. ■

helpful hint

When we get $2 = (1.02)^n$, we can use the definition of log as in Example 8 or take the natural log of each side:

$$\begin{aligned} \ln(2) &= \ln(1.02^n) \\ \ln(2) &= n \cdot \ln(1.02) \\ n &= \frac{\ln(2)}{\ln(1.02)} \end{aligned}$$

In either way we arrive at the same solution.

WARM-UPS

True or false? Explain your answer.

1. If $\log(x - 2) + \log(x + 2) = 7$, then $\log(x^2 - 4) = 7$.
2. If $\log(3x + 7) = \log(5x - 8)$, then $3x + 7 = 5x - 8$.
3. If $e^{x-6} = e^{x^2-5x}$, then $x - 6 = x^2 - 5x$.
4. If $2^{3x-1} = 3^{5x-4}$, then $3x - 1 = 5x - 4$.
5. If $\log_2(x^2 - 3x + 5) = 3$, then $x^2 - 3x + 5 = 8$.
6. If $2^{2x-1} = 3$, then $2x - 1 = \log_2(3)$.
7. If $5^x = 23$, then $x \cdot \ln(5) = \ln(23)$.
8. $\log_3(5) = \dfrac{\ln(3)}{\ln(5)}$
9. $\dfrac{\ln(2)}{\ln(6)} = \dfrac{\log(2)}{\log(6)}$
10. $\log(5) = \ln(5)$

12.4 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What exponential equation is equivalent to $\log_a(x) = y$?
2. How can you find a logarithm with a base other than 10 or e using a calculator?

Solve each equation. See Examples 1 and 2.

3. $\log_2(x + 1) = 3$
4. $\log_3(x^2) = 4$
5. $3 \log_2(x + 1) - 2 = 13$
6. $4 \log_3(2x) - 1 = 7$
7. $12 + 2 \ln(x) = 14$

8. $23 = 3\ln(x - 1) + 14$

9. $\log(x) + \log(5) = 1$

10. $\ln(x) + \ln(3) = 0$

11. $\log_2(x - 1) + \log_2(x + 1) = 3$

12. $\log_3(x - 4) + \log_3(x + 4) = 2$

13. $\log_2(x - 1) - \log_2(x + 2) = 2$

14. $\log_4(8x) - \log_4(x - 1) = 2$

15. $\log_2(x - 4) + \log_2(x + 2) = 4$

16. $\log_6(x + 6) + \log_6(x - 3) = 2$

Solve each equation. See Example 3.

17. $\ln(x) + \ln(x + 5) = \ln(x + 1) + \ln(x + 3)$

18. $\log(x) + \log(x + 5) = 2 \cdot \log(x + 2)$

19. $\log(x + 3) + \log(x + 4) = \log(x^3 + 13x^2) - \log(x)$

20. $\log(x^2 - 1) - \log(x - 1) = \log(6)$

21. $2 \cdot \log(x) = \log(20 - x)$

22. $2 \cdot \log(x) + \log(3) = \log(2 - 5x)$

Solve each equation. See Examples 4 and 5.

23. $3^x = 7$

24. $2^{x-1} = 5$

25. $e^{2x} = 7$

26. $e^{x+3} = 2$

27. $2^{3x+4} = 4^{x-1}$

28. $9^{2x-1} = 27^{1/2}$

29. $\left(\frac{1}{3}\right)^x = 3^{1+x}$

30. $4^{3x} = \left(\frac{1}{2}\right)^{1-x}$

Find the exact solution and approximate solution to each equation. See Example 6.

31. $2^x = 3^{x+5}$

32. $e^x = 10^x$

33. $5^{x+2} = 10^{x-4}$

34. $3^{2x} = 6^{x+1}$

35. $8^x = 9^{x-1}$

36. $5^{x+1} = 8^{x-1}$

Use the base-change formula to find each logarithm to four decimal places. See Example 7.

37. $\log_2(3)$

38. $\log_3(5)$

39. $\log_3\left(\frac{1}{2}\right)$

40. $\log_5(2.56)$

41. $\log_{1/2}(4.6)$

42. $\log_{1/3}(3.5)$

43. $\log_{0.1}(0.03)$

44. $\log_{0.2}(1.06)$

For each equation, find the exact solution and an approximate solution when appropriate. Round approximate answers to three decimal places.

