58. Find all points of intersection of the parabolas $y = x^2$ and $y = (x - 3)^2$.

GETTING MORE INVOLVED

59. ***Exploration.*** Consider the parabola with focus $(p, 0)$ and directrix $x = -p$ for $p > 0$. Let (x, y) be an arbitrary point on the parabola. Write an equation expressing the fact that the distance from (x, y) to the focus is equal to the distance from (x, y) to the directrix. Rewrite the equation in the form $x = ay^2$, where $a = \frac{1}{4p}$.

60. ***Exploration.*** In general, the graph of $x = a(y - h)^2 + k$ for $a \neq 0$ is a parabola opening left or right with vertex at (k, h).

a) For which values of a does the parabola open to the right, and for which values of a does it open to the left?

b) What is the equation of its axis of symmetry?

c) Sketch the graphs $x = 2(y - 3)^2 + 1$ and $x = -(y + 1)^2 + 2$.

GRAPHING CALCULATOR EXERCISES

61. Graph $y = x^2$ using the viewing window with $-1 \leq x \leq 1$ and $0 \leq y \leq 1$. Next graph $y = 2x^2 - 1$ using the viewing window $-2 \leq x \leq 2$ and $-1 \leq y \leq 7$. Explain what you see.

62. Graph $y = x^2$ and $y = 6x - 9$ in the viewing window $-5 \leq x \leq 5$ and $-5 \leq y \leq 20$. Does the line appear to be tangent to the parabola? Solve the system $y = x^2$ and $y = 6x - 9$ to find all points of intersection for the parabola and the line.

13.3 THE CIRCLE

In this section

- Developing the Equation
- Equations Not in Standard Form
- Systems of Equations

In this section we continue the study of the conic sections with a discussion of the circle.

Developing the Equation

A circle is obtained by cutting a cone, as was shown in Fig. 13.3. We can also define a circle using points and distance, as we did for the parabola.

Circle

A **circle** is the set of all points in a plane that lie a fixed distance from a given point in the plane. The fixed distance is called the **radius,** and the given point is called the **center.**

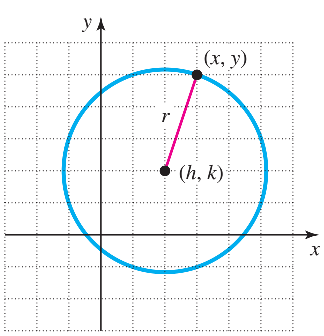


FIGURE 13.12

We can use the distance formula of Section 9.5 to write an equation for the circle with center (h, k) and radius r, shown in Fig. 13.12. If (x, y) is a point on the circle, its distance from the center is r. So

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

We square both sides of this equation to get the **standard form** for the equation of a circle.

Standard Equation for a Circle

The graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

with $r > 0$, is a circle with center (h, k) and radius r.

Note that a circle centered at the origin with radius r ($r > 0$) has the standard equation

$$x^2 + y^2 = r^2.$$

EXAMPLE 1

Finding the equation, given the center and radius

Write the standard equation for the circle with the given center and radius.

a) Center (0, 0), radius 2 **b)** Center $(-1, 2)$, radius 4

Solution

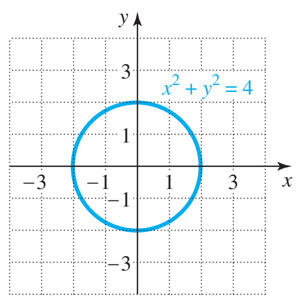


FIGURE 13.13

a) The center at (0, 0) means that $h = 0$ and $k = 0$ in the standard equation. So the equation is $(x - 0)^2 + (y - 0)^2 = 2^2$, or $x^2 + y^2 = 4$. The circle with radius 2 centered at the origin is shown in Fig. 13.13.

b) The center at $(-1, 2)$ means that $h = -1$ and $k = 2$. So

$$[x - (-1)]^2 + [y - 2]^2 = 4^2.$$

Simplify this equation to get

$$(x + 1)^2 + (y - 2)^2 = 16.$$

The circle with center $(-1, 2)$ and radius 4 is shown in Fig. 13.14.

