Appendix I

GEOMETRY AND TRIGONOMETRY

In your previous mathematical training, you probably had separate courses in geometry and trigonometry. Here we describe them together because they both deal with the analysis of shape.

The analysis of shape is critical to engineering. For example, the value of a well-engineered automobile is determined largely by its shape, not the raw materials of which it is composed. In fact, the metal, plastic, and paint are worth only a few hundred dollars, which is but a small fraction of the multithousand dollar selling price. If the engineers have properly shaped the doors, they close nicely and do not leak in a rain storm. If the builders have properly shaped the engine, it will be efficient, powerful, quiet, and vibration free. If they properly shaped the body, it will have a low drag coefficient so it will slice through the air with minimal friction allowing the automobile to be faster and more fuel efficient.

All engineering disciplines require the proper specification of shape. Civil engineers specify the shape of buildings and the beams that support them. Chemical engineers specify the shape of reactors and distillation columns. Industrial engineers specify the shape and layout of factory floors. Electrical engineers specify the shape and layout of computer chip circuits. Mechanical engineers specify the shape of gears and shafts. Shape is what defines the functionality and aesthetics of engineered products.

We will start our discussion of shape with some very simple ideas.

I.1 Angles

A *ray* is a line segment emanating from a point called the *vertex* (see Figure I.1).

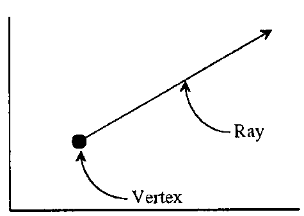


Figure I.1 Ray.

Two rays emanating from a common point form an *angle* (see Figure I.2).

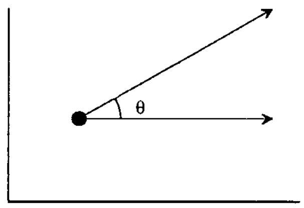


Figure I.2 Angle.

An angle is measured by placing the vertex at the center of a circle (see Figure I.3).

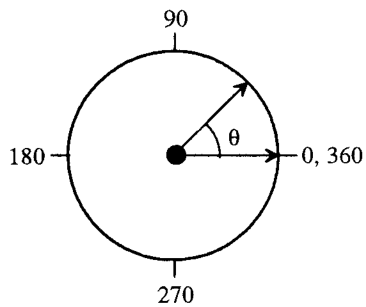


Figure I.3 Circle divided into 360°.

The circumference of the circle is divided into 360 segments, or *degrees* (°). The degree is further divided in 60 *minutes* (') and minutes are divided into 60 *seconds* ("). The angle 43°52'12" is converted to a decimal angle as follows:

$$43+\frac{52}{60}+\frac{12}{3600}=43.87^{\circ}$$

An *acute* angle is between 0° and 90°, a *right* angle is 90°, and an *obtuse* angle is between 90° and 180° (see Figure I.4).

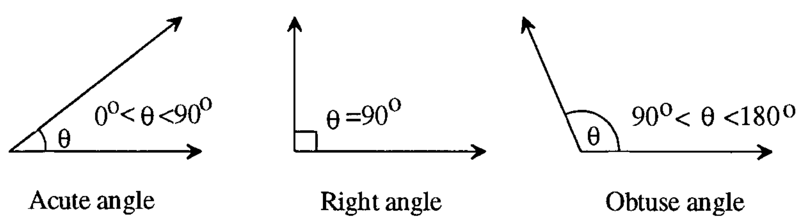


Figure I.4 Types of angles.

The degree is actually an arbitrary angle measurement. The reason there are 360° in a circle is that an integer results when it is divided by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, etc. Although the degree is used to measure an angle in everyday life, it is not a basic unit. The basic unit of an angle is the *radian.*

A *radian* is the ratio of the circumference swept by the angle divided by the radius:

$$\text{Radian}=\frac{\text{swept circumference length}}{\text{radius}} \tag{I-1}$$

Because both the numerator and denominator have dimensions of length, the radian is actually a dimensionless number (i.e., a pure number). This equation can be manipulated to give the swept circumference length:

$$\text{Swept circumference length} = (\text{radian})(\text{radius}) \tag{I-2}$$

Therefore, a radian can be visualized as the pure number by which the radius is multiplied to measure the swept circumference. This can be better understood by studying Figure I.5.

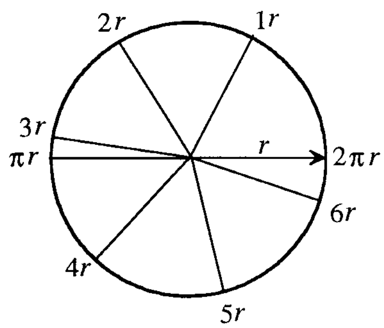


Figure I.5 Circle divided into 2π radians.

Table I.1 shows some radian measurements and their corresponding degree measurements.

Table I.1 Corresponding angles measured in degrees and radians				
Degree	Radian		Degree	Radian
45	$\frac{1}{4}\pi$		225	$\frac{5}{4}\pi$
90	$\frac{1}{2}\pi$		270	$\frac{3}{2}\pi$
135	$\frac{3}{4}\pi$		315	$\frac{7}{4}\pi$
180	π		360	2π

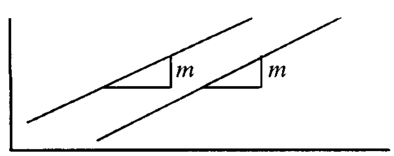
The following example shows how to convert an arbitrary degree angle into a radian angle:

$$36°17'\,51'' = 36 + \frac{17}{60} + \frac{51}{3600} = 36.2975°$$

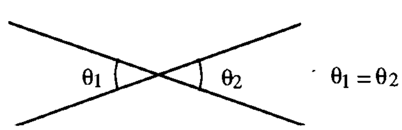
$$\frac{36.2975°}{180°} = \frac{\theta}{\pi} \Rightarrow \theta = \frac{36.2975°}{180°}\pi = 0.6335 \text{ radians}$$

Some theorems on angles follow:

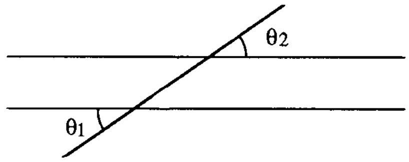
- Lines are parallel if their slopes are identical



- Angles on opposite sides of crossed lines are equal

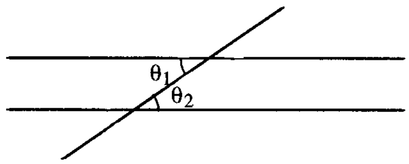


- Exterior angles of a line that crosses two parallel lines are equal



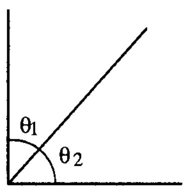
$\theta_1 = \theta_2$

- Interior angles of a line that crosses two parallel lines are equal



$\theta_1 = \theta_2$

- The two angles formed by dividing a right angle add to 90° (π/2 radians)



$\theta_1 + \theta_2 = 90^{\circ} = \pi/2$ radians

I.2 Triangles

A *triangle* is a figure formed by connecting three noncollinear points by three line segments.

