13.1 NONLINEAR SYSTEMS OF EQUATIONS

In this section

- Solving by Elimination
- Applications

We studied systems of linear equations in Chapter 8. In this section we turn our attention to nonlinear systems of equations.

Solving by Elimination

Equations such as

$$y = x^2, \qquad y = \sqrt{x}, \qquad y = |x|, \qquad y = 2^x, \qquad \text{and} \qquad y = \log_2(x)$$

are **nonlinear equations** because their graphs are not straight lines. We say that a system of equations is nonlinear if at least one equation in the system is nonlinear. We solve a nonlinear system just like a linear system, by elimination of variables. However, because the graphs of nonlinear equations may intersect at more than one point, there may be more than one ordered pair in the solution set to the system.

EXAMPLE 1 **A parabola and a line**

Solve the system of equations and draw the graph of each equation on the same coordinate system:

$$y = x^2 - 1$$
$$x + y = 1$$

Solution

We can eliminate y by substituting $y = x^2 - 1$ into $x + y = 1$:

$$x + y = 1$$
$$x + (x^2 - 1) = 1 \quad \text{Substitute } x^2 - 1 \text{ for } y.$$
$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 1 \quad \text{or} \quad x = -2$$

Replace x by 1 and -2 in $y = x^2 - 1$ to find the corresponding values of y:

$$y = (1)^2 - 1 \qquad y = (-2)^2 - 1$$
$$y = 0 \qquad y = 3$$

Check that each of the points (1, 0) and $(-2, 3)$ satisfies both of the original equations. The solution set is $\{(1, 0), (-2, 3)\}$. If we solve $x + y = 1$ for y, we get $y = -x + 1$. The line $y = -x + 1$ has y-intercept (0, 1) and slope -1. The graph of $y = x^2 - 1$ is a parabola with vertex $(0, -1)$. Of course, (1, 0) and $(-2, 3)$ are on both graphs. The two graphs are shown in Fig. 13.1.

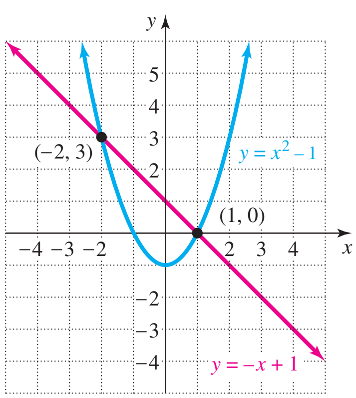


FIGURE 13.1

Graphing is not an accurate method for solving any system of equations. However, the graphs of the equations in a nonlinear system help us to understand how many solutions we should have for the system. It is not necessary to graph a system to solve it. Even when the graphs are too difficult to sketch, we can solve the system.

EXAMPLE 2 **Solving a system algebraically with substitution**

Solve the system:

$$x^2 + y^2 + 2y = 3$$
$$x^2 - y = 5$$

Solution

If we substitute $y = x^2 - 5$ into the first equation to eliminate y, we will get a fourth-degree equation to solve. Instead, we can eliminate the variable x by writing $x^2 - y = 5$ as $x^2 = y + 5$. Now replace x^2 by $y + 5$ in the first equation:

$$x^2 + y^2 + 2y = 3$$
$$(y + 5) + y^2 + 2y = 3$$
$$y^2 + 3y + 5 = 3$$
$$y^2 + 3y + 2 = 0$$
$$(y + 2)(y + 1) = 0 \quad \text{Solve by factoring.}$$
$$y + 2 = 0 \quad \text{or} \quad y + 1 = 0$$
$$y = -2 \quad \text{or} \quad y = -1$$

Let $y = -2$ in the equation $x^2 = y + 5$ to find the corresponding x:

$$x^2 = -2 + 5$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

Now let $y = -1$ in the equation $x^2 = y + 5$ to find the corresponding x:

$$x^2 = -1 + 5$$
$$x^2 = 4$$
$$x = \pm 2$$

Check these values in the original equations. The solution set is

$$\{(\sqrt{3}, -2), (-\sqrt{3}, -2), (2, -1), (-2, -1)\}.$$

The graphs of these two equations intersect at four points. ■

EXAMPLE 3

Solving a system with the addition method

Solve each system:

a) $x^2 - y^2 = 5$
$x^2 + y^2 = 7$

b) $\dfrac{2}{x} + \dfrac{1}{y} = \dfrac{1}{5}$
$\dfrac{1}{x} - \dfrac{3}{y} = \dfrac{1}{3}$

study tip

When you take notes leave space. Go back later and fill in more details, make corrections, or work another problem of the same type.

