Working in groups, simplify this equation. First get the radicals on opposite sides of the equation, then square both sides twice to eliminate the square roots. Finally, let $b^2 = a^2 - c^2$ to get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

56. ***Cooperative learning.*** Let (x, y) be an arbitrary point on a hyperbola with foci $(c, 0)$ and $(-c, 0)$ for $c > 0$. The following equation expresses the fact that the distance from (x, y) to $(c, 0)$ minus the distance from (x, y) to $(-c, 0)$ is the constant value $2a$ (for $a > 0$):

$$\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x-(-c))^2 + (y-0)^2} = 2a$$

Working in groups, simplify the equation. You will need to square both sides twice to eliminate the square roots. Finally, let $b^2 = c^2 - a^2$ to get the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

GRAPHING CALCULATOR EXERCISES

57. Graph $y_1 = \sqrt{x^2 - 1}$, $y_2 = -\sqrt{x^2 - 1}$, $y_3 = x$, and $y_4 = -x$ to get the graph of the hyperbola $x^2 - y^2 = 1$ along with its asymptotes. Use the viewing window $-3 \le x \le 3$ and $-3 \le y \le 3$. Notice how the branches of the hyperbola approach the asymptotes.

58. Graph the same four functions in Exercise 57, but use $-30 \le x \le 30$ and $-30 \le y \le 30$ as the viewing window. What happened to the hyperbola?

13.5 SECOND-DEGREE INEQUALITIES

In this section we graph second-degree inequalities and systems of inequalities involving second-degree inequalities.

In this section

- Graphing a Second-Degree Inequality
- Systems of Inequalities

Graphing a Second-Degree Inequality

A second-degree inequality is an inequality involving squares of at least one of the variables. Changing the equal sign to an inequality symbol for any of the equations of the conic sections gives us a second-degree inequality. Second-degree inequalities are graphed in the same manner as linear inequalities.

EXAMPLE 1

A second-degree inequality

Graph the inequality $y < x^2 + 2x - 3$.

Solution

We first graph $y = x^2 + 2x - 3$. This parabola has x-intercepts at $(1, 0)$ and $(-3, 0)$, y-intercept at $(0, -3)$, and vertex at $(-1, -4)$. The graph of the parabola is drawn with a dashed line, as shown in Fig. 13.29. The graph of the parabola divides the plane into two regions. Every point on one side of the parabola satisfies the inequality $y < x^2 + 2x - 3$, and every point on the other side satisfies the inequality $y > x^2 + 2x - 3$. To determine which side is which, we test a point that is not on the parabola, say $(0, 0)$. Because

$$0 < 0^2 + 2 \cdot 0 - 3$$

is false, the region not containing the origin is shaded, as in Fig. 13.29.

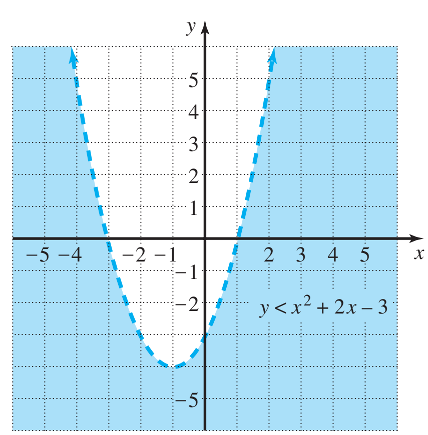

FIGURE 13.29

EXAMPLE 2 **A second-degree inequality**

Graph the inequality $x^2 + y^2 < 9$.

Solution

The graph of $x^2 + y^2 = 9$ is a circle of radius 3 centered at the origin. The circle divides the plane into two regions. Every point in one region satisfies $x^2 + y^2 < 9$, and every point in the other region satisfies $x^2 + y^2 > 9$. To identify the regions, we pick a point and test it. Select (0, 0). The inequality

$$0^2 + 0^2 < 9$$

is true. Because (0, 0) is inside the circle, all points inside the circle satisfy the inequality $x^2 + y^2 < 9$, as shown in Fig. 13.30. The points outside the circle satisfy the inequality $x^2 + y^2 > 9$.