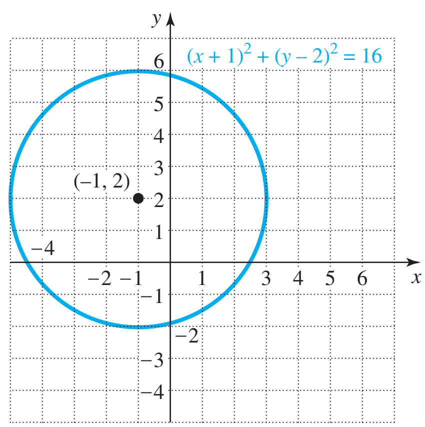


FIGURE 13.14

CAUTION The equations $(x - 1)^2 + (y + 3)^2 = -9$ and $(x - 1)^2 + (y + 3)^2 = 0$ might look like equations of circles, but they are not. The first equation is not satisfied by any ordered pair of real numbers because the left-hand side is nonnegative for any x and y. The second equation is satisfied only by the point $(1, -3)$.

EXAMPLE 2

Finding the center and radius, given the equation

Determine the center and radius of the circle $x^2 + (y + 5)^2 = 2$.

Solution

We can write this equation as

$$(x - 0)^2 + [y - (-5)]^2 = (\sqrt{2})^2.$$

In this form we see that the center is $(0, -5)$ and the radius is $\sqrt{2}$.

EXAMPLE 3

Graphing a circle

Find the center and radius of $(x - 1)^2 + (y + 2)^2 = 9$, and sketch the graph.

Solution

The graph of this equation is a circle with center at $(1, -2)$ and radius 3. See Fig. 13.15 for the graph. ■

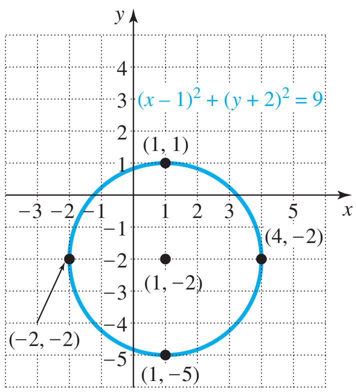


FIGURE 13.15

calculator close-up

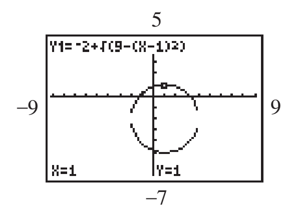
To graph the circle in Example 3, graph

$$y_1 = -2 + \sqrt{9 - (x - 1)^2}$$

and

$$y_2 = -2 - \sqrt{9 - (x - 1)^2}.$$

To get the circle to look round, you must use the same unit length on each axis. Most calculators have a *square* feature that automatically adjusts the window to use the same unit length on each axis.



Equations Not in Standard Form

It is not easy to recognize that $x^2 - 6x + y^2 + 10y = -30$ is the equation of a circle, but it is. In the next example we convert this equation into the standard form for a circle by completing the squares for the variables x and y.

EXAMPLE 4

Converting to standard form

Find the center and radius of the circle given by the equation

$$x^2 - 6x + y^2 + 10y = -30.$$

Solution

To complete the square for $x^2 - 6x$, we add 9, and for $y^2 + 10y$, we add 25. To get an equivalent equation, we must add on both sides:

$$x^2 - 6x + \qquad y^2 + 10y \qquad = -30$$

$$x^2 - 6x + 9 + y^2 + 10y + 25 = -30 + 9 + 25 \quad \text{Add 9 and 25 to both sides.}$$

$$(x - 3)^2 + (y + 5)^2 = 4 \quad \text{Factor the trinomials on the left-hand side.}$$

From the standard form we see that the center is $(3, -5)$ and the radius is 2. ■

helpful hint

What do circles and lines have in common? They are the two simplest graphs to draw. We have compasses to make our circles look good and rulers to make our lines look good.

Systems of Equations

We first solved systems of nonlinear equations in two variables in Section 13.1. We found the points of intersection of two graphs without drawing the graphs. Here we will solve systems involving circles, parabolas, and lines. In the next example we find the points of intersection of a line and a circle.

