Appendix H

EXPONENTS, LOGARITHMS, AND ROOTS

Just as multiplication indicates multiple additions, *exponents* indicate multiple multiplications. The symbol for "multiple multiplications" is

$$y = x^n \tag{H-1}$$

where n indicates the number of times the multiplication is to be performed. In this case, x and n are known and y is unknown. If we know x and y and are trying to find n, then we are seeking a *logarithm*. If we know n and y and are trying to find x, then we are finding the *root*. This is summarized in the following table:

	y	n	x
Exponents	**Unknown**	Known	Known
Logarithms	Known	**Unknown**	Known
Roots	Known	Known	**Unknown**

H.1 Integer Exponents

If n is an integer, an exponent is defined as follows:

$$x^n \equiv x \cdot x \cdot x \cdots x \ (n \text{ times}) \tag{H-2}$$

$$x^{-n} \equiv \frac{1}{x \cdot x \cdot x \cdots x} \ (n \text{ times}) \tag{H-3}$$

For the equation

$$y = x^n \tag{H-1}$$

x is called the *base*. The most common bases are 2, 10, and $e = 2.718281828459045 \cdots$.

The Euler (pronounce "oiler") number e is an irrational number. Like π, it requires an infinite number of digits to represent e exactly. You will encounter e throughout your engineering career. It has almost mystical properties that will become more apparent as you further your studies. Because it is sometimes inconvenient to raise e to an exponent, an alternative notation may be used

$$e^n = \exp(n) \quad \text{(H-4)}$$

The following table shows some integer exponents of the three most commonly encountered bases:

		x^n		
n	exponent	x=2	x=e=2.718 ⋯	x=10
-2	⇒	1/4	1/7.389 ⋯	1/100
-1	⇒	1/2	1/2.718 ⋯	1/10
0	⇒	1	1	1
1	⇒	2	2.718 ⋯	10
2	⇒	4	7.389 ⋯	100

Exponents have a number of useful properties described in Table H.1:

Table H.1 Properties of Exponents	
Rule	Example
$(x^m)(x^n) = x^{m+n}$	$5^3 \cdot 5^2 = 125 \cdot 25 = 3125 = 5^{3+2} = 5^5$
$\dfrac{x^m}{x^n} = x^{m-n}$	$\dfrac{5^3}{5^2} = \dfrac{125}{25} = 5 = 5^{3-2} = 5^1$
$\left(\dfrac{x}{z}\right)^m = \dfrac{x^m}{z^m}$	$\left(\dfrac{4}{2}\right)^2 = (2)^2 = 4 = \dfrac{4^2}{2^2} = \dfrac{16}{4}$
$(x^m)^n = x^{mn}$	$(5^3)^2 = (125)^2 = 15625 = 5^{3 \cdot 2} = 5^6$
$(x^m)(z^m) = (xz)^m$	$(5^3)(4^3) = (125)(64) = 8000 = (5 \cdot 4)^3 = 20^3$
$1^m = 1$	
$x^0 = 1$	

H.2 Logarithms

A *logarithm* is the antioperation of an exponent. If we know x and y in the equation

$$y = x^n \quad \text{(H-1)}$$

the unknown n can be found using the logarithm

$$n = \log_x y \quad \text{(H-5)}$$

where the subscript x is the base and y is called the *argument*. The argument must be positive and nonzero for the logarithm to produce a real number. (*Note*: A negative argument gives a complex number, which is discussed later. An argument of zero gives negative infinity.)

The following table shows some values of n for the three common bases:

$\log_x y$				
n	logarithm	x=2	x=e=2.718 ···	x=10
-2	⇐	1/4	1/7.389 ···	1/100
-1	⇐	1/2	1/2.718 ···	1/10
0	⇐	1	1	1
1	⇐	2	2.718 ···	10
2	⇐	4	7.389 ···	100

Base 10 is called the *common* or *Briggsian* logarithm. It is written as follows:

$$n = \log_{10} 1000 = \log 1000 = 3 \tag{H-6}$$

Base e is called the *natural* or *Napierian* logarithm. It is written as follows:

$$n = \log_e 7.389 = \ln 7.389 = \log 7.389 = 2 \tag{H-7}$$

Note that the same notation sometimes is used for both. Most commonly, log is used for base 10; however, it is sometimes used for base e. The proper interpretation of log must be taken from the context. For example, in the *CRC Handbook of Chemistry and Physics*, the solutions to integral equations use log. Because the Napierian logarithm naturally (excuse the pun) occurs as the solution to integral equations and the Briggsian logarithm never does, then we know from the context that log must represent the Napierian logarithm. Base 2 is often encountered in computer applications. It is written as follows:

$$n = \log_2 16 = 4 \tag{H-8}$$

Logarithms have a number of useful properties, as shown in Table H.2.

