

14.1 SEQUENCES

In this section

- Definition
- Finding a Formula for the n th Term

The word “sequence” is a familiar word. We may speak of a sequence of events or say that something is out of sequence. In this section we give the mathematical definition of a sequence.

Definition

In mathematics we think of a sequence as a list of numbers. Each number in the sequence is called a **term** of the sequence. There is a first term, a second term, a third term, and so on. For example, the daily high temperature readings in Minot, North Dakota, for the first 10 days in January can be thought of as a finite sequence with 10 terms:

$$-9, -2, 8, -11, 0, 6, 14, 1, -5, -11$$

The set of all positive even integers,

$$2, 4, 6, 8, 10, 12, 14, \dots,$$

can be thought of as an infinite sequence.

To give a precise definition of sequence, we use the terminology of functions. The list of numbers is the range of the function.

Sequence

A **finite sequence** is a function whose domain is the set of positive integers less than or equal to some fixed positive integer. An **infinite sequence** is a function whose domain is the set of all positive integers.

When the domain is apparent, we will refer to either a finite sequence or an infinite sequence simply as a sequence. For the independent variable of the function we will usually use n (for natural number) rather than x . For the dependent variable we write a_n (read “ a sub n ”) rather than y . We call a_n the **n th term**, or the **general term** of the sequence. Rather than use the $f(x)$ notation for functions, we will define sequences with formulas. When n is used as a variable, we will assume it represents natural numbers only.

EXAMPLE 1

Listing terms of a finite sequence

List all of the terms of each finite sequence.

a) $a_n = n^2$ for $1 \leq n \leq 5$

b) $a_n = \frac{1}{n+2}$ for $1 \leq n \leq 4$

Solution

a) Using the natural numbers from 1 through 5 in $a_n = n^2$, we get

$$a_1 = 1^2 = 1,$$

$$a_2 = 2^2 = 4,$$

$$a_3 = 3^2 = 9,$$

$$a_4 = 4^2 = 16,$$

and

$$a_5 = 5^2 = 25.$$

calculator

close-up

Because a sequence is a function whose domain is the set of positive integers, we can define the sequence with the Y= key and make a list of the terms.

X	Y1
1	1
2	4
3	9
4	16
5	25
6	36
7	49

Y1 X2

The five terms of this sequence are 1, 4, 9, 16, and 25. We often refer to the listing of the terms of the sequence as the sequence.

b) Using the natural numbers from 1 through 4 in $a_n = \frac{1}{n+2}$, we get the terms

$$a_1 = \frac{1}{1+2} = \frac{1}{3},$$

$$a_2 = \frac{1}{2+2} = \frac{1}{4},$$

$$a_3 = \frac{1}{5},$$

and

$$a_4 = \frac{1}{6}.$$

The four terms of the sequence are $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$. ■

EXAMPLE 2

Listing terms of an infinite sequence

List the first three terms of the infinite sequence whose n th term is

$$a_n = \frac{(-1)^n}{2^{n+1}}.$$

Solution

Using the natural numbers 1, 2, and 3 in the formula for the n th term yields

$$a_1 = \frac{(-1)^1}{2^{1+1}} = -\frac{1}{4}, \quad a_2 = \frac{(-1)^2}{2^{2+1}} = \frac{1}{8}, \quad \text{and} \quad a_3 = \frac{(-1)^3}{2^{3+1}} = -\frac{1}{16}.$$

We write the sequence as follows:

$$-\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots$$

Finding a Formula for the n th Term

We often know the terms of a sequence and want to write a formula that will produce those terms. To write a formula for the n th term of a sequence, examine the terms and look for a pattern. Each term is a function of the term number. The first term corresponds to $n = 1$, the second term corresponds to $n = 2$, and so on.