EXAMPLE 5

Intersection of a line and a circle

Graph both equations of the system

$$(x - 3)^2 + (y + 1)^2 = 9$$

$$y = x - 1$$

on the same coordinate axes, and solve the system by elimination of variables.

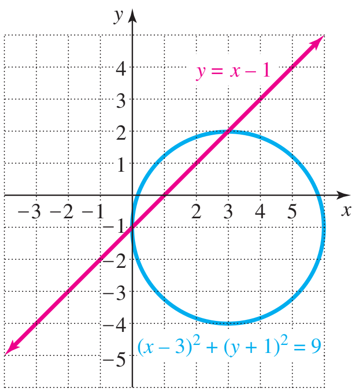


FIGURE 13.16

Solution

The graph of the first equation is a circle with center at $(3, -1)$ and radius 3. The graph of the second equation is a straight line with slope 1 and y-intercept $(0, -1)$. Both graphs are shown in Fig. 13.16. To solve the system by elimination, we substitute $y = x - 1$ into the equation of the circle:

$$
\begin{aligned}
(x - 3)^2 + (x - 1 + 1)^2 &= 9 \\
(x - 3)^2 + x^2 &= 9 \\
x^2 - 6x + 9 + x^2 &= 9 \\
2x^2 - 6x &= 0 \\
x^2 - 3x &= 0 \\
x(x - 3) &= 0 \\
x = 0 \quad \text{or} \quad x &= 3 \\
y = -1 \qquad\quad y &= 2 \qquad \text{Because } y = x - 1
\end{aligned}
$$

Check $(0, -1)$ and $(3, 2)$ in the original system and with the graphs in Fig. 13.16. The solution set is $\{(0, -1), (3, 2)\}$.

MATH AT WORK

EARLY MUSICAL INSTRUMENT MAKER

Friedrich von Huene, a flautist and recorder player, has been crafting woodwind instruments in his family business for over 30 years. Because it is best to play music of earlier centuries on the instruments of their time, von Huene is using many originals as models for his flutes, recorders, and oboes of different sizes.

Because museum instruments have many different pitch standards, their dimensions frequently have to be changed to accommodate pitch standards that musicians use today. For a lower pitch the length of the instrument as well as the inside diameter must be increased. For a higher pitch the length has to be shortened and the diameter decreased. However, the factor for changing the length will be different from the factor for changing the diameter. A row of organ pipes demonstrates that the larger and longer pipes are proportionately more slender than the shorter high-pitched pipes. A pipe an octave higher in pitch is about half as long as the pipe an octave lower, but its diameter will be about 0.6 as large as the diameter of the lower pipe.

When making a very large recorder, von Huene carefully chooses the length, the position of the tone holes, and the bore to get the proper volume of air inside the instrument. In Exercises 57 and 58 of this section you will make the kinds of calculations von Huene makes when he crafts a modern reproduction of a Renaissance flute.

WARM-UPS

True or false? Explain your answer.

1. The radius of a circle can be any nonzero real number.
2. The coordinates of the center must satisfy the equation of the circle.
3. The circle $x^2 + y^2 = 4$ has its center at the origin.
4. The graph of $x^2 + y^2 = 9$ is a circle centered at (0, 0) with radius 9.
5. The graph of $(x - 2)^2 + (y - 3)^2 + 4 = 0$ is a circle of radius 2.
6. The graph of $(x - 3) + (y + 5) = 9$ is a circle of radius 3.
7. There is only one circle centered at $(-3, -1)$ passing through the origin.
8. The center of the circle $(x - 3)^2 + (y - 4)^2 = 10$ is $(-3, -4)$.
9. The center of the circle $x^2 + y^2 + 6y - 4 = 0$ is on the y-axis.
10. The radius of the circle $x^2 - 3x + y^2 = 4$ is 2.

13.3 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the definition of a circle?
2. What is the standard equation of a circle?

Write the standard equation for each circle with the given center and radius. See Example 1.