Table H.2 Properties of Logarithms	
Rule	Example
$\log(xz) = \log x + \log z$	$\log(10 \cdot 100) = \log(1000) = 3 = \log 10 + \log 100 = 1 + 2$
$\log \frac{x}{z} = \log x - \log z$	$\log \frac{100}{10} = \log(10) = 1 = \log 100 - \log 10 = 2 - 1$
$\log x^m = m \log x$	$\log 100^2 = \log 10000 = 4 = 2 \log 100 = 2(2)$
$\log 1 = 0$	

For historical reasons, it should be noted that the first two logarithm rules were extremely useful to engineers prior to the mid-1970s. Calculators were not commonly available, so many calculations were done on a slide rule. A slide rule has two logarithmic scales; one is stationary and the other is movable. Moving one scale relative to the other would perform the operation of addition (subtraction) if the scales were linear, but because they are logarithmic, the operation of multiplication (division) is performed.

Because exponents and logarithms are the antioperations of each other, the following relationships are true

$$x = e^{\ln x} \qquad x = 10^{\log x} \tag{H-9}$$

and

$$x^n = (e^{\ln x})^n = e^{n \ln x} \qquad x^n = (10^{\log x})^n = 10^{n \log x} \tag{H-10}$$

H.3 Noninteger Exponents

So far, we have only considered integer values of n. The function

$$y = e^n \tag{H-11}$$

is plotted for integer values in Figure H.1.

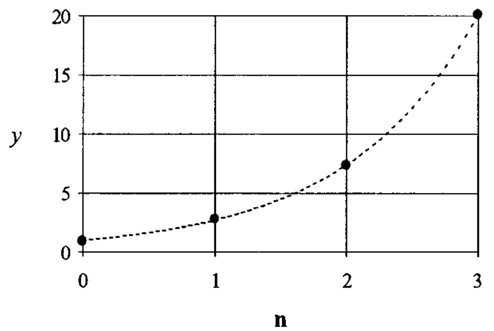


Figure H.1 The function $y=e^n$ for integer values of n.

Because the curve is smooth, it is possible to mathematically "connect the dots." The method is provided by a power series, a topic that is discussed later. Here, we will simply give you the result:

$$y = e^n = \sum_{i=0}^{\infty} \frac{n^i}{i!} = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \cdots + \frac{n^i}{i!} \qquad (-\infty < n < \infty) \tag{H-12}$$

where the exponent n is allowed to be any value from minus infinity to plus infinity. To exactly represent e^n requires all the terms of the series, that is, an infinite number of terms. In practice we find that the first twenty or so terms are sufficient, depending on the desired accuracy of the final answer. The later terms in the series become smaller and smaller because the denominator becomes so large (recall that factorials get big very quickly). For example, let's calculate e^2 using this formula. We know the answer is 7.38905609894 simply by multiplying e twice; i.e.,

$$y = e^2 = e \cdot e = (2.718281828459)(2.718281828459) = 7.38905609894 \tag{H-13}$$

The following table shows the contribution of each successive term in the formula:

Term number	Value of the term	Summation
1	1.00000	1.00000
2	2.00000	3.00000
3	2.00000	5.00000
4	1.33333	6.33333
5	0.66666	7.00000
6	0.26666	7.26667
7	0.08889	7.35556
8	0.02540	7.38095
9	0.00635	7.38730
⇓	⇓	⇓
∞	0.00000	7.38906

The sum of the first nine terms is within 0.18% of the correct answer.

So far we have provided a method to "connect the dots" for nonintegers n, provided we are in base e. What do we do for bases other than e? The following relationship from Equations H-10 and H-12 provides the answer.

$$x^n = (e^{\ln x})^n = e^{(n \ln x)} = 1 + n\ln x + \frac{(n\ln x)^2}{2!} + \frac{(n\ln x)^3}{3!} + \cdots \quad (x>0 \text{ and } -\infty<n<\infty) \tag{H-14}$$

where x is an arbitrary positive base (recall that logarithms are meaningless for negative arguments) and n may be any number from minus infinity to positive infinity.