I.2.1 Types of Triangles

All the angles are less than 90° in *acute* triangles, *right* triangles have one angle equal to 90°, and *obtuse* triangles have one angle greater than 90°. An *equilateral* triangle has all three sides equal whereas an *isosceles* triangle has two sides that are equal. These triangles are illustrated in Figure I.6.

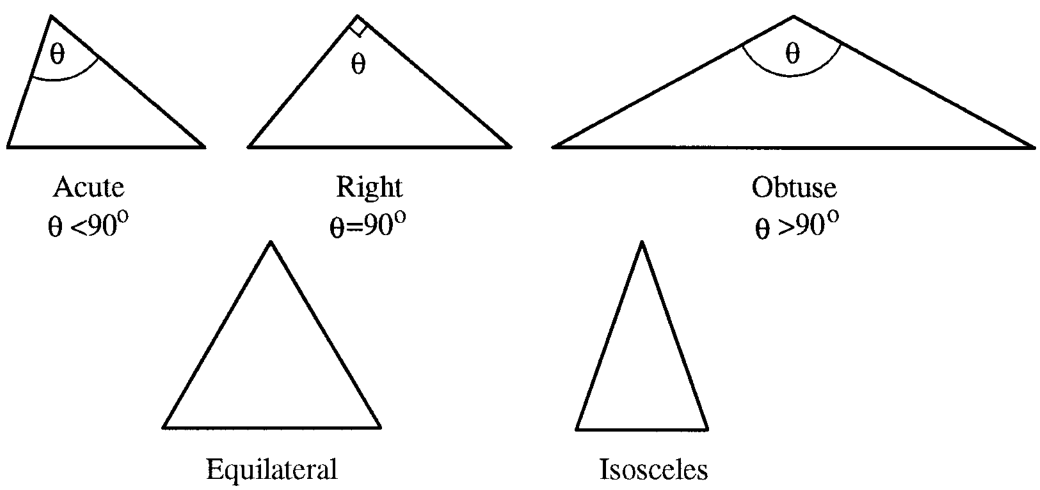
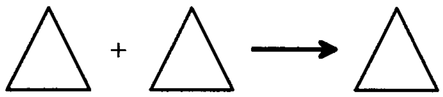
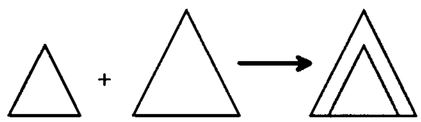


Figure I.6 Types of triangles.

Congruent triangles are identical when superimposed,



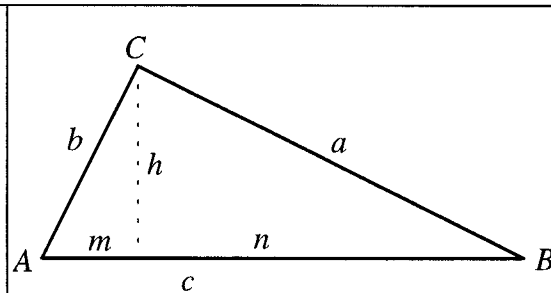
whereas *similar* triangles have the same interior angles, but do not superimpose.



The three interior angles in a triangle must sum to 180°.

I.2.2 Right Triangles

Table I.2 shows some mathematical relationships for right triangles. Note that line segments are indicated with lower case letters and angles are indicated with uppercase letters. An *inscribed* circle is the largest circle that fits inside the triangle. A *circumscribed* circle is the smallest circle that completely contains the triangle.

Table I.2 Relationships for right triangles (reference)	
$A+B=C=90°$ $c^2=a^2+b^2$ (Pythagorean theorem) Area = $\frac{1}{2}ab=\frac{1}{2}ch$ $h=\frac{ab}{c}$, $m=\frac{b^2}{c}$, $n=\frac{a^2}{c}$ Radius inscribed circle = $\frac{ab}{a+b+c}$ Radius circumscribed circle = $\frac{1}{2}c$	

I.2.3 Trigonometry

Trigonometry is the study of the relationships between angles and the length of sides in triangles. The following *trigonometric functions* are defined for **right** triangles (see Table I.2) and by Figure I.7 which allows for angles greater than 90°. (*Note*: The four quadrants in Figure I.7 are identified as I, II, III, and IV.)

$$\sin A=\frac{a}{c}=\frac{\text{opposite}}{\text{hypotenuse}}=\frac{y}{r}\ \text{(sine)} \qquad \csc A=\frac{c}{a}=\frac{\text{hypotenuse}}{\text{opposite}}=\frac{r}{y}\ \text{(cosecant)} \tag{I-3}$$

$$\cos A=\frac{b}{c}=\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{x}{r}\ \text{(cosine)} \qquad \sec A=\frac{c}{b}=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{r}{x}\ \text{(secant)} \tag{I-4}$$

$$\tan A=\frac{a}{b}=\frac{\text{opposite}}{\text{adjacent}}=\frac{y}{x}\ \text{(tangent)} \qquad \cot A=\frac{b}{a}=\frac{\text{adjacent}}{\text{opposite}}=\frac{x}{y}\ \text{(cotangent)} \tag{I-5}$$

The functions in the right column are the inverse of the functions on the left.

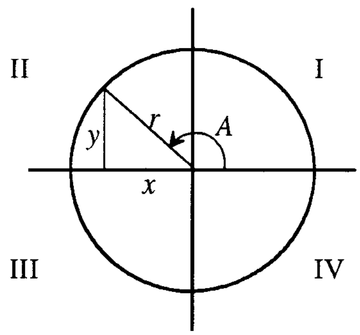


Figure I.7 Definition of trigonometric functions for angles greater than 90°.

The trigonometric functions are *transcendental functions*, much like ln x and e^x. Recall that the definition of a transcendental function is that it cannot be described by simple algebraic relationships of the argument and constants. Although the trigonometric functions seem to be defined using simple algebraic relationships, the definition does not include the argument (i.e., the angle), so they are transcendental functions. The only way to calculate these transcendental functions is through a series, such as the power series shown below:

$$\sin\theta = \sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i+1}}{(2i+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \quad (-\infty < \theta < \infty) \tag{I-6}$$

$$\cos\theta = \sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \quad (-\infty < \theta < \infty) \tag{I-7}$$

For these equations to be valid, the argument, θ, *must* be in radians. From these two power series, all the other trigonometric functions can be calculated. For example, tan θ is

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{I-8}$$

and the other functions, csc θ, sec θ, and cot θ, are the inverse of sin θ, cos θ, and tan θ, respectively. Figure I.8 shows graphs of these trigonometric functions.