45. $x \cdot \ln(2) = \ln(7)$

46. $x \cdot \log(3) = \log(5)$

47. $3x - x \cdot \ln(2) = 1$

48. $2x + x \cdot \log(5) = \log(7)$

49. $3^x = 5$

50. $2^x = \frac{1}{3}$

51. $2^{x-1} = 9$

52. $10^{x-2} = 6$

53. $3^x = 20$

54. $2^x = 128$

55. $\log_3(x) + \log_3(5) = 1$

56. $\log(x) - \log(3) = \log(6)$

57. $8^x = 2^{x+1}$

58. $2^x = 5^{x+1}$

In Exercises 59–70, solve each problem. See Example 8.

59. ***Finding the time.*** How many months does it take for \$1000 to grow to \$1500 in an account paying 12% compounded monthly?

60. ***Finding the time.*** How many years does it take for \$25 to grow to \$100 in an account paying 8% compounded annually?

61. ***Going with the flow.*** The flow y [in cubic feet per second (ft^3/sec)] of the Tangipahoa River at Robert, Louisiana, is modeled by the exponential function $y = 114.308e^{0.265x}$, where x is the depth in feet. Find the flow when the depth is 15.8 feet.

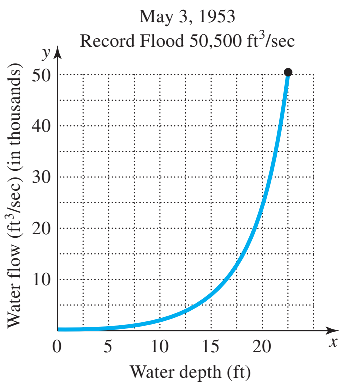


FIGURE FOR EXERCISES 61 AND 62

62. ***Record flood.*** Use the formula of the previous exercise to find the depth of the Tangipahoa River at Robert, Louisiana, on May 3, 1953 when the flow reached an all-time record of 50,500 ft^3/sec (U.S.G.S., waterdata.usgs.gov).

63. ***Above the poverty level.*** In a certain country the number of people above the poverty level is currently 28 million and growing 5% annually. Assuming the population is growing continuously, the population P (in millions), t years from now, is determined by the formula $P = 28e^{0.05t}$. In how many years will there be 40 million people above the poverty level?

64. ***Below the poverty level.*** In the same country as in Exercise 63, the number of people below the poverty level is currently 20 million and growing 7% annually. This population (in millions), t years from now, is determined by the formula $P = 20e^{0.07t}$. In how many years will there be 40 million people below the poverty level?

65. ***Fifty-fifty.*** For this exercise, use the information given in Exercises 63 and 64. In how many years will the number of people above the poverty level equal the number of people below the poverty level?

66. ***Golden years.*** In a certain country there are currently 100 million workers and 40 million retired people. The population of workers is decreasing according to the formula $W = 100e^{-0.01t}$, where t is in years and W is in millions. The population of retired people is increasing according to the formula $R = 40e^{0.09t}$, where t is in years and R is in millions. In how many years will the number of workers equal the number of retired people?

67. ***Ions for breakfast.*** Orange juice has a pH of 3.7. What is the hydrogen ion concentration of orange juice? (See Exercises 83–86 of Section 12.2.)

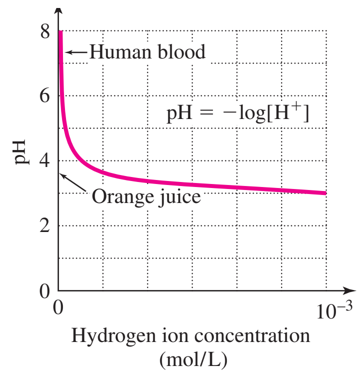


FIGURE FOR EXERCISES 67 AND 68

68. ***Ions in your veins.*** Normal human blood has a pH of 7.4. What is the hydrogen ion concentration of normal human blood?