If you have a real exponent, Equation H.14 requires that you evaluate the Napierian logarithm. Fortunately there is a power series expression for this:

$$\ln x = 2\left\{\sum_{i=0}^{\infty}\left[\frac{1}{2i+1}\left(\frac{x-1}{x+1}\right)^{2i+1}\right]\right\} = 2\left\{\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right\} \quad (x>0) \tag{H-15}$$

where x must be greater than zero, as for all logarithms.

H.4 Conversion of Bases (Advanced Topic)

Equation H-15 helps us evaluate logarithms for bases other than e. Assume we have the expression

$$y = x^n \tag{H-1}$$

Then by definition, the following is true:

$$n = \log_x y \tag{H-5}$$

If we take the Napierian logarithm of both sides of Equation H-1,

$$\ln y = \ln x^n \tag{H-16}$$

and use the logarithm rule for exponents

$$\ln y = n \ln x \quad (x \neq 0 \text{ and } x \neq 1) \tag{H-17}$$

and substitute Equation H-5

$$\ln y = (\log_x y)\ln x \tag{H-18}$$

we can solve explicitly for the $\log_x y$:

$$\log_x y = \frac{\ln y}{\ln x} \tag{H-19}$$

Therefore, the logarithm of any arbitrary positive base may be evaluated as the ratio of the Napierian logarithms of the argument (y) and the base (x) using the power series shown in Equation H-15. This formula may also be used to convert from logarithms of one base to another. For example, when slide rules were used for calculations, it was much easier to use base 10. However, the solution to many practical engineering problems requires base e. Converting from one base to another is done as follows:

$$\log_{10} y = \frac{1}{\ln 10}\ln y = 0.434294 \ln y \tag{H-20}$$

$$\ln y = \ln 10 \log_{10} y = 2.302585 \log_{10} y \tag{H-21}$$

The conversion factor 2.302585 frequently is seen in older engineering books, written when slide rules were used.

H.5 Roots

If we know y and n in the equation

$$y = x^n \qquad \text{(H-1)}$$

the unknown x is found using the *root*. The notation for a root is

$$x = \sqrt[n]{y} \qquad \text{(H-22)}$$

Some common roots are square roots and cube roots:

$$x = \sqrt{y} = \sqrt[2]{y} \qquad \text{Square root} \qquad \text{(H-23)}$$

$$x = \sqrt[3]{y} \qquad \text{Cube root} \qquad \text{(H-24)}$$

The rules for roots are shown in Table H.3.

Table H.3 Properties of roots	
Rule	Example
$x^{m/n} = \sqrt[n]{x^m}$	$10^{4/2} = 10^2 = 100 = \sqrt[2]{10^4} = \sqrt[2]{10000}$
$x^{m/n} = (\sqrt[n]{x})^m$	$100^{4/2} = 100^2 = 10000 = \left(\sqrt[2]{100}\right)^4 = (10)^4$
$\sqrt[n]{xz} = \sqrt[n]{x}\,\sqrt[n]{z}$	$\sqrt[2]{4 \cdot 9} = \sqrt[2]{36} = 6 = \sqrt[2]{4}\,\sqrt[2]{9} = 2 \cdot 3$
$\sqrt[n]{\frac{x}{z}} = \frac{\sqrt[n]{x}}{\sqrt[n]{z}} \ (z \neq 0)$	$\sqrt[2]{\frac{16}{4}} = \sqrt[2]{4} = 2 = \frac{\sqrt[2]{16}}{\sqrt[2]{4}} = \frac{4}{2}$
$\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$	$\sqrt[3]{\sqrt[2]{1000000}} = \sqrt[3]{1000} = 10 = \sqrt[6]{1000000}$

This notation for roots is commonly found in math textbooks, but is not often used with computers and calculators. These machines indicate a root using an exponent, which is explained in the following paragraph.

