14.2 SERIES

In this section

- Summation Notation
- Series
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If you make a sequence of bank deposits, then you might be interested in the total value of the terms of the sequence. Of course, if the sequence has only a few terms, you can simply add them. In Sections 14.3 and 14.4 we will develop formulas that give the sum of the terms for certain finite and infinite sequences. In this section you will first learn a notation for expressing the sum of the terms of a sequence.

Summation Notation

To describe the sum of the terms of a sequence, we use **summation notation.** The Greek letter Σ (sigma) is used to indicate sums. For example, the sum of the first five terms of the sequence $a_n = n^2$ is written as

$$\sum_{n=1}^{5} n^2.$$

You can read this notation as "the sum of n^2 for n between 1 and 5, inclusive." To find the sum, we let n take the values 1 through 5 in the expression n^2:

$$\begin{aligned}\sum_{n=1}^{5} n^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2\\ &= 1 + 4 + 9 + 16 + 25\\ &= 55\end{aligned}$$

In this context the letter n is the **index of summation.** Other letters may also be used. For example, the expressions

$$\sum_{n=1}^{5} n^2, \quad \sum_{j=1}^{5} j^2, \quad \text{and} \quad \sum_{i=1}^{5} i^2$$

all have the same value. Note that i is used as a variable here and not as an imaginary number.

EXAMPLE 1 **Evaluating a sum in summation notation**

Find the value of the expression

$$\sum_{i=1}^{3} (-1)^i(2i + 1).$$

Solution

Replace i by 1, 2, and 3, and then add the results:

$$\begin{aligned}\sum_{i=1}^{3} (-1)^i(2i + 1) &= (-1)^1[2(1) + 1] + (-1)^2[2(2) + 1] + (-1)^3[2(3) + 1]\\ &= -3 + 5 - 7\\ &= -5\end{aligned}$$

Series

The sum of the terms of the sequence 1, 4, 9, 16, 25 is written as

$$1 + 4 + 9 + 16 + 25.$$

This expression is called a *series*. It indicates that we are to add the terms of the given sequence. The sum, 55, is the sum of the series.

Series

The indicated sum of the terms of a sequence is called a **series.**

Just as a sequence may be finite or infinite, a series may be finite or infinite. In this section we discuss finite series only. In Section 14.4 we will discuss one type of infinite series.

Summation notation is a convenient notation for writing a series.

EXAMPLE 2

Converting to summation notation

Write the series in summation notation:

$$2 + 4 + 6 + 8 + 10 + 12 + 14$$

Solution

The general term for the sequence of positive even integers is $2n$. If we let n take the values from 1 through 7, then $2n$ ranges from 2 through 14. So

$$2 + 4 + 6 + 8 + 10 + 12 + 14 = \sum_{n=1}^{7} 2n.$$

EXAMPLE 3

helpful hint

A series is called an *indicated sum* because the addition is indicated but not actually being performed. The sum of a series is the real number obtained by actually performing the indicated addition.

Converting to summation notation

Write the series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots + \frac{1}{50}$$

in summation notation.

Solution

For this series we let n be 2 through 50. The expression $(-1)^n$ produces alternating signs. The series is written as

$$\sum_{n=2}^{50} \frac{(-1)^n}{n}.$$

Changing the Index

In Example 3 we saw the index go from 2 through 50, but this is arbitrary. A series can be written with the index starting at any given number.

EXAMPLE 4

Changing the index

Rewrite the series

$$\sum_{i=1}^{6} \frac{(-1)^i}{i^2}$$

with an index j, where j starts at 0.

Solution

Because i starts at 1 and j starts at 0, we have $i = j + 1$. Because i ranges from 1 through 6 and $i = j + 1$, j must range from 0 through 5. Now replace i by $j + 1$ in the summation notation:

$$\sum_{j=0}^{5} \frac{(-1)^{j+1}}{(j + 1)^2}$$

Check that these two series have exactly the same six terms.

WARM-UPS

True or false? Explain your answer.

1. A series is the indicated sum of the terms of a sequence.
2. The sum of a series can never be negative.
3. There are eight terms in the series $\sum_{i=2}^{10} i^3$.
4. The series $\sum_{i=1}^{9} (-1)^i i^2$ and $\sum_{j=0}^{8} (-1)^j (j + 1)^2$ have the same sum.
5. The ninth term of the series $\sum_{i=1}^{100} \frac{(-1)^i}{(i + 1)(i + 2)}$ is $\frac{1}{110}$.
6. $\sum_{i=1}^{2} (-1)^i 2^i = 2$
7. $\sum_{i=1}^{5} 3i = 3\left(\sum_{i=1}^{5} i\right)$
8. $\sum_{i=1}^{5} 4 = 20$
9. $\sum_{i=1}^{5} 2i + \sum_{i=1}^{5} 7i = \sum_{i=1}^{5} 9i$
10. $\sum_{i=1}^{3} (2i + 1) = \left(\sum_{i=1}^{3} 2i\right) + 1$

14.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is summation notation?
2. What is the index of summation?
3. What is a series?
4. What is a finite series?

