Appendix G

PROBABILITY

Many future events cannot be known with 100% certainty. For example, if you were to ask a meteorologist "Will it rain today?," she might respond "There is an 80% chance of rain." This means that according to historical data, today's meteorological conditions have resulted in rain four times out of five.

Much of our understanding of probability has resulted from studying games of chance; for this reason, we will highlight them in our discussion of probability.

G.1 Probability

Figure G.1(a) shows a wheel that spins about a central axis. It is divided into three equal sections, each of which is numbered. At the top is a stationary indicator. Initially, the wheel is spun at a high speed. Eventually, because of friction, it stops and the indicator points to one of the sections. Provided the wheel is *fair*, meaning it is perfectly balanced and the three sections have identical shapes, there is an equal chance it could stop on any one of the numbers. For example, in 999 spins, each number would be selected about 333 times.

A single spinning of the wheel can be considered an *event*. Figure G.1(b) shows an *event tree* in which each of the three possible outcomes is shown as a branch of the tree. Assuming the wheel is fair, p (the *probability of occurrence*) is h (the number of ways a particular occurence can happen during an event) divided by n (the number of equally likely ways the event could occur).

$$p = \frac{h}{n} = \frac{1}{3} \qquad \text{(G-1)}$$

The *probability of nonoccurrence* q is simply

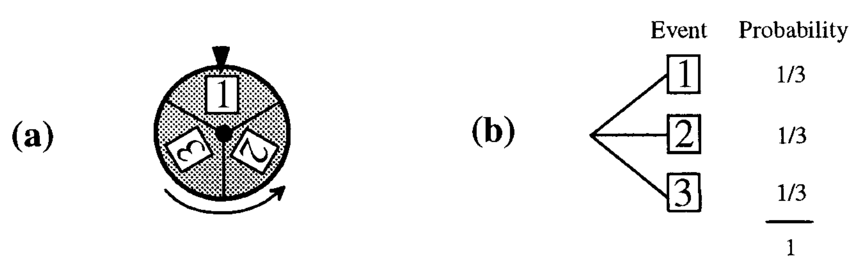

Figure G.1 Game of chance involving a spinning wheel.

$$q = \frac{n-h}{n} = 1 - \frac{h}{n} = 1 - p = 1 - \frac{1}{3} = \frac{2}{3} \quad \text{(G-2)}$$

Clearly,

$$p + q = 1 \quad \text{(G-3)}$$

For this game, the *odds of occurrence* are expressed as the following ratio:

$$p : q = \frac{1}{3} : \frac{2}{3} = 1 : 2 \quad \text{(G-4)}$$

The *odds of nonoccurrence* are expressed as follows:

$$q : p = \frac{2}{3} : \frac{1}{3} = 2 : 1 \quad \text{(G-5)}$$

G.2 Independent Events

In an *independent event*, the outcome of one event does not affect other events. For example, suppose you spin the wheel multiple times. If the chance of getting a 1 on the second spin is the same as it was on the first spin, we would say that each spin was an independent event.

In a *dependent event*, the outcome of one event does affect other events. For example, suppose that the rules of the game state that if you obtain a 1 on the first spin, then you must remove the 1 and replace it with a 0. Clearly, your chance of getting a 1 on the second spin depends on what occurred during the first spin.

Figure G.2(a) shows that the wheel is spun twice. In this case, each spin is assumed to be an independent event. The event tree (Figure G.2(b)) clearly shows that the probability of obtaining a 1 on the first spin and a 1 on the second spin is 1/9. This probability is easily calculated as follows:

$$p = p_1 \times p_2 = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \quad \text{(G-6)}$$

In general, the probability p of obtaining a particular outcome from n independent events is

$$p = p_1 \times p_2 \times \cdots \times p_n = \prod_{i=1}^{n} p_i \quad \text{(G-7)}$$