Both sides of Equation H-1 may be raised to the power $1/n$:

$$(y)^{1/n} = (x^n)^{1/n} \qquad \text{(H-25)}$$

The rule for the exponent of an exponent says multiply the exponents together, which in this case gives one

$$y^{1/n} = x \qquad \text{(H-26)}$$

Thus, we have solved explicitly for x, the root. Therefore an alternate notation for square root and cube root is

$$y^{1/2} = \sqrt{y} \qquad \text{Square root} \qquad \text{(H-27)}$$

$$y^{1/3} = \sqrt[3]{y} \qquad \text{Cube root} \qquad \text{(H-28)}$$

History of Logarithms

The sequence

$$\cdots x^{-3}\ x^{-2}\ x^{-1}\ x^{0}\ x^{1}\ x^{2}\ x^{3} \cdots$$

is a *geometric progression* in which x is the *base*. (*Note*: The middle term x^0 is interpreted as 1.) Adjacent terms of the geometric progression are in a fixed ratio of 1:x. For example, the following numbers are a geometric progression:

x^n	$\cdots 10^{-3}$	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	$10^{3}\cdots$
y	$\cdots$ 0.001	0.01	0.1	1	10	100	1000 $\cdots$

in which $x = 10$. Adjacent terms are in the ratio of 1:10.

In 1544, Michael Stifel (1487 – 1567) published *Arithemetica integra* in which he noted that $x^m \cdot x^n = x^{(m+n)}$ and $x^m/x^n = x^{(m-n)}$. Using these relationships, the operation of multiplication can be replaced by addition and division can be replaced by subtraction. Because addition (subtraction) is easier than multiplication (division), this observation had the potential to simplify calculations. Unfortunately, Stifel considered only integer values of the exponents, so the full utility of this observation was not realized.

In 1588, Jobst Bürgi (1552 – 1632), a Swiss watchmaker, created a table relating n and y using base $x = 1.0001$. This table was not published until 1620, too late to receive credit for his creation; this honor has been bestowed upon John Napier.

From 1594 to 1614, John Napier (ca. 1550 – 1617) – a Scottish mathematician, inventor, and religious activist – worked on a series of tables relating n and y. He used bases of 0.9999999, 0.99999, 0.9995, and 0.99. (*Note*: Napier did *not* use base e; the designation of base-e logarithms as Napierian logarithms is purely honorary.) Both Napier and Bürgi used integer exponents n; their bases x were approximately 1 to closely space the y values. At first, Napier christened the exponent n as the "artificial number," but later called it *logarithm*, meaning *ratio number*.

In 1614, Napier published his mathematical innovation in *Mirifici logarithmorum canonis descriptio* (*Descriptions of the wonderful canon of logarithms*). It was widely praised because it helped relieve drudgery in calculations; scientists and mathematicians throughout Europe and even China quickly adopted it.

Henry Briggs (1561 – 1631), a British geometry professor, was so impressed by Napier's invention he went to Scotland to meet him. He suggested a number of modifications, including using base 10, which Napier readily accepted. In 1624, Briggs published base-10 logarithms in his *Arithmetica logarithmica* to an accuracy of 14 decimal places. It was not until 1924 that this was improved when 20-place tables were published. Base-10 logarithm tables were an important calculating tool until the invention of the hand-held calculator in the 1970s.

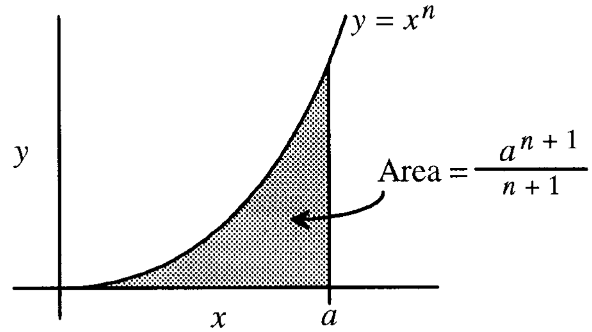
Compared to other bases, base-10 logarithms make calculations fairly easy. For example, suppose we wish to multiply 63 by 128. Given that 63 is between 10 and 100, the logarithm must be between 1 and 2. Let's write it as 1.*abc* where *abc* are the unknown digits behind the decimal. Similarly, the logarithm of 128 could be written as 2.*xyz*. The digits before the decimal point are called the *characteristic* and the digits after the decimal point are called the *mantissa*. (*Note*: Both these terms were coined by Briggs. *Mantissa* is Latin for "makeweight," the small weights used to balance scales.) The characteristic is easily determined from the order of magnitude of the number and the mantissa is determined from logarithm tables for numbers ranging from 0 to 10. From the tables, the logarithm of 6.3 is 0.799, therefore the logarithm of 63 is 1.799. The logarithm of 1.28 is 0.107, therefore the logarithm of 128 is 2.107. By adding the two logarithms, we obtain 3.906. To find our answer, we find the antilogarithm of 3.906. From the mantissa (0.906), we determine the antilog is 8.064. Using the characteristic (3), we