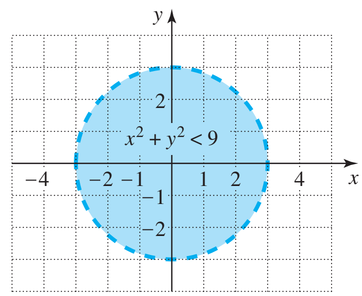

FIGURE 13.30

EXAMPLE 3 **A second-degree inequality**

Graph the inequality $\frac{x^2}{4} - \frac{y^2}{9} > 1$.

Solution

First graph the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$. Because the hyperbola shown in Fig. 13.31 divides the plane into three regions, we select a test point in each region and check to see whether it satisfies the inequality. Testing the points $(-3, 0)$, (0, 0), and (3, 0) gives us the inequalities

$$\frac{(-3)^2}{4} - \frac{0^2}{9} > 1, \qquad \frac{0^2}{4} - \frac{0^2}{9} > 1, \qquad \text{and} \qquad \frac{3^2}{4} - \frac{0^2}{9} > 1.$$

Because only the first and third inequalities are correct, we shade only the regions containing (3, 0) and $(-3, 0)$, as shown in Fig. 13.31.

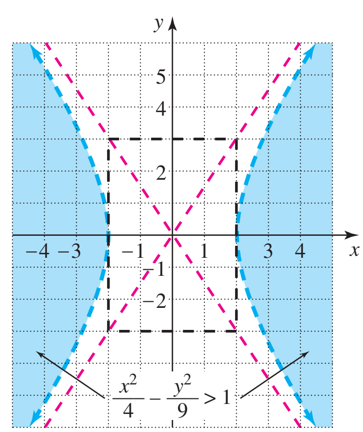

FIGURE 13.31

Systems of Inequalities

A point is in the solution set to a system of inequalities if it satisfies all inequalities of the system. We graph a system of inequalities by first determining the graph of each inequality and then finding the intersection of the graphs.

EXAMPLE 4 **Systems of second-degree inequalities**

Graph the system of inequalities:

$$\frac{y^2}{4} - \frac{x^2}{9} > 1$$
$$\frac{x^2}{9} + \frac{y^2}{16} < 1$$

Solution

Figure 13.32(a) shows the graph of the first inequality. In Fig. 13.32(b) we have the graph of the second inequality. In Fig. 13.32(c) we have shaded only the points that satisfy both inequalities. Figure 13.32(c) is the graph of the system.

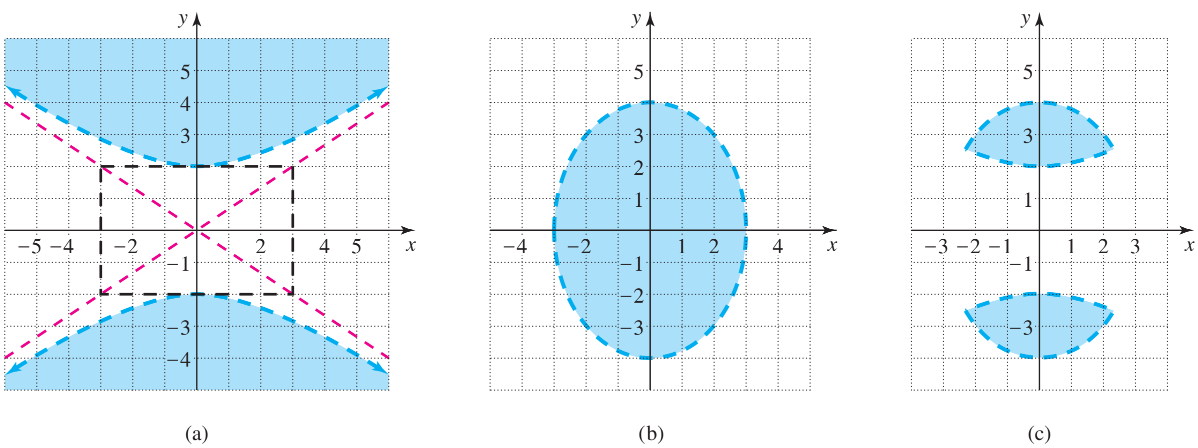

FIGURE 13.32

WARM-UPS

True or false? Explain your answer.