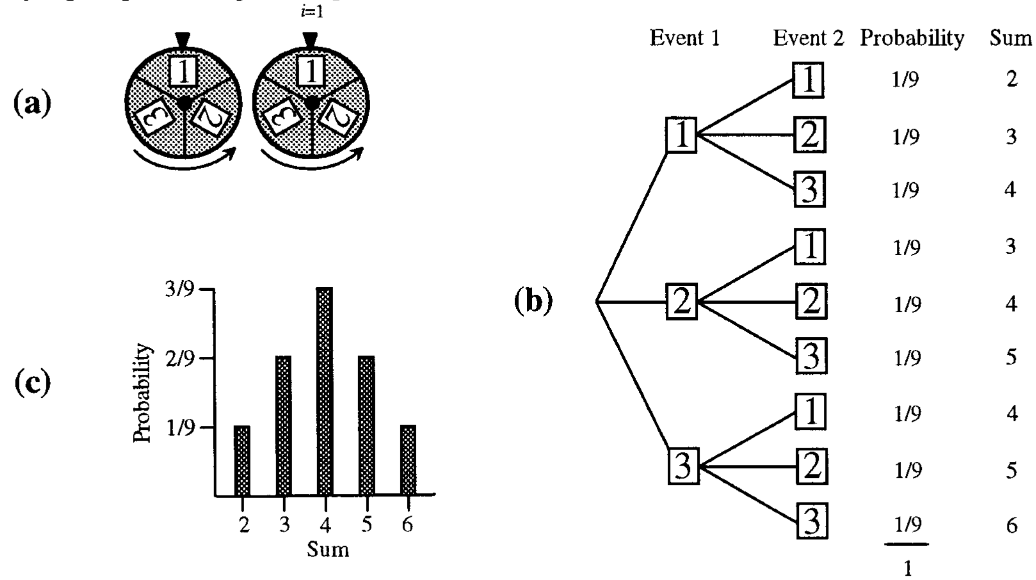

Figure G.2 Results from spinning the wheel twice.

Example G.1

Problem Statement: Figure G.3(a) shows a nuclear reactor with fuel rods submerged in a pool of water. The fuel rods contain uranium-235, which fissions into lighter elements and releases thermal energy. When a uranium-235 atom fissions spontaneously, it releases approximately 2 neutrons, which can hit other uranium-235 atoms causing them to fission. There is the potential for a runaway nuclear reaction (*meltdown*) because one fission event can cause two fission events, which can cause four fission events, which can cause eight fission events, etc. To prevent meltdown, control rods regulate the rate of the nuclear reaction. When the control rods are lowered into the pool of water, they absorb neutrons and inhibit the nuclear reaction. When the control rods are raised from the pool of water, they no longer absorb neutrons and the nuclear reaction is able to proceed uninhibited.

A safety backup system consists of a cooling pump that can circulate the hot water in the pool through the cooling tower. If the control rods fail to lower into the pool, the cooling pump is turned on to remove thermal energy generated by the nuclear reaction, thereby preventing meltdown.

For meltdown to occur, both the control rods and cooling pump must fail. Acme makes nuclear equipment. Suppose that its control rod system fails once during 20 years of operation and that its cooling pump fails once during 10 years of operation.

a. For a single nuclear power plant, if an Acme control rod system and cooling pump were installed, how long must it operate for a meltdown to probably occur?

b. If the Acme control rod system and cooling pump were installed in 100 plants, how long would it take for a meltdown to probably occur?

c. Would you recommend that Acme equipment be installed in nuclear power plants?

Solution:

Figure G.3(b) shows the event tree. For meltdown to occur, both the control rods and pump must fail. The probability of this occurence is

$$p = p_1 \times p_2 = \frac{1 \text{ failure}}{20 \text{ years} \cdot \text{Acme system}} \times \frac{1 \text{ failure}}{10 \text{ years} \cdot \text{Acme system}} = \frac{1 \text{ failure}}{200 \text{ years} \cdot \text{Acme system}}$$

a. For one plant

$$p = 1 \text{ plant} \times \frac{1 \text{ Acme system}}{\text{plant}} \times \frac{1 \text{ failure}}{200 \text{ years} \cdot \text{Acme system}} = \frac{1 \text{ failure}}{200 \text{ years}}$$

b. For 100 plants

$$p = 100 \text{ plants} \times \frac{1 \text{ Acme system}}{\text{plant}} \times \frac{1 \text{ failure}}{200 \text{ years} \cdot \text{Acme system}} = \frac{1 \text{ failure}}{2 \text{ years}}$$

c. Acme equipment is not reliable enough.