History of Logarithms (con't)

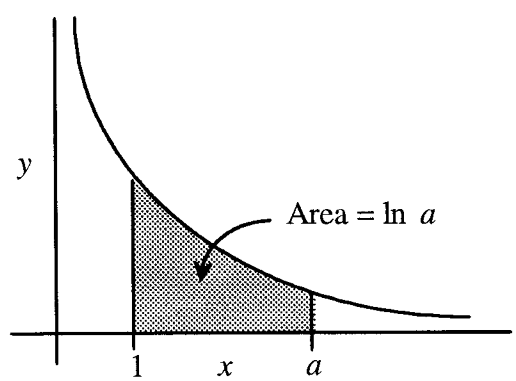
know the final answer is 8064. Although cumbersome, this procedure was often quicker than direct multiplication. Because older engineers had to keep track of the characteristic, they often had a better feeling for the order of magnitude of the final answer than modern engineers who rely on calculators and computers.

Because logarithms reduce multiplication/division to addition/subtraction, it allows simple instruments to perform multiplication/division. In 1620, English minister Edmund Gunter (1581 – 1626) devised a single logarithmic scale and accompanying pair of dividers that performed multiplication/division. In 1622, clergyman William Oughtred (1574 – 1660) simplified the operation by using *two* logarithmic scales that slide past each other; this *slide rule* was an important calculating device for centuries until it was displaced by the hand-held calculator in the 1970s.

Near 1640, prior to the invention of calculus, Pierre Fermat (1601 – 1665) showed that the area under the function $y = x^n$ from 0 to a is



This formula worked well except for $n = -1$ because the denominator is zero. Shortly afterward, Grégoire Saint-Vincent (1584 – 1667) and his student Alfonso Anton de Sarasa (1618 to 1667) showed that for $y = x^{-1}$, the area under the curve from 1 to a is



Thus, they solved a problem that had been attempted by the ancient Greeks 2000 years earlier. Their solution was the first use of a logarithmic *function*, prior to that time, logarithms were regarded strictly as a computational tool.

Eli Maor, *e: The Story of a Number*, Princeton University Press, Princeton,1994.

A Brief History of *e*

Money is used by civilization as a unit of exchange for wealth. Its use extends back thousands of years to ancient civilizations. Those individuals who were successful at accumulating money soon realized that they could lend this money to others and charge a fee called *interest*.

As early as 5000 BC, records indicate that food was borrowed; the interest was paid using food rather than money. In ancient Babylon, the interest rate was 20 to 35% for cereals and 10 to 25% for silver. Ancient Mesopotamians, Hittites, Phoenicians, and Egyptians all borrowed money with interest rates often set by the state.

A Mesopotamian clay tablet from 1700 B.C., now in the Louvre, recorded the solution to the following problem: "How long does it take to double one's money earning 20% annual interest compounded annually?" In modern notation, this would be stated "Find t that solves the equation $1.2^t = 2$." The solution to this problem requires an understanding of logarithms which were not yet discovered. Nonetheless, using linear interpolation, the Mesopotamians estimated the answer as 3.7870 years, which is close to the correct answer of 3.8018 years.

In 1179, usury was banned among Christians as justified by the following Biblical verse:

> *You shall not charge interest to your brother – interest on money or food or anything that is lent out at interest. To a foreigner you may charge interest, but to your brother you shall not charge interest.*
>
> Deuteronomy 23:19–20 (New King James Version)

This same law forbade Jews from lending to fellow Jews, but did not prevent them from lending to gentiles. Thus, many Jews became moneylenders, a hated profession that led to their persecution.