Solution

a) We can eliminate y by adding the equations:

$$x^2 - y^2 = 5$$
$$x^2 + y^2 = 7$$
$$2x^2 = 12$$
$$x^2 = 6$$
$$x = \pm\sqrt{6}$$

Since $x^2 = 6$, the second equation yields $6 + y^2 = 7$, $y^2 = 1$, and $y = \pm 1$. If $x^2 = 6$ and $y^2 = 1$, then both of the original equations are satisfied. The solution set is

$$\{(\sqrt{6}, 1)(\sqrt{6}, -1), (-\sqrt{6}, 1), (-\sqrt{6}, -1)\}$$

b) Usually with equations involving rational expressions we first multiply by the least common denominator (LCD), but this would make the given system more complicated. So we will just use the addition method to eliminate y:

$$\frac{6}{x} + \frac{3}{y} = \frac{3}{5} \quad \text{Eq. (1) multiplied by 3}$$
$$\frac{1}{x} - \frac{3}{y} = \frac{1}{3} \quad \text{Eq. (2)}$$
$$\frac{7}{x} = \frac{14}{15} \quad \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$$
$$14x = 7 \cdot 15$$
$$x = \frac{7 \cdot 15}{14} = \frac{15}{2}$$

To find y, substitute $x = \frac{15}{2}$ into Eq. (1):

$$\frac{2}{\frac{15}{2}} + \frac{1}{y} = \frac{1}{5}$$
$$\frac{4}{15} + \frac{1}{y} = \frac{1}{5} \quad \frac{2}{\frac{15}{2}} = 2 \cdot \frac{2}{15} = \frac{4}{15}$$
$$15y \cdot \frac{4}{15} + 15y \cdot \frac{1}{y} = 15y \cdot \frac{1}{5} \quad \text{Multiply each side by the LCD, } 15y.$$
$$4y + 15 = 3y$$
$$y = -15$$

Check that $x = \frac{15}{2}$ and $y = -15$ satisfy both original equations. The solution set is $\left\{\left(\frac{15}{2}, -15\right)\right\}$.

A system of nonlinear equations might involve exponential or logarithmic functions. To solve such systems, you will need to recall some facts about exponents and logarithms.

EXAMPLE 4 **A system involving logarithms**

Solve the system

$$y = \log_2(x + 28)$$
$$y = 3 + \log_2(x)$$

Solution

Eliminate y by substituting $\log_2(x + 28)$ for y in the second equation:

$$\log_2(x + 28) = 3 + \log_2(x) \quad \text{Eliminate } y.$$
$$\log_2(x + 28) - \log_2(x) = 3 \quad \text{Subtract } \log_2(x) \text{ from each side.}$$
$$\log_2\left(\frac{x + 28}{x}\right) = 3 \quad \text{Quotient rule for logarithms}$$
$$\frac{x + 28}{x} = 8 \quad \text{Definition of logarithm}$$
$$x + 28 = 8x \quad \text{Multiply each side by } x.$$
$$28 = 7x \quad \text{Subtract } x \text{ from each side.}$$
$$4 = x \quad \text{Divide each side by 7.}$$

If $x = 4$, then $y = \log_2(4 + 28) = \log_2(32) = 5$. Check (4, 5) in both equations. The solution to the system is $\{(4, 5)\}$. ■

Applications

The next example shows a geometric problem that can be solved with a system of nonlinear equations.

EXAMPLE 5

Nonlinear equations in applications

A 15-foot ladder is leaning against a wall so that the distance from the bottom of the ladder to the wall is one-half of the distance from the top of the ladder to the ground. Find the distance from the top of the ladder to the ground.