The trigonometric functions described thus far operate on angles and produce the ratio of two sides. The antioperators below produce an angle when given the ratio of two sides.

$$\arcsin\frac{a}{c} = \arcsin\frac{y}{r} = A \text{ (arcsine)} \qquad \operatorname{arccsc}\frac{c}{a} = \operatorname{arccsc}\frac{r}{y} = A \text{ (arccosecant)} \tag{I-9}$$

$$\arccos\frac{b}{c} = \arccos\frac{x}{r} = A \text{ (arccosine)} \qquad \operatorname{arcsec}\frac{c}{b} = \operatorname{arcsec}\frac{r}{x} = A \text{ (arcsecant)} \tag{I-10}$$

$$\arctan\frac{a}{b} = \arctan\frac{y}{x} = A \text{ (arctangent)} \qquad \operatorname{arccot}\frac{b}{a} = \operatorname{arccot}\frac{x}{y} = A \text{ (arccotangent)} \tag{I-11}$$

An alternative mathematical notation that is sometimes used for these antioperators is $\sin^{-1}$ for arcsine, $\cos^{-1}$ for arccosine, etc. This notation does *not* mean take the inverse of the function. For example,

$$\sin^{-1}\theta \neq (\sin\theta)^{-1} \tag{I-12}$$

Figure I.9 shows plots of these antioperators.

Mathematics' Most Famous Formula

As shown by Isaac Newton (1642 – 1727), the power series for e^x *is*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

where x is a real number. Leonhard Euler (1707 – 1783), in a playful move, decided to substitute an imaginary number for x

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

where i is the square root of negative one. Noting that $i^2 = -1$, this becomes

$$e^{i\theta} = 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots$$

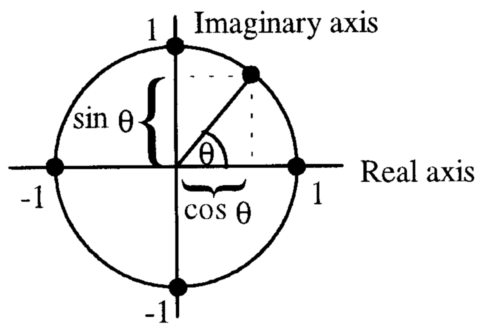
By rearranging the terms of this infinite series (which requires caution because it potentially leads to errors), it becomes

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

The first bracketed term is the power series for cosine and the second bracketed term is the power series for sine:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus, Euler established a link between exponents and trigonometry. The following figure provides a graphical interpretation of this equation



For $\theta = \pi$,

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1 + i0$$

$$\boxed{e^{i\pi} = -1}$$

This single equation incorporates some of the most important mathematical concepts: exponents, negative numbers, imaginary numbers, and geometry.

Eli Maor, *e, The Story of a Number*, Princeton University Press, Princeton, 1995.

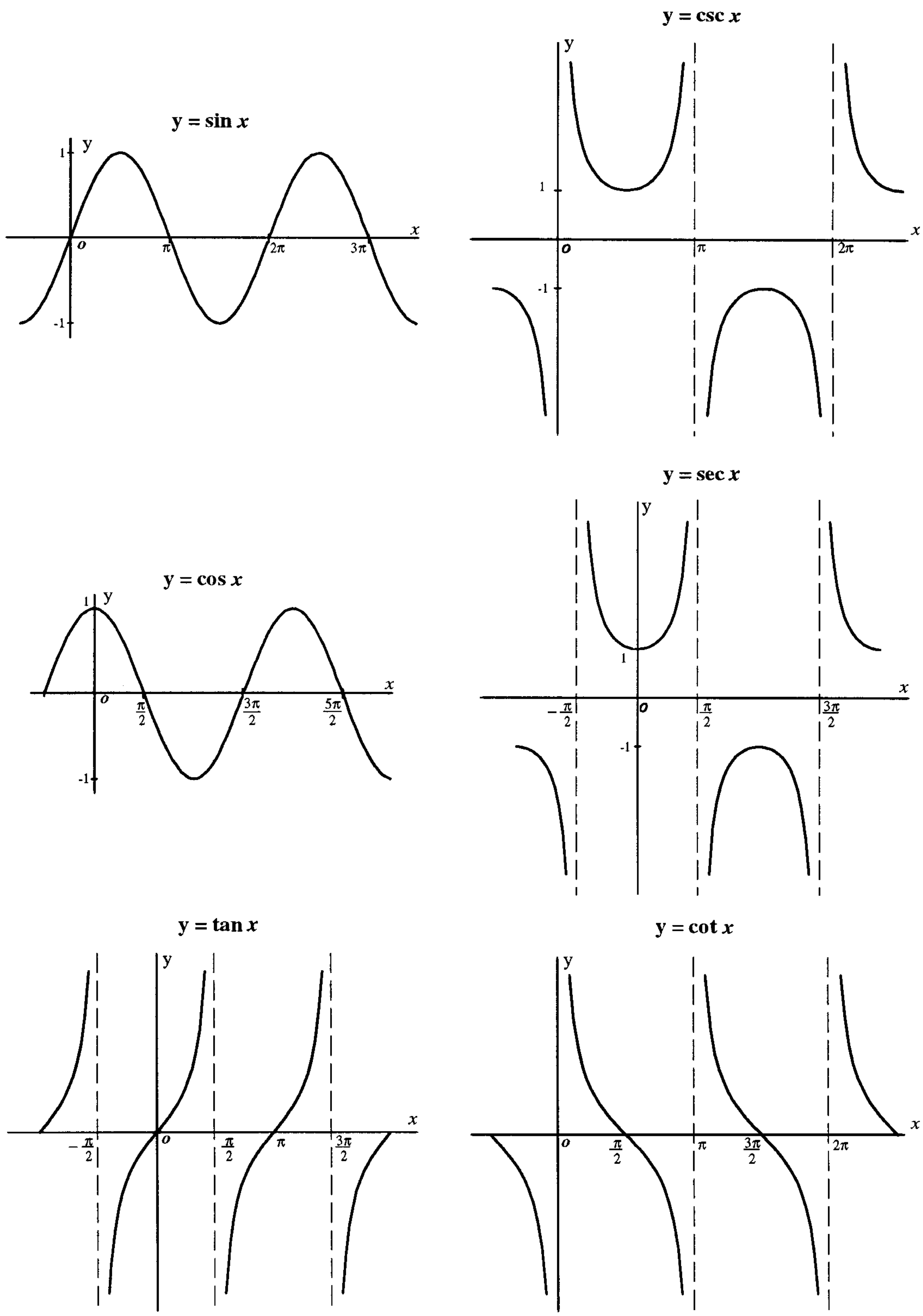


Figure I.8 The trigonometric functions.