In the early 1600s, international trade and finance saw explosive growth. The following formula is closely associated with compound interest:

$$\left(1+\frac{1}{n}\right)^n$$

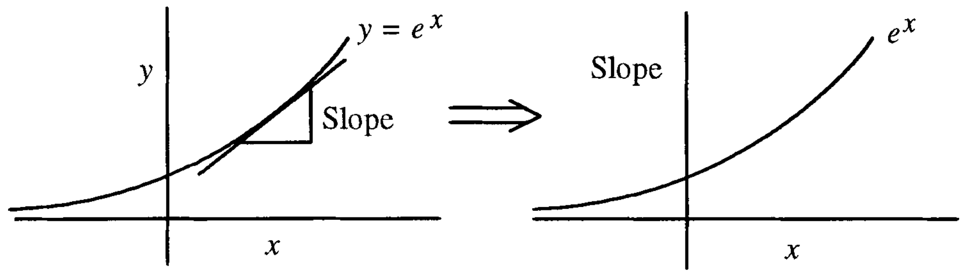
Mathematicians observed that as n became very large, this formula became $2.71828\cdots$, a number we now call e. The exact date of this discovery is unknown.

In 1618, the first record of e appeared in Edward Wright's translation of John Napier's *Mirifici logarithmorum canonis descriptio*, in which appeared a statement equivalent to $\log_e 10 = 2.302585$.

In 1665, Isaac Newton (1642 – 1727) obtained the following expression for e using the binomial expansion of $(1 + 1/n)^n$ by letting $n \to \infty$

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

In 1665 to 1675, Isaac Newton and Gottfried Wilhelm Leibniz (1646 – 1716) were developing calculus. A central idea of calculus is that of slopes. It soon became apparent that the slope of $y = e^x$ is e^x. To better understand this statement, study the following figure:



This property of e makes it extraordinarily important and powerful.

Although the remarkable number 2.71828 · · · was known since the early 1600s, it had no standardly used symbol. In 1727, Leonhard Euler (1707 – 1783) – who also gave us the modern symbols for π, i, and $f(x)$ – assigned this number the symbol e in his manuscript *Meditation upon Experiments made recently on the firing of Cannon*. It is not known why he selected the letter e. Perhaps it is from the first letter of the word "exponential." Perhaps it was the first unused letter of the alphabet, for a, b, c, and d were already commonly used symbols. It is unlikely that it comes from the first letter of his name, for Euler was known to be a very modest individual.

Because π is a simpler concept than e, π was known to the ancients. The great fame of π has lead to international competitions to calculate π to ever larger numbers of significant figures. Often, students memorize the first few digits of π; 3.14159 easily rolls off the tongue. Even though it could be argued the e is a more important mathematical concept than π, students rarely memorize this number. The following pneumonic device should help you remember it. The first ten digits of e are 2.718281828. Note that the last eight digits happen to repeat 1828, the year Andrew Jackson was elected president of the United States. If you are a good history student, you can remember e as 2.7(Andrew Jackson)^2.

Eli Maor, *e: The Story of a Number*, Princeton University Press, Princeton,1994.
Paul Johnson, *A History of the Jews*, Harper & Row, New York, 1987.

H.6 Transcendental Functions

Exponents, logarithms, and roots are all examples of *transcendental functions*. The dictionary defines *transcendental functions* as "functions that cannot be given by algebraic expressions consisting only of the argument and constants." The following is **not** a transcendental function:

$$y = f(x) = 3x + 4 \tag{H-29}$$

because $f(x)$ **can** be represented by an algebraic expression consisting of constants and the argument x. In contrast, transcendental functions can only be represented by an infinite series, such as the power series. Other examples of transcendental functions include the trigonometric functions [e.g., sin (x), arcsin (x)] and hyperbolic functions [e.g., sinh (x)]. These functions will be discussed in later sections.

H.7 Hierarchy of Mathematical Operations

Consider the following equation

$$x = [4 + (3 \cdot 2 + 7^2)]6 - \frac{4}{2} \tag{H-30}$$

Different values for x will be calculated depending on the sequence in which the operations are performed. The hierarchy shown in Table H.4 has been established so the same answer is obtained regardless of who performs the calculation.