Find the sum of each series. See Example 1.

5. $\sum_{i=1}^{4} i^2$
6. $\sum_{j=0}^{3} (j + 1)^2$
7. $\sum_{j=0}^{5} (2j - 1)$
8. $\sum_{i=1}^{6} (2i - 3)$
9. $\sum_{i=1}^{5} 2^{-i}$
10. $\sum_{i=1}^{5} (-2)^{-i}$
11. $\sum_{i=1}^{10} 5i^0$
12. $\sum_{j=1}^{20} 3$
13. $\sum_{i=1}^{3} (i - 3)(i + 1)$
14. $\sum_{i=0}^{5} i(i - 1)(i - 2)(i - 3)$
15. $\sum_{j=1}^{10} (-1)^j$
16. $\sum_{j=1}^{11} (-1)^j$

Write each series in summation notation. Use the index i, and let i begin at 1 in each summation. See Examples 2 and 3.

17. $1 + 2 + 3 + 4 + 5 + 6$
18. $2 + 4 + 6 + 8 + 10$
19. $-1 + 3 - 5 + 7 - 9 + 11$
20. $1 - 3 + 5 - 7 + 9$
21. $1 + 4 + 9 + 16 + 25 + 36$
22. $1 + 8 + 27 + 64 + 125$
23. $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$
24. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

25. $\ln(2) + \ln(3) + \ln(4)$

26. $e^1 + e^2 + e^3 + e^4$

27. $a_1 + a_2 + a_3 + a_4$

28. $a^2 + a^3 + a^4 + a^5$

29. $x_3 + x_4 + x_5 + \cdots + x_{50}$

30. $y_1 + y_2 + y_3 + \cdots + y_{30}$

31. $w_1 + w_2 + w_3 + \cdots + w_n$

32. $m_1 + m_2 + m_3 + \cdots + m_k$

Complete the rewriting of each series using the new index as indicated. See Example 4.

33. $\sum_{i=1}^{5} i^2 = \sum_{j=0}$

34. $\sum_{i=1}^{6} i^3 = \sum_{j=0}$

35. $\sum_{i=0}^{12} (2i - 1) = \sum_{j=1}$

36. $\sum_{i=1}^{3} (3i + 2) = \sum_{j=0}$

37. $\sum_{i=4}^{8} \frac{1}{i} = \sum_{j=1}$

38. $\sum_{i=5}^{10} 2^{-i} = \sum_{j=1}$

39. $\sum_{i=1}^{4} x^{2i+3} = \sum_{j=0}$

40. $\sum_{i=0}^{2} x^{3-2i} = \sum_{j=1}$

41. $\sum_{i=1}^{n} x^i = \sum_{j=0}$

42. $\sum_{i=0}^{n} x^{-i} = \sum_{j=1}$

Write out the terms of each series.

43. $\sum_{i=1}^{6} x^i$

44. $\sum_{i=1}^{5} (-1)^i x^{i-1}$

45. $\sum_{j=0}^{3} (-1)^j x_j$

46. $\sum_{j=1}^{5} \frac{1}{x_j}$

47. $\sum_{i=1}^{3} i x^i$

48. $\sum_{i=1}^{5} \frac{x}{i}$

A series can be used to model the situation in each of the following problems.

49. ***Leap frog.*** A frog with a vision problem is 1 yard away from a dead cricket. He spots the cricket and jumps halfway to the cricket. After the frog realizes that he has not reached the cricket, he again jumps halfway to the cricket. Write a series in summation notation to describe how far the frog has moved after nine such jumps.

50. ***Compound interest.*** Cleo deposited \$1000 at the beginning of each year for 5 years into an account paying 10% interest compounded annually. Write a series using summation notation to describe how much she has in the account at the end of the fifth year. Note that the first \$1000 will receive interest for 5 years, the second \$1000 will receive interest for 4 years, and so on.

51. ***Total economic impact.*** In Exercise 43 of Section 14.1 we described a factory that spends \$1 million annually in a community in which 80% of all money received in the community is respent in the community. Use summation notation to write the sum of the first four terms of the economic impact sequence for the factory.

52. ***Total spending.*** Suppose you earn \$1 on January 1, \$2 on January 2, \$3 on January 3, and so on. Use summation notation to write the sum of your earnings for the entire month of January.

GETTING MORE INVOLVED

53. ***Discussion.*** What is the difference between a sequence and a series?

54. ***Discussion.*** For what values of n is $\sum_{i=1}^{n} \frac{1}{i} > 4$?