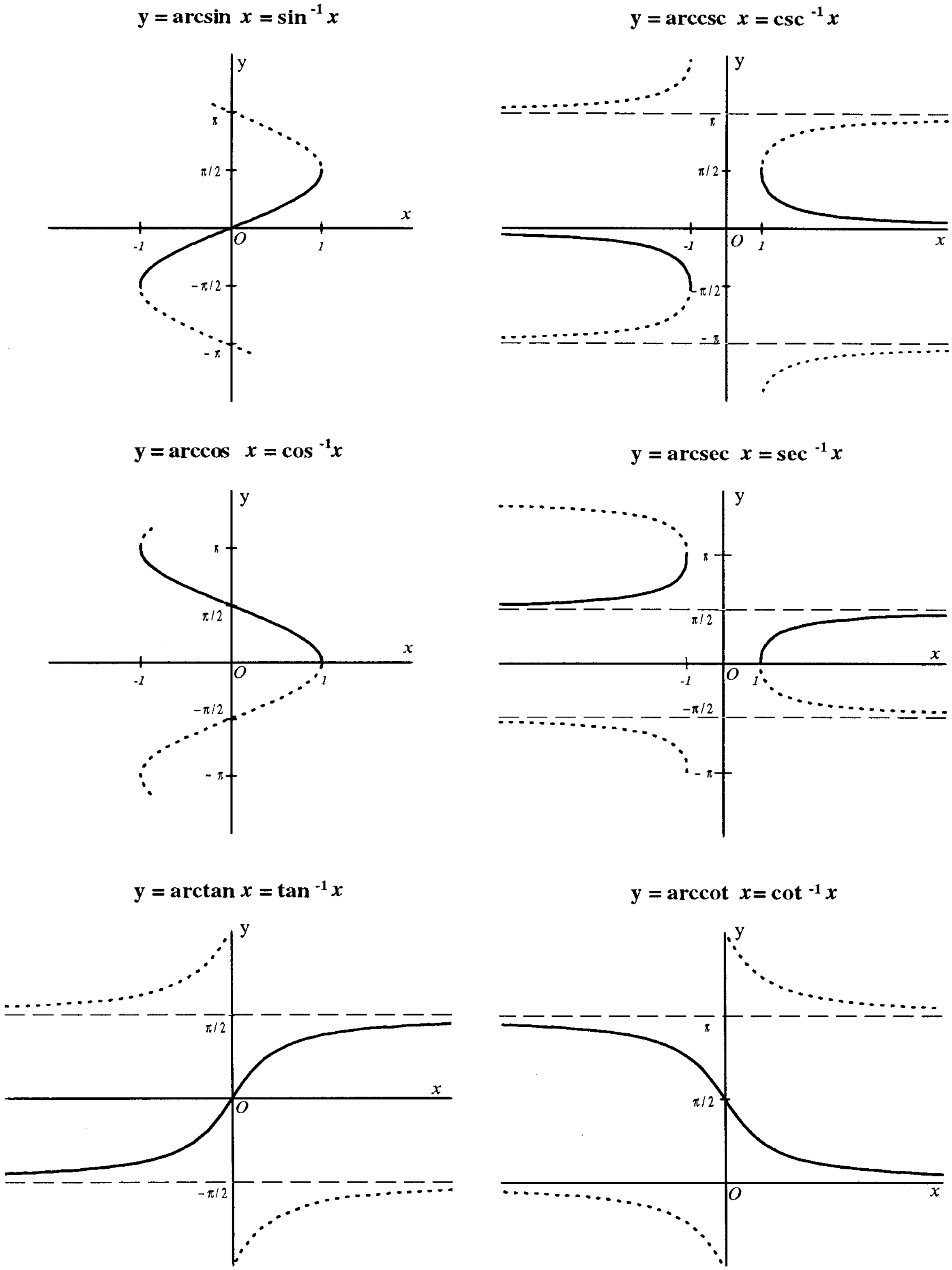


Figure I.9 The trigonometric antioperators.

From the definitions of the trigonometric functions, there are many *identities* that can be derived (see Table I.3). In some cases, trigonometric functions with exponents are presented. Provided the exponent is not -1, it is interpreted as you would expect. For example,

$$\sin^2\theta = (\sin\theta)^2 \tag{I-13}$$

Table I.3 Trigonometric identities (reference)

*	$\tan A = \frac{\sin A}{\cos A}$		$\sin^2 A + \cos^2 A = 1$
*	$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$		$\sec^2 A - \tan^2 A = 1$
*	$\sec A = \frac{1}{\cos A}$		$\csc^2 A - \cot^2 A = 1$
*	$\csc A = \frac{1}{\sin A}$		
*	$\sin(-A) = -\sin A$	*	$\sec(-A) = \sec A$
*	$\csc(-A) = -\csc A$	*	$\tan(-A) = -\tan A$
*	$\cos(-A) = \cos A$	*	$\cot(-A) = -\cot A$
*	$\sin 2A = 2\sin A\cos A$		$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$
	$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$		
	$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$ + if $A/2$ is in quadrant I or II; − if $A/2$ is in quadrant III or IV		
	$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ + if $A/2$ is in quadrant I or IV; − if $A/2$ is in quadrant II or III		
	$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ + if $A/2$ is in quadrant I or III; − if $A/2$ is in quadrant II or IV		
*	$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$		$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \csc A - \cot A$
	$\sin 3A = 3\sin A - 4\sin^3 A$		$\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$
*	$\cos 3A = 4\cos^3 A - 3\cos A$		$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A$
	$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$		$\tan 5A = \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}$
	$\sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$		$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$
*	$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$		

Table I.3 (continued)

	$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$	*	$\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$
*	$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$	*	$\cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$
	$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$		$\sin^5 A = \frac{5}{8}\sin A - \frac{5}{16}\sin 3A + \frac{1}{16}\sin 5A$
*	$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$		$\cos^5 A = \frac{5}{8}\cos A + \frac{5}{16}\cos 3A + \frac{1}{16}\cos 5A$
	$\sin(A+B) = \sin A\cos B + \cos A\sin B$		$\sin(A-B) = \sin A\cos B - \cos A\sin B$
	$\cos(A+B) = \cos A\cos B - \sin A\sin B$		$\cos(A-B) = \cos A\cos B + \sin A\sin B$
	$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$		$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
	$\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$		$\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
*	$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$		$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
*	$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$		$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
*	$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$		$\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
	$\cos A - \cos B = 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(B-A)$		
*	An asterisk means this relationship also applies to hyperbolic functions		

I.2.4 Hyperbolic Functions (Advanced Topic)

Hyperbolic functions are sometimes encountered in engineering problems (e.g., heat transfer, electrical circuit analysis, and dynamic mechanical systems). The definition of the hyperbolic functions follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ (hyperbolic sine)} \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}} \text{ (hyperbolic cosecant)} \tag{I-14}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ (hyperbolic cosine)} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}} \text{ (hyperbolic secant)} \tag{I-15}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ (hyperbolic tangent)} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \text{ (hyperbolic cotangent)} \tag{I-16}$$

Many of the relationships shown in Table I.3 are also valid for the hyperbolic functions. The asterisks in Table I.3 indicate those relationships that apply to hyperbolic functions. For example,

$$\tan A = \frac{\sin A}{\cos A} \tag{I-8}$$

is presented in Table I.3 for trigonometric functions. Because there is an asterisk next to it, the following relationship is also valid:

$$\tanh x = \frac{\sinh x}{\cosh x} \tag{I-17}$$

Many of the trigonometric relationships that are not marked by an asterisk differ from the corresponding hyperbolic relationship by only a sign.

The hyperbolic functions are *transcendental functions,* much like the trigonometric functions. They can be expressed only by an infinite series, such as the power series shown below:

$$\sinh x = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad (-\infty < x < \infty) \tag{I-18}$$

$$\cosh x = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad (-\infty < x < \infty) \tag{I-19}$$

As with the trigonometric functions, there are also corresponding antioperations for the hyperbolic functions.