Solution

Let x be the number of feet from the bottom of the ladder to the wall and y be the number of feet from the top of the ladder to the ground (see Fig. 13.2). We can write two equations involving x and y:

$$x^2 + y^2 = 15^2 \quad \text{Pythagorean theorem}$$
$$y = 2x$$

Solve by substitution:

$$x^2 + (2x)^2 = 225 \quad \text{Replace } y \text{ by } 2x.$$
$$x^2 + 4x^2 = 225$$
$$5x^2 = 225$$
$$x^2 = 45$$
$$x = \pm\sqrt{45} = \pm 3\sqrt{5}$$

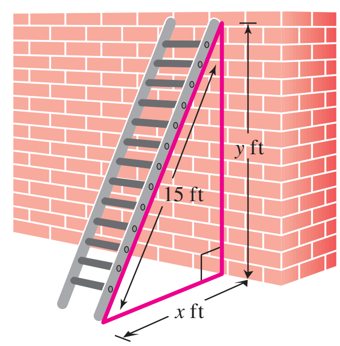


FIGURE 13.2

Because x represents distance, x must be positive. So $x = 3\sqrt{5}$. Because $y = 2x$, we get $y = 6\sqrt{5}$. The distance from the top of the ladder to the ground is $6\sqrt{5}$ feet. ■

calculator close-up

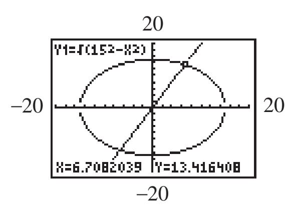
To see the solutions, graph

$y_1 = \sqrt{15^2 - x^2}$,

$y_2 = -\sqrt{15^2 - x^2}$,

and $y_3 = 2x$.

The line intersects the circle twice.



The next example shows how a nonlinear system can be used to solve a problem involving work.

EXAMPLE 6

Nonlinear equations in applications

A large fish tank at the Gulf Aquarium can usually be filled in 10 minutes using pumps A and B. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself?

Solution

Let a represent the number of minutes that it takes pump A to fill the tank alone and b represent the number of minutes it takes pump B to fill the tank alone. The rate at which pump A fills the tank is $\frac{1}{a}$ of the tank per minute, and the rate at which pump B fills the tank is $\frac{1}{b}$ of the tank per minute. Because the work completed is the product of the rate and time, we can make the following table when the pumps work together to fill the tank:

	Rate	Time	Work
Pump A	$\frac{1}{a}\ \frac{\text{tank}}{\text{min}}$	10 min	$\frac{10}{a}$ tank
Pump B	$\frac{1}{b}\ \frac{\text{tank}}{\text{min}}$	10 min	$\frac{10}{b}$ tank

Note that each pump fills a fraction of the tank and those fractions have a sum of 1:

$$(1) \qquad \frac{10}{a} + \frac{10}{b} = 1$$

helpful hint

Note that we could write equations about the rates. Pump A's rate is $\frac{1}{a}$ tank per minute, B's rate is $\frac{1}{b}$ tank per minute, and together their rate is $\frac{1}{10}$ tank per minute or $\frac{1}{30}$ tank per minute.

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{10}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{30}$$

In the 30 minutes in which pump B is working in reverse, A puts in $\frac{30}{a}$ of the tank whereas B takes out $\frac{30}{b}$ of the tank. Since the tank still gets filled, we can write the following equation:

$$(2) \qquad \frac{30}{a} - \frac{30}{b} = 1$$

Multiply Eq. (1) by 3 and add the result to Eq. (2) to eliminate b:

$$\frac{30}{a} + \frac{30}{b} = 3 \quad \text{Eq. (1) multiplied by 3}$$

$$\frac{30}{a} - \frac{30}{b} = 1 \quad \text{Eq. (2)}$$

$$\frac{60}{a} = 4$$

$$4a = 60$$

$$a = 15$$

Use $a = 15$ in Eq. (1) to find b:

$$\frac{10}{15} + \frac{10}{b} = 1$$

$$\frac{10}{b} = \frac{1}{3} \quad \text{Subtract } \frac{10}{15} \text{ from each side.}$$

$$b = 30$$

So pump A fills the tank in 15 minutes working alone, and pump B fills the tank in 30 minutes working alone. ■

WARM-UPS

True or false? Explain your answer.