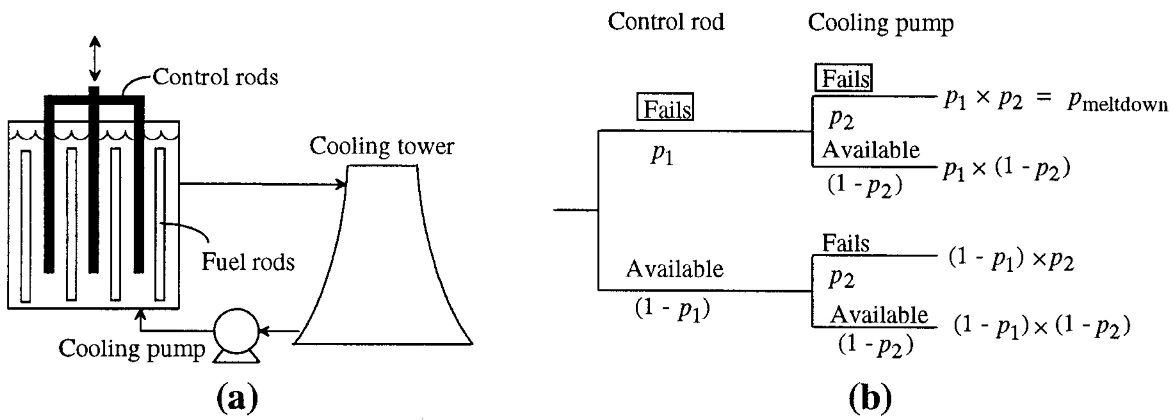

Figure G.3 Analysis of failure modes for a nuclear reactor.

G.3 Mutually Exclusive Events

Two or more events are *mutually exclusive* if the occurence of one precludes the other(s) from happening. For example, if a player obtains a 1 on the first spin of the wheel shown in Figure G.1, then the possibility of obtaining a 2 or 3 on the first spin is precluded. The wheel is designed so that only one result may be achieved on a given spin; therefore, spinning the wheel is a mutally exclusive event.

Suppose you were to ask "What is the probability of obtaining a 2 or 3 from a single spin of the wheel shown in Figure G.1?" In this case, you are asking about the probability of going down one or the other branches of the event tree. The probability of going down a single branch is 1/3, so the probability of going down one or the other branches is

$$p = p_1 + p_2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \qquad \text{(G-8)}$$

In general, the probability that any one of n mutually exclusive events will occur is

$$p = p_1 + p_2 + \cdots + p_n = \sum_{i=1}^{n} p_i \qquad \text{(G-9)}$$

G.4 Probability Distribution

The last column of Figure G.2(b) shows the sum of the numbers from two spins of the wheel. Obtaining a sum of 2 can be achieved from only one branch of the event tree (1 + 1), so the probability is

$$\text{Probability of obtaining a sum of 2} = \frac{1}{9}$$

Obtaining a sum of 3 can be achieved from two branches of the event tree (1 + 2 or 2 + 1). Each spinning event is mutually exclusive; therefore,

$$\text{Probability of obtaining a sum of 3} = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

The probability of obtaining other sums (4, 5, or 6) can be calculated by using similar logic. Figure G.2(c) shows the results of this calculation as a bar graph. The graph indicates the probability of obtaining each sum and is termed a *probability distribution.*

G.5 Expectation

An *expectation* E is simply the probability p of a particular event occurring times the result X if the event actually occurs. For example, if a game pays \$100 from obtaining a 1 on the first spin of the wheel shown in Figure G.1, then the expectation is

$$E = pX = \frac{1}{3}(\$100) = \$33.33$$

For n mutually exclusive events, the following equation is valid:

$$E = p_1X_1 + p_2X_2 + \cdots + p_nX_n = \sum_{i=1}^{n} p_iX_i \quad \text{(G-10)}$$