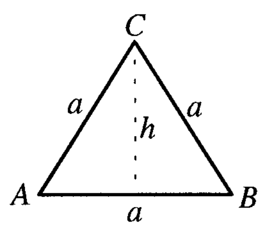
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \quad (-\infty < x < \infty) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) \quad (x \neq 0) \tag{I-20}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geq 1) \qquad \operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right) \quad (0 < x \leq 1) \tag{I-21}$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (-1 < x < 1) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \quad (x > 1 \text{ or } x < -1) \tag{I-22}$$

I.2.5 Equilateral Triangles

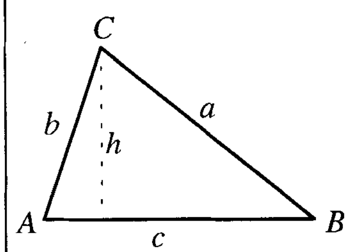
Table I.4 shows relationships for an equilateral triangle, that is, a triangle in which all three sides are identical.

Table I.4 Relationships for equilateral triangles (reference)	
$A = B = C = 60°$ $h = \frac{\sqrt{3}}{2}a$ Area $= \frac{1}{2}ah = \frac{\sqrt{3}}{4}a^2$ Inscribed circle radius $= \frac{\sqrt{3}}{6}a$ Circumscribed circle radius $= \frac{\sqrt{3}}{3}a$	

I.2.6 General Triangles

Table I.5 shows relationships for a general triangle in which the angles and lengths of sides are arbitrary.

Table I.5 Relationships for general triangle (reference)

$A+B+C=180°$	
$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ (law of sines)	
$c^2=a^2+b^2-2ab\cos C$ (law of cosines)	
$\dfrac{a+b}{a-b}=\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$ (law of tangents)	
$h=a\sin B=b\sin A=\dfrac{2(\text{area})}{c}$	
Perimeter $=P=a+b+c$	
Semiperimeter $=s=\dfrac{a+b+c}{2}$	
Inscribed circle radius $=r=c\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sec\dfrac{C}{2}=\dfrac{ab\sin C}{P}$	
$r=(s-c)\tan\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}=\dfrac{ch}{P}$	
Circumscribed circle radius $=R=\dfrac{c}{2\sin C}=\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\dfrac{abc}{4(\text{Area})}=\dfrac{ab}{2h}$	
Area $=\dfrac{1}{2}hc=\dfrac{1}{2}ab\sin C=\dfrac{c^2\sin A\sin B}{2\sin C}=rs=\dfrac{abc}{4R}=\sqrt{s(s-a)(s-b)(s-c)}$	

I.3 Scalars and Vectors

A *scalar* is a quantity that is independent of direction. It is measured with a single number and often has an accompanying unit. Scalars are used to measure quantities such as mass, volume, time, and length. The following are examples of scalars: 7, 5 kg, 3 in, 4.3 ft^3, 58 min.

Vectors have a size (or *magnitude*) and an associated direction. They are used to measure quantities such as force, velocity, and acceleration. The following are examples of vectors: 7 mph velocity in the easterly direction, 9.8 m/s^2 acceleration in the downward direction, 8.2 newtons force in the upward direction.

To properly specify the vector direction, a *coordinate system* must be established. The type of coordinate system depends on how many dimensions are needed to describe the problem at hand. Figure I.10 shows some typical coordinate systems for one, two, and three dimensions.

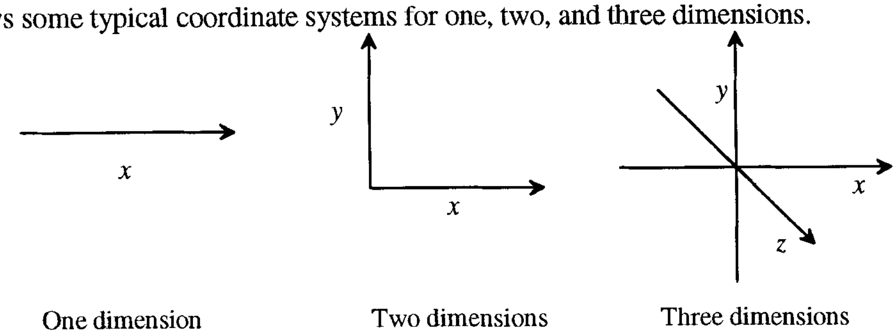
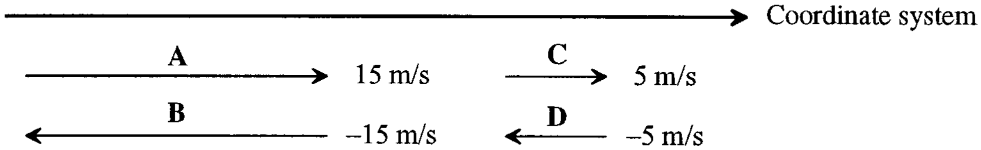


Figure I.10 Coordinate systems.

All of the coordinate systems must establish a positive direction (indicated by the arrow). The vector is also drawn with an arrow to indicate its direction. A short arrow indicates that the vector quantity is small and a long arrow indicates the vector quantity is large. If the vector arrow is in the same direction as the coordinate system, the vector is positive. Conversely, if the vector arrow is opposite the coordinate system, the vector is negative. To illustrate these ideas, examine the following figure:



To distinguish vectors from scalars, they are indicated by either a boldface or a conventional letter with an arrow over it

$\mathbf{A}$ $\quad$ $\vec{A}$

Vector notation

Here we will use bold letters to indicate vectors.

The *magnitude* of a vector may be interpreted as the length of the arrow used to represent the vector quantity. Because length has no direction associated with it, magnitude is a scalar quantity. Magnitude is represented by enclosing the vector in absolute value symbols, e.g., $|\mathbf{A}|$. When considering the four vectors shown in the above figure, their magnitude is

$$a = |\mathbf{A}| = 15 \text{ m/s} \qquad b = |\mathbf{B}| = 15 \text{ m/s} \qquad c = |\mathbf{C}| = 5 \text{ m/s} \qquad d = |\mathbf{D}| = 5 \text{ m/s}$$

where a, b, c, and d are scalars representing the length, or magnitude, of the vectors. Notice that even though the vectors **B** and **D** are negative (because their direction is opposite the coordinate system), their magnitudes are positive because length is independent of direction.

Vectors may be multiplied by scalars in order to increase their length. For example, vector **A** is three times longer than vector **C**. This is expressed as

$$\mathbf{A} = 3\mathbf{C}$$

where 3 is a scalar.

Vectors are added by placing them head to tail to produce the *resultant* vector. For example, Figure I.11 shows that vector **C** is the resultant when vectors **A** and **B** are added.

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

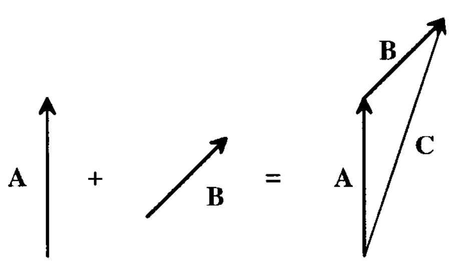


Figure I.11 Vectors **A** and **B** are added by placing them head to tail to find resultant vector **C**.