3. Center (0, 3), radius 5
4. Center (2, 0), radius 3
5. Center $(1, -2)$, radius 9
6. Center $(-3, 5)$, radius 4
7. Center (0, 0), radius $\sqrt{3}$
8. Center (0, 0), radius $\sqrt{2}$
9. Center $(-6, -3)$, radius $\frac{1}{2}$
10. Center $(-3, -5)$, radius $\frac{1}{4}$
11. Center $\left(\frac{1}{2}, \frac{1}{3}\right)$, radius 0.1
12. Center $\left(-\frac{1}{2}, 3\right)$, radius 0.2

Find the center and radius for each circle. See Example 2.

13. $(x - 3)^2 + (y - 5)^2 = 2$
14. $(x + 3)^2 + (y - 7)^2 = 6$
15. $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$
16. $5x^2 + 5y^2 = 5$
17. $4x^2 + 4y^2 = 9$
18. $9x^2 + 9y^2 = 49$
19. $3 - y^2 = (x - 2)^2$
20. $9 - x^2 = (y + 1)^2$

Sketch the graph of each equation. See Example 3.

21. $x^2 + y^2 = 9$
22. $x^2 + y^2 = 16$

23. $x^2 + (y - 3)^2 = 9$

24. $(x - 4)^2 + y^2 = 16$

25. $(x + 1)^2 + (y - 1)^2 = 2$

26. $(x - 2)^2 + (y + 2)^2 = 8$

27. $(x - 4)^2 + (y + 3)^2 = 16$

28. $(x - 3)^2 + (y - 7)^2 = 25$

29. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$

30. $\left(x + \frac{1}{3}\right)^2 + y^2 = \frac{1}{9}$

Rewrite each equation in the standard form for the equation of a circle, and identify its center and radius. See Example 4.

31. $x^2 + 4x + y^2 + 6y = 0$

32. $x^2 - 10x + y^2 + 8y = 0$

33. $x^2 - 2x + y^2 - 4y - 3 = 0$

34. $x^2 - 6x + y^2 - 2y + 9 = 0$

35. $x^2 + y^2 = 8y + 10x - 32$

36. $x^2 + y^2 = 8x - 10y$

37. $x^2 - x + y^2 + y = 0$

38. $x^2 - 3x + y^2 = 0$

39. $x^2 - 3x + y^2 - y = 1$

40. $x^2 - 5x + y^2 + 3y = 2$

41. $x^2 - \frac{2}{3}x + y^2 + \frac{3}{2}y = 0$

42. $x^2 + \frac{1}{3}x + y^2 - \frac{2}{3}y = \frac{1}{9}$

Graph both equations of each system on the same coordinate axes. Solve the system by elimination of variables to find all points of intersection of the graphs. See Example 5.

43. $x^2 + y^2 = 10$
$y = 3x$

44. $x^2 + y^2 = 4$
$y = x - 2$

45. $x^2 + y^2 = 9$
$y = x^2 - 3$

46. $x^2 + y^2 = 4$
$y = x^2 - 2$

47. $(x - 2)^2 + (y + 3)^2 = 4$
$y = x - 3$

48. $(x + 1)^2 + (y - 4)^2 = 17$
$y = x + 2$

In Exercises 49–58, solve each problem.

49. Determine all points of intersection of the circle $(x - 1)^2 + (y - 2)^2 = 4$ with the y-axis.

50. Determine the points of intersection of the circle $x^2 + (y - 3)^2 = 25$ with the x-axis.

51. Find the radius of the circle that has center $(2, -5)$ and passes through the origin.

52. Find the radius of the circle that has center $(-2, 3)$ and passes through $(3, -1)$.

53. Determine the equation of the circle that is centered at $(2, 3)$ and passes through $(-2, -1)$.

54. Determine the equation of the circle that is centered at $(3, 4)$ and passes through the origin.

55. Find all points of intersection of the circles $x^2 + y^2 = 9$ and $(x - 5)^2 + y^2 = 9$.

56. A donkey is tied at the point $(2, -3)$ on a rope of length 12. Turnips are growing at the point $(6, 7)$. Can the donkey reach them?

57. ***Volume of a flute.*** The volume of air in a flute is a critical factor in determining its pitch. A cross section of a Renaissance flute in C is shown in the accompanying figure. If the length of the flute is 2874 millimeters, then what is the volume of air in the flute (to the nearest cubic millimeter (mm^3))? (*Hint:* Use the formula for the volume of a cylinder.)