1. The graph of $y = x^2$ is a parabola.
2. The graph of $y = |x|$ is a straight line.
3. The point $(3, -4)$ satisfies both $x^2 + y^2 = 25$ and $y = \sqrt{5x + 1}$.
4. The graphs of $y = \sqrt{x}$ and $y = -x - 2$ do not intersect.
5. Substitution is the only method for eliminating a variable when solving a nonlinear system.
6. If Bob paints a fence in x hours, then he paints $\frac{1}{x}$ of the fence per hour.
7. In a triangle whose angles are 30°, 60°, and 90°, the length of the side opposite the 30° angle is one-half the length of the hypotenuse.
8. The formula $V = LWH$ gives the volume of a rectangular box in which the sides have lengths L, W, and H.
9. The surface area of a rectangular box is $2LW + 2WH + 2LH$.
10. The area of a right triangle is one-half the product of the lengths of its legs.

13.1 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. Why are some equations called nonlinear?
2. Why do we graph the equations in a nonlinear system?
3. Why don't we solve systems by graphing?
4. What techniques do we use to solve nonlinear systems?

Solve each system and graph both equations on the same set of axes. See Example 1.

5. $y = x^2$
   $x + y = 6$
6. $y = x^2 - 1$
   $x + y = 11$
7. $y = |x|$
   $2y - x = 6$
8. $y = |x|$
   $3y = x + 6$
9. $y = \sqrt{2x}$
   $x - y = 4$
10. $y = \sqrt{x}$
    $x - y = 6$

11. $4x - 9y = 9$
$xy = 1$

12. $2x + 2y = 3$
$xy = -1$

13. $y = -x^2 + 1$
$y = x^2$

14. $y = x^2$
$y = \sqrt{x}$

Solve each system. See Examples 2 and 3.

15. $x^2 + y^2 = 25$
$y = x^2 - 5$

16. $x^2 + y^2 = 25$
$y = x + 1$

17. $xy - 3x = 8$
$y = x + 1$

18. $xy + 2x = 9$
$x - y = 2$

19. $xy - x = 8$
$xy + 3x = -4$

20. $2xy - 3x = -1$
$xy + 5x = -7$

21. $x^2 + y^2 = 8$
$x^2 - y^2 = 2$

22. $y^2 - 2x^2 = 1$
$y^2 + 2x^2 = 5$

23. $x^2 + 2y^2 = 8$
$2x^2 - y^2 = 1$

24. $2x^2 + 3y^2 = 8$
$3x^2 + 2y^2 = 7$

25. $\frac{1}{x} - \frac{1}{y} = 5$
$\frac{2}{x} + \frac{1}{y} = -3$

26. $\frac{2}{x} - \frac{3}{y} = \frac{1}{2}$
$\frac{3}{x} + \frac{1}{y} = \frac{1}{2}$

27. $\frac{2}{x} - \frac{1}{y} = \frac{5}{12}$
$\frac{1}{x} - \frac{3}{y} = -\frac{5}{12}$

28. $\frac{3}{x} - \frac{2}{y} = 5$
$\frac{4}{x} + \frac{3}{y} = 18$

29. $x^2y = 20$
$xy + 2 = 6x$

30. $y^2x = 3$
$xy + 1 = 6x$

31. $x^2 + xy - y^2 = -11$
$x + y = 7$

32. $x^2 + xy + y^2 = 3$
$y = 2x - 5$

33. $3y - 2 = x^4$
$y = x^2$

34. $y - 3 = 2x^4$
$y = 7x^2$

Solve the following systems involving logarithmic and exponential functions. See Example 4.