Figure I.11 also shows the relationship between vectors that are subtracted. That is,

$$\mathbf{A} = \mathbf{C} - \mathbf{B}$$

$$\mathbf{B} = \mathbf{C} - \mathbf{A}$$

The *components* of a vector are found by *projecting* the vector onto the coordinate axis. For example, the vector **V** projects onto $x = 7$ and $y = 4$ as shown in the Figure I.12.

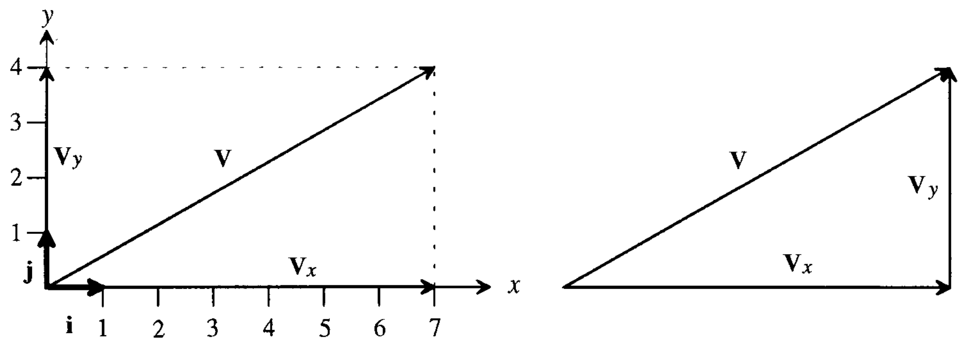


Figure I.12 The components of vector **V** are found by projecting onto the coordinate axis.

Therefore, the two components of vector **V** are vector $\mathbf{V}_x$ of length 7 in the x direction and vector $\mathbf{V}_y$ of length 4 in the y direction. These component vectors may be rearranged slightly into a right triangle (see Figure I.12). The magnitude of **V** is therefore easily calculated using the Pythagorean theorem:

$$|\mathrm{V}| = \sqrt{|\mathbf{V}_x|^2 + |\mathbf{V}_y|^2} \qquad \text{(I-23)}$$

The *unit vectors* **i** and **j** have a magnitude of 1 and point in the x-direction and y-direction, respectively. Vector $\mathbf{V}_x$ may be represented as the scalar 7 times the unit vector **i** and $\mathbf{V}_y$ may be represented as the scalar 4 times the unit vector **j**. Figure I.12 shows that **V** is the vector sum of $\mathbf{V}_x$ and $\mathbf{V}_y$; therefore, the following equation may be written

$$\mathbf{V} = 7\mathbf{i} + 4\mathbf{j}$$

Expressing vectors as the sum of their component vectors provides a convenient way to add two vectors. As shown in Figure I.13, vector **C** is the sum of vectors **A** and **B**. The summation may be performed as follows:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) + (4\mathbf{i} + 1\mathbf{j}) = (2+4)\mathbf{i} + (3+1)\mathbf{j} = 6\mathbf{i} + 4\mathbf{j}$$

The magnitude of **C** is

$$|\mathbf{C}| = \sqrt{6^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52} = 7.2111$$

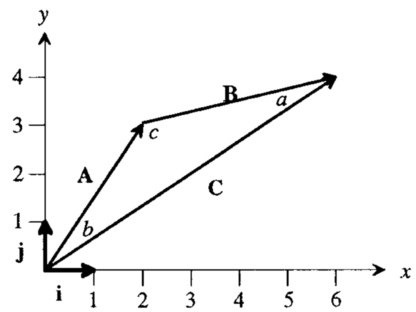


Figure I.13 Vector **C** is the sum of vectors **A** and **B**.

In this particular example, the component vectors were easily read from the figure. In most situations, trigonometry is needed to determine the magnitude of the component vectors.

If a vector is multiplied by a scalar, then each component of the vector is multiplied by that scalar. For example,

$$\mathbf{D} = 3\mathbf{A} = 3(2\mathbf{i} + 3\mathbf{j}) = (3 \times 2)\mathbf{i} + (3 \times 3)\mathbf{j} = 6\mathbf{i} + 9\mathbf{j}$$

Thus, a vector **D** that is 3 times longer than the vector **A** shown in Figure I.1 has a component of 6 in the x direction and a component of 9 in the y direction.

Rather than use components to add vectors, some people prefer to use the relationships for general triangles, such as the law of sines, law of cosines, or law of tangents

$$\frac{|\mathbf{A}|}{\sin a} = \frac{|\mathbf{B}|}{\sin b} = \frac{|\mathbf{C}|}{\sin c} \qquad \text{(Law of sines)} \qquad \text{(I-24)}$$

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos c \qquad \text{(Law of cosines)} \qquad \text{(I-25)}$$

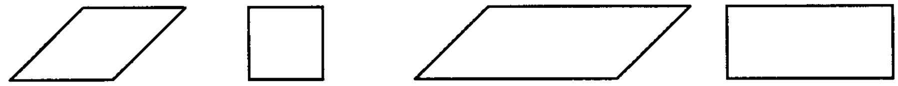
$$\frac{|\mathbf{A}| + |\mathbf{B}|}{|\mathbf{A}| - |\mathbf{B}|} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)} \qquad \text{(Law of tangents)} \qquad \text{(I-26)}$$

where a, b, and c are the angles shown in Figure I.13.

I.4 Quadrilaterals

A *quadrilateral* is a geometric figure bounded by four straight line segments (called sides), each of which intersects each of the adjacent sides at points called *vertices*, but fails to intersect the opposite side.

Parallelograms are quadrilaterals with opposite sides that are parallel. A *rhombus* is a parallelogram in which opposite sides are of equal length. A *square* is a rhombus with 90° angles. A *rhomboid* is a parallelogram in which adjacent sides are of unequal length. A rectangle is a rhomboid with 90° angles.



Rhombus Square Rhomboid Rectangle

Trapezoids are quadrilaterals with only two parallel lines. *Isosceles trapezoids* have nonparallel sides that are of equal length. A *kite* has adjacent sides of equal length.

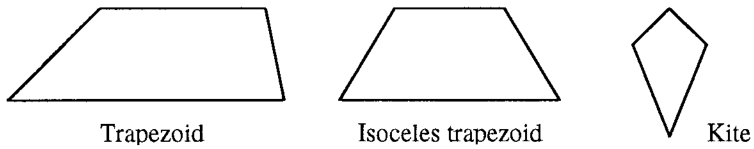
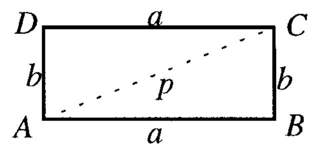
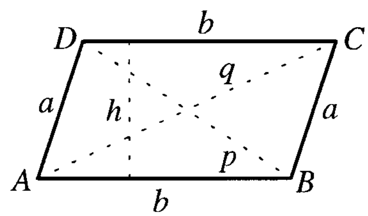
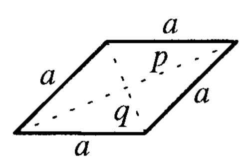
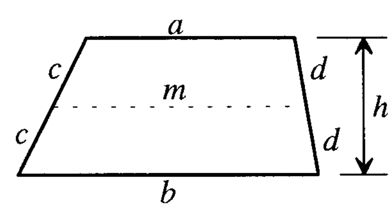
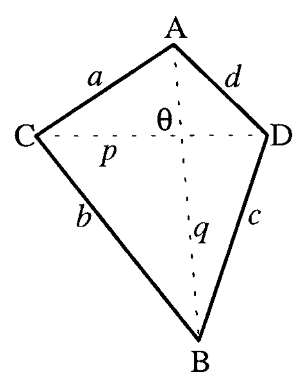


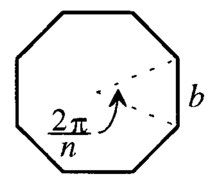
Table I.6 shows some relationships for quadrilaterals.