Table H.4 Hierarchy of mathematical operations	
Hierarchy	Operation
1st	Parenthesis
2nd	Exponents
3rd	Multiplication/division
4th	Addition/subtraction

The operations in the first hierarchy are performed from left to right. Then, the operations in the second hierarchy are performed from left to right, etc. For example, Equation H-30 is solved in the following order:

$$x = [4 + (3 \cdot 2 + 49)]6 - \frac{4}{2}$$

$$x = [4 + (6 + 49)]6 - \frac{4}{2}$$

$$x = [4 + (55)]6 - \frac{4}{2}$$

$$x = (59)6 - \frac{4}{2}$$

$$x = 354 - 2$$

$$x = 352$$

H.8 Summary

The equation $y = x^n$ relates the three variables x, y, and n. If any two of them are known, the third can be determined. The root allows x to be found from y and n, the logarithm allows n to be found from x and y, and the exponent allows y to be found from x and n. If n is an integer, the exponent is evaluated by multiplying x by itself n times. For noninteger values of n, the exponent is evaluated by using a power series.

In the expression $y = x^n$, x is the base. When x is 10 and y is known, n is found by using the common logarithm. When x is e (i.e., $2.718 \cdots$) and y is known, n is found by using the natural logarithm.

The root x is found by raising y to the $1/n$ power. A common notation for the nth root is $x = \sqrt[n]{y}$.

When evaluating complex mathematical expressions, the mathematical operations must be performed in proper sequence so that a unique answer is obtained regardless of who evaluates the expression. The hierarchy of mathematical operations establishes the sequence; the innermost parentheses have highest priority, followed by exponents, then multiplication/division, and lastly addition/subtraction.

Further Readings

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline Series in Mathematics, McGraw-Hill, New York, 1968.

Problems

H.1 In chemistry, pH is defined as

$$pH \equiv \log_{10}\frac{1}{[H^+]} = \log_{10}[H^+]^{-1} = -\log_{10}[H^+]$$

where $[H^+]$ is the proton concentration expressed in moles per liter. Recall that pH = 7 is neutral, pH < 7 is acidic, and pH > 7 is alkaline. Calculate the proton concentration for the following pHs:

a. 7.0
b. 3.0
c. 8.0
d. 0.5
e. –1.0 (yes, the pH can be negative)
f. 16.0 (yes, the pH can be greater than 14)

H.2 Given that $y = 9.85$, what is n in the following equation:

$$y = x^n$$

when x is 2, 10, e, and π?

H.3 Given that $y = 9.85$, what is x in the following equation:

$$y = x^n$$

when n is 1, 2, 3, 2/3, and -0.5?

H.4 Solve for x in the following equations:

a. $\sqrt{x+5} = 3$
b. $(x+10)^{2/3} = 15$
c. $\dfrac{(x-3)^4}{(x-3)^2} = 11$
d. $(x+1)^{0.463}(x+1)^{-1.5} = 3$
e. $\ln(x+1) = 3$
f. $\ln(x+5) - \ln(x+3) = 4$
g. $\ln(x+6)^2 = 5$
h. $\ln(3(x+4)) = 10$

H.5 Solve for x in the following equations:

a. $x = (3+5^2)(4 \times 8^{2/3})^2$
b. $x = \dfrac{(\frac{2}{3} + \sqrt{6} \times 4^2)^{0.5}}{(6^3 - (\sqrt{5} + 3^2) \times 4)}$
c. $x = 4^2 \cdot 3 + \frac{1}{8} - \sqrt{3}\,(\sqrt{5} - 3^2)$
d. $x = \left[\dfrac{5^2 + \sqrt{3^{4.5}}}{(8 - \sqrt[3]{5})(2/3 + 7^{2.5})}\right]^{\frac{2}{3}}$

H.6 Write a computer program that calculates x for Equation d in Problem H.5.

H.7 Write a computer program to calculate e^n using the power series (Equation H-12). Stop the calculation when an additional term changes the value by less than 0.0001% (i.e., 1 part in a million). The power series will be performed by using a loop structure. The factorial term is to be calculated by using a loop, so this program will have a loop within a loop. Compare this value to the one obtained by calling the intrinsic function for e^n.

H.8 Write a computer program that performs the calculation described in Problem H.7. Rather than use a loop within a loop, call a function subprogram to calculate the factorial.

H.9 Write a computer program to calculate the Napierian logarithm $\ln x$ using the power series (Equation H-15). Stop the calculation when an additional term changes the value by less than 0.0001% (i.e., 1 part in a million). Compare this value to the one obtained by calling the intrinsic function $\ln x$.

H.10 Transform the computer program you wrote in Problem H.9 into a function subprogram. In your main program, call this subprogram to evaluate the logarithm of an arbitrary base x according to Equation H-19. The main program must ask the user to specify y and the base x.