35. $y = \log_2(x - 1)$
$y = 3 - \log_2(x + 1)$

36. $y = \log_3(x - 4)$
$y = 2 - \log_3(x + 4)$

37. $y = \log_2(x - 1)$
$y = 2 + \log_2(x + 2)$

38. $y = \log_4(8x)$
$y = 2 + \log_4(x - 1)$

39. $y = 2^{3x+4}$
$y = 4^{x-1}$

40. $y = 4^{3x}$
$y = \left(\frac{1}{2}\right)^{1-x}$

Solve each problem by using a system of two equations in two unknowns. See Examples 5 and 6.

41. ***Known hypotenuse.*** Find the lengths of the legs of a right triangle whose hypotenuse is $\sqrt{15}$ feet and whose area is 3 square feet.

42. ***Known diagonal.*** A small television is advertised to have a picture with a diagonal measure of 5 inches and a viewing area of 12 square inches (in.^{2}). What are the length and width of the screen?

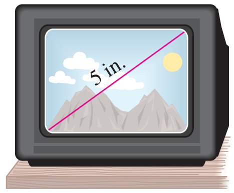


FIGURE FOR EXERCISE 42

43. ***House of seven gables.*** Vincent has plans to build a house with seven gables. The plans call for an attic vent in the shape of an isosceles triangle in each gable. Because of the slope of the roof, the ratio of the height to the base of each triangle must be 1 to 4. If the vents are to provide a total ventilating area of 3500 in.^{2}, then what should be the height and base of each triangle?

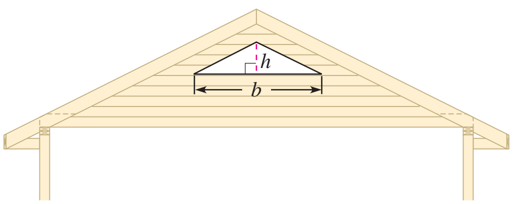


FIGURE FOR EXERCISE 43

44. ***Known perimeter.*** Find the lengths of the sides of a triangle whose perimeter is 6 feet (ft) and whose angles are 30°, 60°, and 90° (see Appendix A).

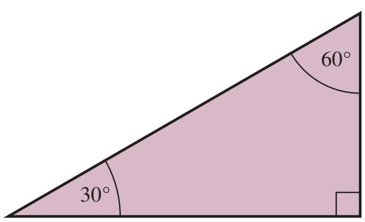


FIGURE FOR EXERCISE 44

45. ***Filling a tank.*** Pump A can either fill a tank or empty it in the same amount of time. If pump A and pump B are working together, the tank can be filled in 6 hours. When pump A was inadvertently left in the drain position while pump B was trying to fill the tank, it took 12 hours to fill the tank. How long would it take either pump working alone to fill the tank?

46. ***Cleaning a house.*** Roxanne either cleans the house or messes it up at the same rate. When Roxanne is cleaning with her mother, they can clean up a completely messed up house in 6 hours. If Roxanne is not cooperating, it takes her mother 9 hours to clean the house, with Roxanne continually messing it up. How long would it take her mother to clean the entire house if Roxanne were sent to her grandmother's house?

47. ***Cleaning fish.*** Jan and Beth work in a seafood market that processes 200 pounds of catfish every morning. On Monday, Jan started cleaning catfish at 8:00 A.M. and finished cleaning 100 pounds just as Beth arrived. Beth then took over and finished the job at 8:50 A.M. On Tuesday they both started at 8 A.M. and worked together to finish the job at 8:24 A.M. On Wednesday, Beth was sick. If Jan is the faster worker, then how long did it take Jan to complete all of the catfish by herself?

FIGURE FOR EXERCISE 47

48. ***Building a patio.*** Richard has already formed a rectangular area for a flagstone patio, but his wife Susan is unsure of the size of the patio they want. If the width is increased by 2 ft, then the area is increased by 30 square feet (ft^2). If the width is increased by 1 ft and the length by 3 ft, then the area is increased by 54 ft^2. What are the dimensions of the rectangle that Richard has already formed?