69. ***Diversity index.*** In Exercise 98 of Section 12.3 we expressed the diversity index d for a certain water sample as

$$d = \log_2\left(\frac{3\sqrt[3]{2}}{2}\right).$$

Use the base-change formula and a calculator to calculate the value of d. Round the answer to four decimal places.

70. ***Quality water.*** In a certain water sample, 5% of the organisms are in one taxon, 10% are in a second taxon, 20% are in a third taxon, 15% are in a fourth taxon, 23% are in a fifth taxon, and the rest are in a sixth taxon. Use the formula given in Exercise 98 of Section 12.3 with $n = 6$ to find the diversity index of the water sample.

GETTING MORE INVOLVED

71. ***Exploration.*** Logarithms were designed to solve equations that have variables in the exponents, but logarithms can be used to solve certain polynomial equations. Consider the following example:

$$x^5 = 88$$
$$5 \cdot \ln(x) = \ln(88)$$
$$\ln(x) = \frac{\ln(88)}{5} \approx 0.895467$$
$$x = e^{0.895467} \approx 2.4485$$

Solve $x^3 = 12$ by taking the natural logarithm of each side. Round the approximate solution to four decimal places. Solve $x^3 = 12$ without using logarithms and compare with your previous answer.

72. ***Discussion.*** Determine whether each logarithm is positive or negative without using a calculator. Explain your answers.

a) $\log_2(0.45)$
b) $\ln(1.01)$
c) $\log_{1/2}(4.3)$
d) $\log_{1/3}(0.44)$

GRAPHING CALCULATOR EXERCISES

73. Graph $y_1 = 2^x$ and $y_2 = 3^{x-1}$ on the same coordinate system. Use the intersect feature of your calculator to find the point of intersection of the two curves. Round to two decimal places.

74. Bob invested \$1000 at 6% compounded continuously. At the same time Paula invested \$1200 at 5% compounded monthly. Write two functions that give the amounts of Bob's and Paula's investments after x years. Graph these functions on a graphing calculator. Use the intersect feature of your graphing calculator to find the approximate value of x for which the investments are equal in value.

75. Graph the functions $y_1 = \log_2(x)$ and $y_2 = 3^{x-4}$ on the same coordinate system and use the intersect feature to find the points of intersection of the curves. Round to two decimal places. (*Hint:* To graph $y = \log_2(x)$, use the base-change formula to write the function as $y = \ln(x)/\ln(2)$.)

COLLABORATIVE ACTIVITIES

In How Much Space Could We Live?

The formula for population growth is

$$P(t) = P_0e^{kt},$$

where $P(t)$ is the population after t years, P_0 is the population initially, k is the growth rate per year, and t is the number of years elapsed. We will find out how long it would take to cover the earth completely if the human population were growing exponentially. For all of your calculations, round your answers to one decimal place unless directed otherwise.

1. The population of the world in 1994 was approximately 5.5×10^9 people. If the population was 2.75×10^9 people in 1964, then what is the current growth rate (express as a percent)? Round your answer to the nearest tenth of a percent.

The earth has a total surface area of approximately 5.1×10^{14} square meters. Seventy percent of this surface area is rock, ice, sand, and open ocean. Another 8% of the total surface area is tundra, lakes and streams, continental shelves, algal beds and reefs, and estuaries. We will consider the remaining area to be suitable for growing food and for living space.

2. Determine the surface area available for growing food and for living space.

Grouping: 2 students per group
Topic: Exponential and logarithmic functions

3. If each person needs 100 square meters of the earth's surface for living space and growing food, then in how many years after 1994 will the livable surface of the earth be used up? (Use the rate and the 1994 population from Question 1 and the surface area from Question 2.)

4. With your partner, think about the following questions and report your conclusions.

- Does 100 square meters per person for living space and growing food seem reasonable? Take into account that many people live in tall apartment buildings and how that translates into surface area used per person.
- How much room do you think it takes to grow animals for food (cows, chickens, pigs, etc.)? What about grains, vegetables, nuts, and fruit? Would food grow as well in desert areas, mountainous areas, or jungle areas?
- Would there be any space left for wild animals or natural plant life? Would there be any space left for shopping malls, movie theaters, concert halls, factories, office buildings, or parking lots?