1. The graph of $x^2 + y = 4$ is a circle of radius 2.
2. The graph of $x^2 + 9y^2 = 9$ is an ellipse.
3. The graph of $y^2 = x^2 + 1$ is a hyperbola.
4. The point (0, 0) satisfies the inequality $2x^2 - y < 3$.
5. The graph of the inequality $y > x^2 - 3x + 2$ contains the origin.
6. The origin should be used as a test point for graphing $x^2 > y$.
7. The solution set to $x^2 + 3x + y^2 + 8y + 3 < 0$ includes the origin.
8. The graph of $x^2 + y^2 < 4$ is the region inside a circle of radius 2.
9. The point (0, 4) satisfies $x^2 - y^2 < 1$ and $y > x^2 - 2x + 3$.
10. The point (0, 0) satisfies $x^2 + y^2 < 1$ and $y < x^2 + 1$.

13.5 EXERCISES

Graph each inequality. See Examples 1–3.

1. $y > x^2$

2. $y \leq x^2 + 1$

3. $y < x^2 - x$

4. $y > x^2 + x$

5. $y > x^2 - x - 2$

6. $y < x^2 + x - 6$

7. $x^2 + y^2 \leq 9$

8. $x^2 + y^2 > 16$

9. $x^2 + 4y^2 > 4$

10. $4x^2 + y^2 \leq 4$

11. $4x^2 - 9y^2 < 36$

12. $25x^2 - 4y^2 > 100$

13. $(x - 2)^2 + (y - 3)^2 < 4$

14. $(x + 1)^2 + (y - 2)^2 > 1$

15. $x^2 + y^2 > 1$

16. $x^2 + y^2 < 25$

17. $4x^2 - y^2 > 4$

18. $x^2 - 9y^2 \leq 9$

19. $y^2 - x^2 \le 1$

20. $x^2 - y^2 > 1$

21. $x > y$

22. $x < 2y - 1$

Graph the solution set to each system of inequalities. See Example 4.

23. $x^2 + y^2 < 9$
$y > x$

24. $x^2 + y^2 > 1$
$x > y$

25. $x^2 - y^2 > 1$
$x^2 + y^2 < 4$

26. $y^2 - x^2 < 1$
$x^2 + y^2 > 9$

27. $y > x^2 + x$
$y < 5$

28. $y > x^2 + x - 6$
$y < x + 3$

29. $y \ge x + 2$
$y \le 2 - x$

30. $y \ge 2x - 3$
$y \le 3 - 2x$

31. $4x^2 - y^2 < 4$
$x^2 + 4y^2 > 4$

32. $x^2 - 4y^2 < 4$
$x^2 + 4y^2 > 4$

33. $x - y < 0$
$y + x^2 < 1$

34. $y + 1 > x^2$
$x + y < 2$

35. $y < 5x - x^2$
$x^2 + y^2 < 9$

36. $y < x^2 + 5x$
$x^2 + y^2 < 16$

37. $y \geq 3$
$x \leq 1$

38. $x > -3$
$y < 2$

39. $4y^2 - 9x^2 < 36$
$x^2 + y^2 < 16$

40. $25y^2 - 16x^2 < 400$
$x^2 + y^2 > 4$

41. $y < x^2$
$x^2 + y^2 < 1$

42. $y > x^2$
$4x^2 + y^2 < 4$

Solve the problem.

43. ***Buried treasure.*** An old pirate on his deathbed gave the following description of where he had buried some treasure on a deserted island: “Starting at the large palm tree, I walked to the north and then to the east, and there I buried the treasure. I walked at least 50 paces to get to that spot, but I was not more than 50 paces, as the crow flies, from the large palm tree. I am sure that I walked farther in the northerly direction than in the easterly direction.” With the large palm tree at the origin and the positive y-axis pointing to the north, graph the possible locations of the treasure.

FIGURE FOR EXERCISE 43

GRAPHING CALCULATOR EXERCISES

44. Use graphs to find an ordered pair that is in the solution set to the system of inequalities:

$$y > x^2 - 2x + 1$$
$$y < -1.1(x - 4)^2 + 5$$

Verify that your answer satisfies both inequalities.

45. Use graphs to find the solution set to the system of inequalities:

$$y > 2x^2 - 3x + 1$$
$$y < -2x^2 - 8x - 1$$