Table I.6 Relationships for quadrilaterals (reference)	
Rectangle	
$A = B = C = D$ Area $= ab$ $p = \sqrt{a^2 + b^2}$	
Parallelogram	
$A = C,\ B = D,\ A + B = 180°$ Area $= bh = ab \sin A = ab \sin B$ $p = \sqrt{a^2 + b^2 - 2ab\cos A}$ $q = \sqrt{a^2 + b^2 - 2ab\cos B} = \sqrt{a^2 + b^2 + 2ab\cos A}$ $h = a \sin A = a \sin B$	
Rhombus	
$p^2 + q^2 = 4a^2$ Area $= \frac{1}{2}pq$	
Trapezoid	
$m = \frac{1}{2}(a + b)$ Area $= \frac{1}{2}(a + b)h = mh$	
General quadrilateral	
Area $= \frac{1}{2}pq \sin\theta$ Area $= \frac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta$ Area $= \frac{1}{4}\sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2}$ Area $= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+B}{2}\right)}$ where $s = \frac{1}{2}(a + b + c + d)$	

I.5 Polygons

Polygons are closed planar figures bounded by three or more line segments. Examples are: triangle (3 sides), quadrangle (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), and octagon (8 sides). All the sides of *regular polygons* are equal lengths. Table I.7 shows some relationships for regular polygons.

Table I.7 Relationships for regular polygons (reference)

Regular polygon of n sides each of length b	
Area $= \frac{1}{4}nb^2 \cot\frac{\pi}{n} = \frac{1}{4}nb^2 \frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$ Perimeter $= nb$	

I.6 Circles

A *circle* is the set of all points in a plane that are equidistant from a fixed point. As shown in Figure I.14, the equation of a circle is

$$r^2 = (x-h)^2 + (y-k)^2 \tag{I-27}$$

which is a restatement of the Pythagorean theorem. In this equation, h, k, and r are constants, and x and y are variables.

Table I.8 shows some relationships for circles.

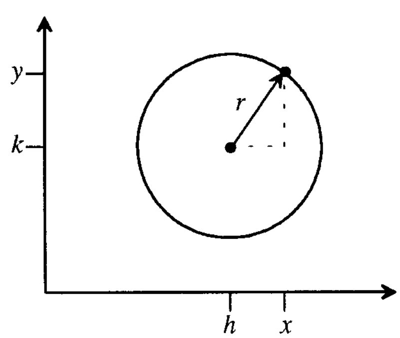


Figure I.14 A circle is the set of all points equidistant from the center (h, k).

Table I.8 Relationships for circles (reference)

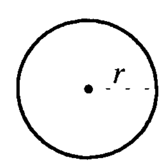
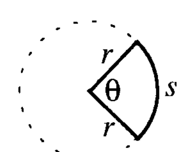
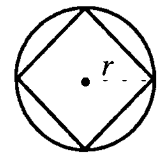
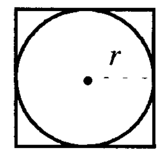
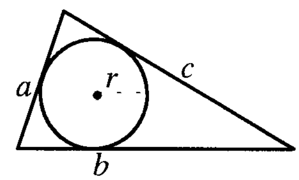
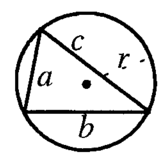
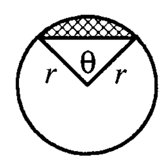
Circle of radius r	
Area $= \pi r^2 = \frac{\pi}{4}d^2$ Perimeter $= 2\pi r = \pi d$ where $d = 2r$	
Sector of circle with radius r	
Area $= \frac{1}{2}r^2\theta$ (θ in radians) Arc Length $s = r\theta$	

Table I.8 (continued)	
Regular polygon of n sides inscribed in a circle of radius r Area $= \frac{1}{2}nr^2\sin\frac{2\pi}{n} = \frac{1}{2}nr^2\sin\frac{360°}{n}$ Perimeter $= 2nr\sin\frac{\pi}{n} = 2nr\sin\frac{180°}{n}$	
Regular polygon of n sides circumscribing a circle of radius r Area $= nr^2\tan\frac{\pi}{n} = nr^2\tan\frac{180°}{n}$ Perimeter $= 2nr\tan\frac{\pi}{n} = 2nr\tan\frac{180°}{n}$	
Radius of circle inscribed in a triangle with sides a, b, c $r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$ where $s = \frac{1}{2}(a+b+c)$	
Radius of circle circumscribing a triangle with sides a, b, c $r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$ where $s = \frac{1}{2}(a+b+c)$	
Segment of circle with radius r Area of shaded part $= \frac{1}{2}r^2(\theta - \sin\theta)$	

I.7 Ellipse

An *ellipse* is the set of all points for which the sum of the distances from two fixed points (called *foci*) is constant (see Figure I.15). The equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(I-28)}$$

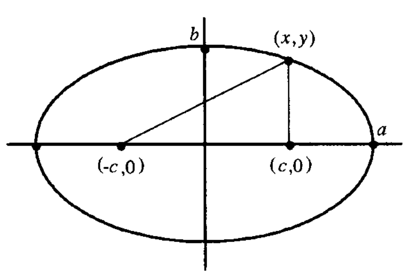
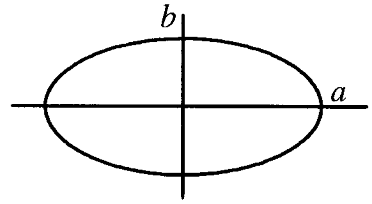


Figure I.15 An ellipse.

The area and perimeter of an ellipse are shown in Table I.9.

Table I.9 Relationships for an ellipse (reference)	
Ellipse of semimajor Axis a and semiminor axis b	
Area $= \pi ab$ Perimeter $= 4a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$ where $k = \frac{\sqrt{a^2 - b^2}}{a}$ Perimeter $\cong 2\pi \sqrt{\frac{1}{2}(a^2 + b^2)}$ (approximate)	

I.8 Solid Figures

Table I.10 shows the surface area and volume of some solid figures.