EXAMPLE 3

A familiar sequence

Write the general term for the infinite sequence

$$3, 5, 7, 9, 11, \dots$$

Solution

The even numbers are all multiples of 2 and can be represented as $2n$. Because each odd number is 1 more than an even number, a formula for the n th term might be

$$a_n = 2n + 1.$$

calculator

close-up

Some calculators have a sequence feature that allows you to specify the formula and which terms to evaluate. We can even get the terms as fractions.

```
seq((-1)^n/2^(n+1),n,1,3)Frac
(-1/4 1/8 -1/16)
```

helpful hint

Finding a formula for a sequence could be extremely difficult. For example, there is no known formula that will produce the sequence of prime numbers:

2, 3, 5, 7, 11, 13, 17, 19, ...

To be sure, we write out a few terms using the formula:

$$a_1 = 2(1) + 1 = 3$$

$$a_2 = 2(2) + 1 = 5$$

$$a_3 = 2(3) + 1 = 7$$

So the general term is $a_n = 2n + 1$. ■

CAUTION There can be more than one formula that produces the given terms of a sequence. For example, the sequence

$$1, 2, 4, \dots$$

could have n th term $a_n = 2^{n-1}$ or $a_n = \frac{1}{2}n^2 - \frac{1}{2}n + 1$. The first three terms for both of these sequences are identical, but their fourth terms are different.

EXAMPLE 4 A sequence with alternating signs

Write the general term for the infinite sequence

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \dots$$

Solution

To obtain the alternating signs, we use powers of -1 . Because any even power of -1 is positive and any odd power of -1 is negative, we use $(-1)^{n+1}$. The denominators are the squares of the positive integers. So the n th term of this infinite sequence is given by the formula

$$a_n = \frac{(-1)^{n+1}}{n^2}.$$

Check this sequence by using this formula to find the first four terms. ■

In the next example we use a sequence to model a physical situation.

EXAMPLE 5 The bouncing ball

Suppose a ball always rebounds $\frac{2}{3}$ of the height from which it falls and the ball is dropped from a height of 6 feet. Write a sequence whose terms are the heights from which the ball falls. What is a formula for the n th term of this sequence?

Solution

On the first fall the ball travels 6 feet (ft), as shown in Fig. 14.1. On the second fall it travels $\frac{2}{3}$ of 6, or 4 ft. On the third fall it travels $\frac{2}{3}$ of 4, or $\frac{8}{3}$ ft, and so on. We write the sequence as follows:

$$6, 4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \dots$$

The n th term can be written by using powers of $\frac{2}{3}$:

$$a_n = 6\left(\frac{2}{3}\right)^{n-1}$$
■

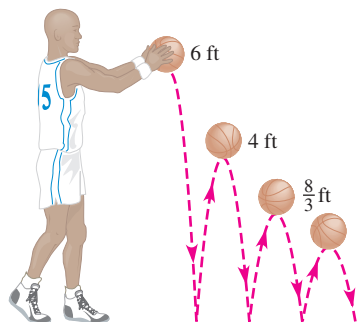


FIGURE 14.1

MATH AT WORK

$$x^2 + |x+1|^2 = 52$$

Most of us find an upholstered chair, sink into it, and remark on the comfort. Before design consultant Audrey Jordan sits down, she often looks at the fabric to observe the color and texture—and especially to see whether the fabric is one of her original designs. Fabric design is more than just an idea that is printed on a piece of cloth. Consideration must be given to the end product, which could be anything from a handbag to a large sofa. Colors and themes must be chosen with both current trends and styles in mind. Sometimes a design will be an overall or nondirectional pattern, such as polka dots, which can be cut randomly. More often, it will have a specific theme, such as fruit, which can be cut and sewn in only one direction.



**FABRIC
DESIGNER**

For all products one of the main considerations is the vertical repeat. A good portion of textile machinery is standardized for vertical repeats of 27 inches or fractions thereof. For example, the vertical repeat could be every $13\frac{1}{2}$ inches or every 9 inches. Even though the horizontal repeat can vary, Ms. Jordan must consider both the horizontal and vertical repeats for a particular end product.

In Exercise 45 of this section you will find the standard vertical repeats for a textile machine.

WARM-UPS

True or false? Explain your answer.