58. ***Flute reproduction.*** To make the smaller C# flute, Friedrich von Huene multiplies the length and cross-sectional area of the flute of Exercise 57 by 0.943. Find the equation for the bore hole (centered at the origin) and the volume of air in the C# flute.

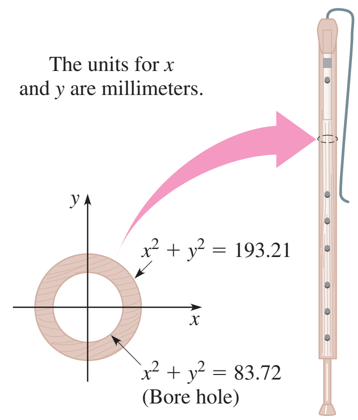


FIGURE FOR EXERCISES 57 AND 58

Graph each equation.

59. $x^2 + y^2 = 0$

60. $x^2 - y^2 = 0$

61. $y = \sqrt{1 - x^2}$

62. $y = -\sqrt{1 - x^2}$

GETTING MORE INVOLVED

63. ***Cooperative learning.*** The equation of a circle is a special case of the general equation $Ax^2 + Bx + Cy^2 + Dy = E$, where A, B, C, D, and E are real numbers. Working in small groups, find restrictions that must be placed on A, B, C, D, and E so that the graph of this equation is a circle. What does the graph of $x^2 + y^2 = -9$ look like?

64. ***Discussion.*** Suppose lighthouse A is located at the origin and lighthouse B is located at coordinates (0, 6). The captain of a ship has determined that the ship's distance from lighthouse A is 2 and its distance from lighthouse B is 5. What are the possible coordinates for the location of the ship?

GRAPHING CALCULATOR EXERCISES

Graph each relation on a graphing calculator by solving for y and graphing two functions.

65. $x^2 + y^2 = 4$

66. $(x - 1)^2 + (y + 2)^2 = 1$

67. $x = y^2$

68. $x = (y + 2)^2 - 1$

69. $x = y^2 + 2y + 1$

70. $x = 4y^2 + 4y + 1$

13.4 THE ELLIPSE AND HYPERBOLA

In this section

- The Ellipse
- The Hyperbola

In this section we study the remaining two conic sections: the ellipse and the hyperbola.

The Ellipse

An ellipse can be obtained by intersecting a plane and a cone, as was shown in Fig. 13.3. We can also give a definition of an ellipse in terms of points and distance.

Ellipse

An **ellipse** is the set of all points in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a **focus** (plural: foci).

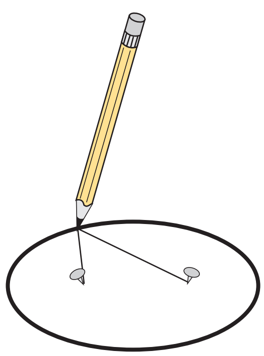


FIGURE 13.17

An easy way to draw an ellipse is illustrated in Fig. 13.17. A string is attached at two fixed points, and a pencil is used to take up the slack. As the pencil is moved around the paper, the sum of the distances of the pencil point from the two fixed points remains constant. Of course, the length of the string is that constant. You may wish to try this.

Like the parabola, the ellipse also has interesting reflecting properties. All light or sound waves emitted from one focus are reflected off the ellipse to concentrate at the other focus (see Fig. 13.18). This property is used in light fixtures where a concentration of light at a point is desired or in a whispering gallery such as Statuary Hall in the U.S. Capitol Building.

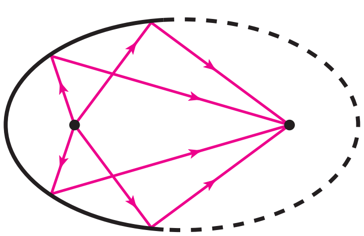


FIGURE 13.18

The orbits of the planets around the sun and satellites around the earth are elliptical. For the orbit of the earth around the sun, the sun is at one focus. For the elliptical path of an earth satellite, the earth is at one focus and a point in space is the other focus.