As an example, for a single spin of the wheel shown in Figure G.1, suppose the game pays \$100 for a 1, \$50 for a 2, and \$10 for a 3, then the expectation is

$$E = \frac{1}{3}(\$100) + \frac{1}{3}(\$50) + \frac{1}{3}(\$10) = \$53.33$$

G.6 Combinatorial Analysis

When assigning probabilities, it is necessary to know how many different ways various events can occur. If Event 1 can occur n_1 different ways and Event 2 can occur n_2 different ways, then the number of ways n both events can occur is

$$n = n_1 \times n_2 \quad \text{(G-11)}$$

In general, for k events

$$n = n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i \quad \text{(G-12)}$$

For example, Figure G.4 shows three wheels, two that have two numbers and one that has three numbers. The number of different ways the spins could result is

$$n = 2 \times 2 \times 3 = 12$$

G.6.1 Permutations

Figure G.5(a) shows a rotating barrel containing uniquely numbered balls. (*Note*: Such barrels are often used in lotteries.) If you were to remove r balls sequentially from a barrel containing n unique balls, the number of *permutations* P (i.e., the number of possible outcomes taking into account the order that the balls are removed) is

$$P = \frac{n!}{(n-r)!} \quad \text{(G-13)}$$

Figure G.5(b) shows the results for $n = 3$ and $r = 1, 2,$ or 3.

G.6.2 Combinations

If you were to remove r balls from a barrel containing n unique balls, the number of *combinations* C (i.e., the number of possible outcomes not taking into account the order that the balls are removed) is

$$C = \frac{n!}{r!(n-r)!} \quad \text{(G-14)}$$

Figure G.5(c) shows the results for $n = 3$ and $r = 1, 2,$ or 3.

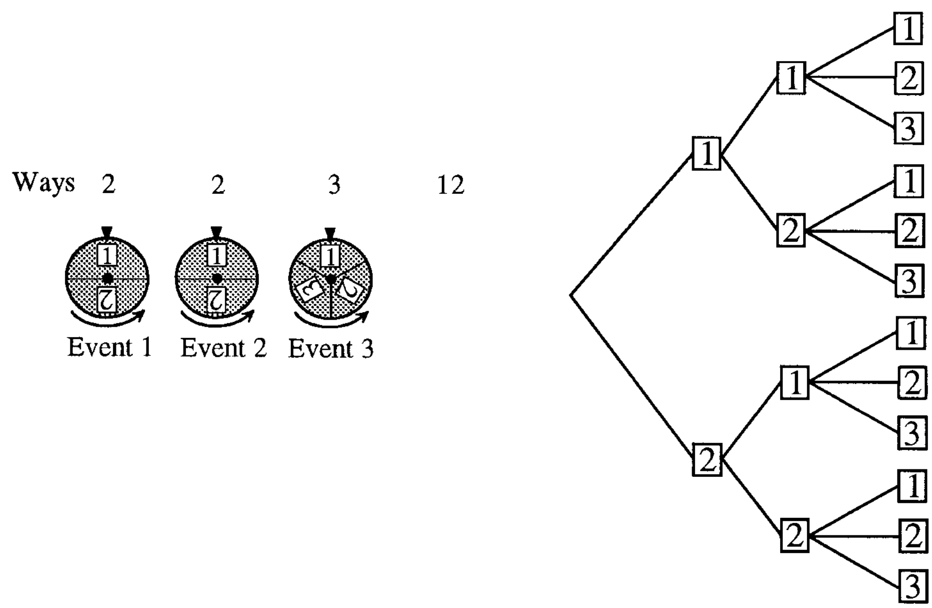

Figure G.4 Event tree for three wheels.