1. The n th term of the sequence 2, 4, 6, 8, 10, . . . is $a_n = 2n$.
2. The n th term of the sequence 1, 3, 5, 7, 9, . . . is $a_n = 2n - 1$.
3. A sequence is a function.
4. The domain of a finite sequence is the set of positive integers.
5. The n th term of $-1, 4, -9, 16, -25, \dots$ is $a_n = (-1)^{n+1}n^2$.
6. For the infinite sequence $b_n = \frac{1}{n}$, the independent variable is $\frac{1}{n}$.
7. For the sequence $c_n = n^3$, the dependent variable is c_n .
8. The sixth term of the sequence $a_n = (-1)^{n+1}2^n$ is 64.
9. The symbol a_n is used for the dependent variable of a sequence.
10. The tenth term of the sequence 2, 4, 8, 16, 32, 64, 128, . . . is 1024.

14.1 EXERCISES

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a sequence?
2. What is a term of a sequence?
3. What is a finite sequence?
4. What is an infinite sequence?

List all terms of each finite sequence. See Example 1.

5. $a_n = n^2$ for $1 \leq n \leq 8$
6. $a_n = -n^2$ for $1 \leq n \leq 4$
7. $b_n = \frac{(-1)^n}{n}$ for $1 \leq n \leq 10$
8. $b_n = \frac{(-1)^{n+1}}{n}$ for $1 \leq n \leq 6$
9. $c_n = (-2)^{n-1}$ for $1 \leq n \leq 5$
10. $c_n = (-3)^{n-2}$ for $1 \leq n \leq 5$
11. $a_n = 2^{-n}$ for $1 \leq n \leq 6$
12. $a_n = 2^{-n+2}$ for $1 \leq n \leq 5$
13. $b_n = 2n - 3$ for $1 \leq n \leq 7$
14. $b_n = 2n + 6$ for $1 \leq n \leq 7$
15. $c_n = n^{-1/2}$ for $1 \leq n \leq 5$
16. $c_n = n^{1/2}2^{-n}$ for $1 \leq n \leq 4$

Write the first four terms of the infinite sequence whose n th term is given. See Example 2.

17. $a_n = \frac{1}{n^2 + n}$
18. $b_n = \frac{1}{(n + 1)(n + 2)}$
19. $b_n = \frac{1}{2n - 5}$
20. $a_n = \frac{4}{2n + 5}$
21. $c_n = (-1)^n(n - 2)^2$
22. $c_n = (-1)^n(2n - 1)^2$
23. $a_n = \frac{(-1)^{2n}}{n^2}$
24. $a_n = (-1)^{2n+1}2^{n-1}$

Write a formula for the general term of each infinite sequence. See Examples 3 and 4.

25. 1, 3, 5, 7, 9, ...
26. 5, 7, 9, 11, 13, ...
27. 1, -1, 1, -1, ...
28. -1, 1, -1, 1, ...
29. 0, 2, 4, 6, 8, ...
30. 4, 6, 8, 10, 12, ...
31. 3, 6, 9, 12, ...
32. 4, 8, 12, 16, ...
33. 4, 7, 10, 13, ...
34. 3, 7, 11, 15, ...
35. -1, 2, -4, 8, -16, ...
36. 1, -3, 9, -27, ...
37. 0, 1, 4, 9, 16, ...
38. 0, 1, 8, 27, 64, ...

Solve each problem. See Example 5.

39. **Football penalties.** A football is on the 8-yard line, and five penalties in a row are given that move the ball half the distance to the (closest) goal. Write a sequence of five terms that specify the location of the ball after each penalty.
40. **Infestation.** Leona planted 9 acres of soybeans, but by the end of each week, insects had destroyed one-third of the acreage that was healthy at the beginning of the week. How many acres does she have left after 6 weeks?
41. **Constant rate of increase.** The MSRP for the 1999 Ford F-250 Lariat 4WD Super Duty Super Cab was \$32,535 (Edmund's New Car Prices, www.edmunds.com). Suppose the price of this truck increases by 5% each year. Find the prices to the nearest dollar for the 2000 through 2005 models.