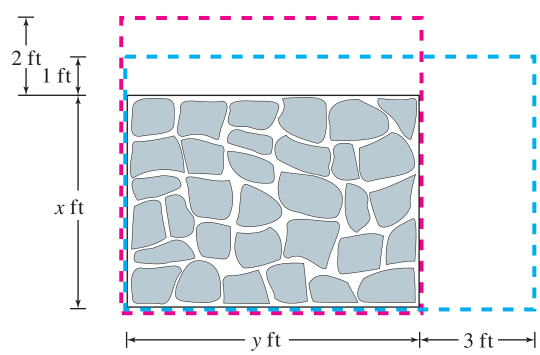


FIGURE FOR EXERCISE 48

49. ***Fencing a rectangle.*** If 34 ft of fencing are used to enclose a rectangular area of 72 ft^2, then what are the dimensions of the area?

50. ***Real numbers.*** Find two numbers that have a sum of 8 and a product of 10.

51. ***Imaginary numbers.*** Find two complex numbers whose sum is 8 and whose product is 20.

52. ***Imaginary numbers.*** Find two complex numbers whose sum is -6 and whose product is 10.

53. ***Making a sign.*** Rico's Sign Shop has a contract to make a sign in the shape of a square with an isosceles triangle on top of it, as shown in the figure. The contract calls for a total height of 10 ft with an area of 72 ft^2. How long should Rico make the side of the square and what should be the height of the triangle?

54. ***Designing a box.*** Angelina is designing a rectangular box of 120 cubic inches that is to contain new Eaties breakfast cereal. The box must be 2 inches thick so that it is easy to hold. It must have 184 square inches of surface area to provide enough space for all of the special offers and coupons. What should be the dimensions of the box?

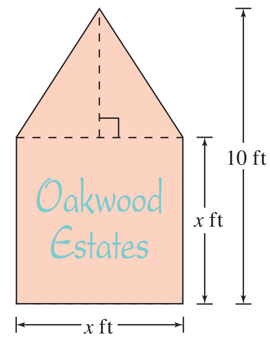


FIGURE FOR EXERCISE 53

GRAPHING CALCULATOR EXERCISES

55. Solve each system by graphing each pair of equations on a graphing calculator and using the intersect feature to estimate the point of intersection. Find the coordinates of each intersection to the nearest hundredth.

a) $y = e^x - 4$
$y = \ln(x + 3)$

b) $3^{y-1} = x$
$y = x^2$

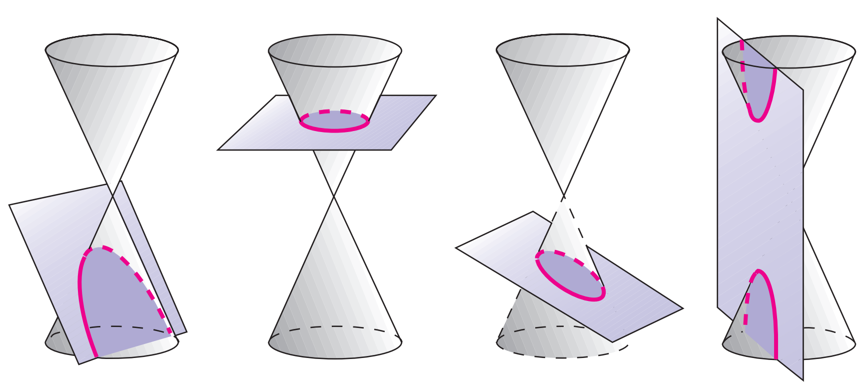
c) $x^2 + y^2 = 4$
$y = x^3$

13.2 THE PARABOLA

In this section

- The Geometric Definition
- Developing the Equation
- Parabolas in the Form $y = a(x - h)^2 + k$
- Finding the Vertex, Focus, and Directrix
- Axis of Symmetry
- Changing Forms

The parabola is one of four different curves that can be obtained by intersecting a cone and a plane as in Fig. 13.3. These curves, called **conic sections,** are the parabola, circle, ellipse, and hyperbola. We graphed parabolas in Sections 10.3 and 11.2. In this section we learn some new facts about parabolas.



Parabola Circle Ellipse Hyperbola

FIGURE 13.3