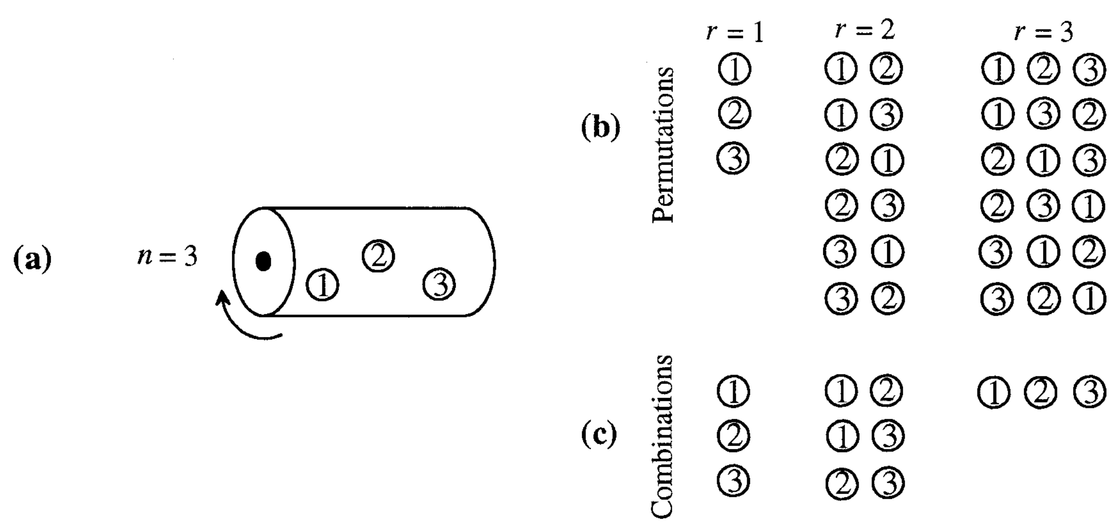

Figure G.5 Permutations and combinations of three numbered balls.

G.7 Summary

Probability is used to quantify the likelihood that an event may, or may not, occur in the future. An *event tree* describes the possible outcomes; each branch has an assigned probability that ranges from zero to one. If the event tree describes all possible outcomes, then the probabilities assigned to each branch sum to one.

Independent events are not affected by previous events, whereas *dependent events* are affected by previous events. Once a *mutully exclusive event* occurs, it precludes other events from occurring.

A *probability distribution* graphically describes the likelihood that particular events may occur in the future. An *expectation* is calculated as the product of the probability that a particular event occurs times the result if that event actually occurs.

Combinatorial analysis is used to calculate the number of possible ways that events can occur. A *permutation* is the number of possible ways that a subset of items can be arranged when selected from a larger set; the order of the subset matters. A *combination* is the number of possible subsets of items that can be selected from a larger set; the order of the subset does not matter.

Further Readings

Spiegel, M. R. *Statistics*. Schaum's Outline Series in Mathematics. New York: McGraw-Hill, 1961.

Problems

G.1 A *fair coin* has an equal probability of coming up heads or tails after being flipped.

a. What is the probability of getting four heads in a row?

b. On the fifth flip, what is the probabilty of getting heads?

G.2 A *die* is a small cube with one to six dots marked on each face. They are used in games, usually as a pair of dice. A *fair die* has equal probability of showing 1, 2, 3, 4, 5, or 6 dots on the upper face after being rolled.

a. After rolling one die, what is the probability of obtaining three dots on the upper face?

b. What are the odds of obtaining three dots on the upper face?

c. After rolling one die, what is the probability of obtaining either three dots or one dot on the upper face?

d. After rolling two dice, what is the probability of obtaining three dots on the upper face of each die?

e. After rolling two dice, what is the probability of obtaining a total of six dots on the upper face of the dice?

f. Prepare a probability distribution showing the probability of obtaining a total of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 dots on the upper faces of two rolled dice.

g. A gambling casino charges $100 to roll a pair of dice. It pays the following for each outcome:

2 or 12	$300
3 or 11	$150
4 or 10	$100
5 or 9	$80
6 or 8	$60
7	$50

For every 100 players, how much profit does the gambling casino make?

G.3 The state operates a lottery in which 20 balls – each uniquely numbered 1 through 20 – are placed in a barrel. At random, six balls are selected.

a. What is the probability that the balls will be drawn in the following order?

2-15-7-18-10-12

b. What is the probability that the following balls will be drawn, irrespective of order?

2-7-10-12-15-18

c. The state operates a "Pick 6" lottery where a player wins if he picks the six numbers drawn from the drum, regardless of the order that they were drawn. One million Pick 6 tickets are sold for $1 each. How many winners are expected? Each winner is awarded $15,000. How much profit should the state expect to make?