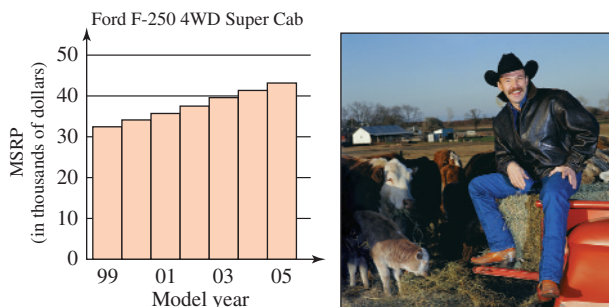


FIGURE FOR EXERCISE 41

42. **Constant increase.** The MSRP for a new 1999 Mercury Cougar was \$21,455 (Edmund's New Car Prices, www.edmunds.com). Suppose the price of this car increases by \$1000 each year. Find the prices of the 2000 through 2005 models.
43. **Economic impact.** To assess the economic impact of a factory on a community, economists consider the annual amount the factory spends in the community, then the portion of the money that is respent in the community, then the portion of the respent money that is respent in the community, and so on. Suppose a garment manufacturer spends \$1 million annually in its community and 80% of all money received in the community is respent in the community. Find the first four terms of the economic impact sequence.

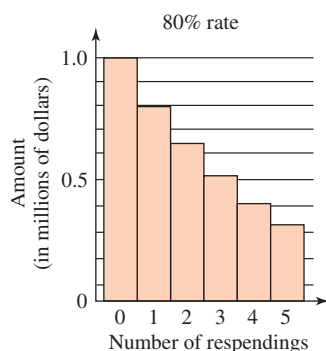


FIGURE FOR EXERCISE 43



44. **Less impact.** The rate at which money is respent in a community varies from community to community. Find the first four terms of the economic impact sequence for the manufacturer in Exercise 43, assuming only 50% of money received in the community is respent in the community.
45. **Fabric design.** A fabric designer must take into account the capability of textile machines to produce material with vertical repeats. A textile machine can be set up for a vertical repeat every $\frac{27}{n}$ inches (in.), where n is a natural number. Write the first five terms of the sequence $a_n = \frac{27}{n}$, which gives the possible vertical repeats for a textile machine.



46. **Musical tones.** The note middle C on a piano is tuned so that the string vibrates at 262 cycles per second, or 262 Hertz (Hz). The C note one octave higher is tuned to 524 Hz. The tuning for the 11 notes in between using the method called *equal temperament* is determined by the sequence $a_n = 262 \cdot 2^{n/12}$. Find the tuning for the 11 notes in between.

GETTING MORE INVOLVED



47. **Discussion.** Everyone has two (biological) parents, four grandparents, eight great-grandparents, 16 great-great-grandparents, and so on. If we put the word “great” in front of the word “grandparents” 35 times, then how many of this type of relative do you have? Is this more or less than the present population of the earth? Give reasons for your answers.



48. **Discussion.** If you deposit 1 cent into your piggy bank on September 1 and each day thereafter deposit twice as much as on the previous day, then how much will you be depositing on September 30? The total amount deposited for the month can be found without adding up all 30 deposits. Look at how the amount on deposit is increasing each day and see whether you can find the total for the month. Give reasons for your answers.



49. **Cooperative learning.** Working in groups, have someone in each group make up a formula for a_n , the n th term of a sequence, but do not show it to the other group members. Write the terms of the sequence on a piece of paper one at a time. After each term is given, ask whether anyone knows the next term. When the group can correctly give the next term, ask for a formula for the n th term.



50. **Exploration.** Find a real-life sequence in which all of the terms are the same. Find one in which each term after the first is one larger than the previous term. Find out what the sequence of fines is on your campus for your first, second, third, and fourth parking ticket.



51. **Exploration.** Consider the sequence whose n th term is $a_n = (0.999)^n$.

a) Calculate a_{100} , a_{1000} , and $a_{10,000}$.

b) What happens to a_n as n gets larger and larger?