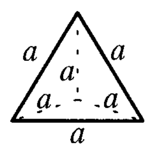
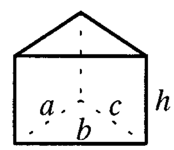
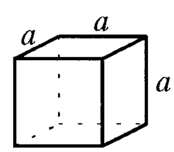
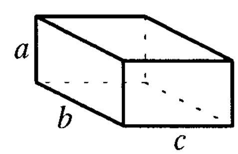
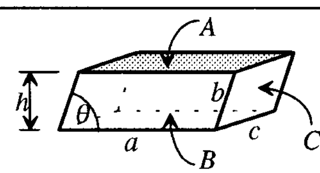
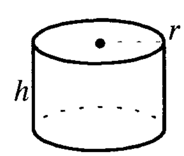
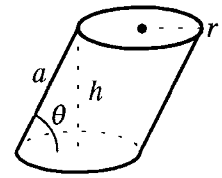
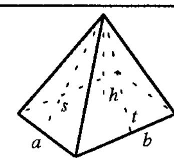
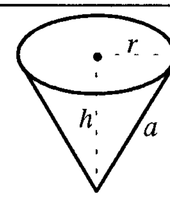
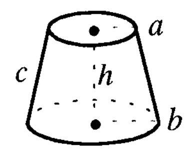
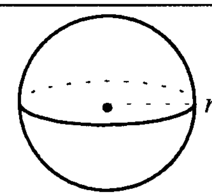
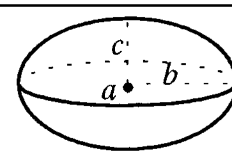
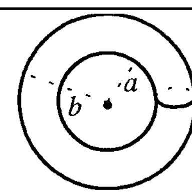
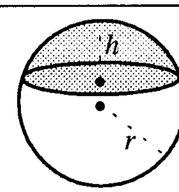
Table I.10 Relationships for solid figures (reference)	
Tetrahedron	
Area $= \sqrt{3}\, a^2$ Volume $= \frac{\sqrt{3}}{16} a^3$	
Right triangular prism	
Area $= 2\sqrt{s(s-a)(s-b)(s-c)} + (a+b+c)h$ Volume $= h\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$	
Cube	
Area $= 6a^2$ Volume $= a^3$	
Rectangular parallelepiped (or box)	
Area $= 2(ab + bc + ac)$ Volume $= abc$	
Parallelepiped of cross-sectional area A	
Area $= 2(A + B + C)$ Volume $= Ah = abc \sin\theta$	
Right circular cylinder	
Area $= 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$ Volume $= \pi r^2 h$	

Table I.10 (continued)	
Slanted circular cylinder Area $= 2\pi r^2 + 2\pi ra = 2\pi r^2 + \dfrac{2\pi rh}{\sin\theta} = 2\pi r\left(r + \dfrac{h}{\sin\theta}\right)$ Volume $= \pi r^2 h = \pi r^2 a \sin\theta$	
Pyramid Area $= ab + as + bt$ Volume $= \dfrac{1}{3}abh$	
Right circular cone Area $= \pi r^2 + \pi r\sqrt{r^2 + h^2} = \pi r^2 + \pi ra = \pi r(r + a)$ Volume$= \dfrac{1}{3}\pi r^2 h$	
Frustum of right circular cone Area $= \pi a^2 + \pi b^2 + \pi(a + b)\sqrt{h^2 + (b - a)^2}$ Area $= \pi a^2 + \pi b^2 + \pi(a + b)c = \pi[a^2 + b^2 + (a + b)c]$ Volume $= \dfrac{1}{3}\pi h(a^2 + ab + b^2)$	
Sphere Area $= 4\pi r^2$ Volume $= \dfrac{4}{3}\pi r^3$	
Ellipsoid Volume $= \dfrac{4}{3}\pi abc$	
Torus Area $= \pi^2(b^2 - a^2)$ Volume $= \dfrac{1}{4}\pi^2(a + b)(b - a)^2$	
Spherical cap Exterior area of shaded cap $= 2\pi rh$ Shaded volume $= \dfrac{1}{3}\pi h^2(3r - h)$	

I.9 Constructions

Constructions are techniques for performing various drafting functions, such as bisecting angles or lines, using only a compass and a straight edge. Historically, constructions were used by ancient Greeks in their study of geometry.

Figure I.16(a) shows line segment *AB* being bisected by a perpendicular line segment *CD*. The compass tip is placed on point *A* to draw circle segment *a*. Then the compass tip is placed on point *B* to draw circle segment *b*. The two circle segments intersect at points *C* and *D* which are used to draw line segment *CD*.

Figure I.16(b) shows angle *ABC* being bisected by line segment *BF*. The compass tip is placed at point *B* to draw circle segment *b*. Points *D* and *E* are formed where the circle segment intersects with the angle. The compass tip then is placed on points *D* and *E* to make circle segments *d* and *e*, respectively. Point *F* is formed at the intersection of circle segments *d* and *e* which allows bisecting line segment *BF* to be drawn.

Figure I.16(c) shows a perpendicular line segment being drawn from line *AB* through an external point *O*. The compass tip is placed on point *O* to make circle segment *o*. This circle segment intersects with line segment *AB* at points *D* and *E*. The compass tip is placed on points *D* and *E* to make circle segments *d* and *e*, respectively. These two circle segments intersect at point *F* which allows perpendicular line segment *FO* to be drawn.

Figure I.16(d) shows a perpendicular line segment being drawn from line *AB* through point *P* that falls on the line. The compass tip is placed at point *P* to draw circle segment *p* which intersects with line segment *AB* at points *C* and *D*. The compass tip is placed on these two points to draw circle segments *c* and *d*. These two circle segments intersect at point *F* which allows line segment *FP* to be drawn.

Figure I.16(e) shows how line segment *AB* may be divided into an arbitrary number of equal parts (in this case, three parts). Circle segments *a* and *b* are drawn by placing the compass tip at points *A* and *B*, respectively. The tangent line to circle segments *a* and *b* is parallel line segment *CG*. The compass tip is placed at point *C* to make circle segment *c*. The intersection of circle segment *c* and line segment *CG* is point *D*. The compass tip is placed at point *D* to make circle segment *d*. This process is repeated the desired number of times (in this case, three times). Line segment *AC* and *BF* intersect at point *P*. Line segments *DP* and *EP* divide line segment *AB* into three equal parts.

Figure I.16(f) shows how to locate the center of a given circle *o*. Two line segments *AB* and *BC* are drawn to intersect with circle *o* at points *A*, *B*, and *C*. The compass tip is placed on these points to draw circle segments *a*, *b*, and *c*. Circle segments *a* and *b* intersect at points *D* and *E* whereas circle segments *b* and *c* intersect at points *F* and *G*. Lines *DE* and *FG* intersect at the circle center point *O*.

I.10 Summary

Geometry is the study of various shapes such as triangles, quadrilaterals, polygons, ellipses, and circles. Two rays emanating from a point form an angle. If the point is located at the center of a circle, the rays will intersect the circle. The circle circumference may be divided into 360 even segments. The magnitude of the angle can be measured in degrees, which is the number of circle segments located between the two rays. A more fundamental method of measuring angles is with radians; a radian is defined as the ratio of the swept circumference length divided by the circle radius.

Trigonometry is the study of the relationship between angles and lengths in a triangle. The trigonometric functions, such as sine and cosine, are defined based on the angles and sides of a right triangle. Trigonometric identities are equalities relating the trigonometric functions. Many of these identities also are valid for hyperbolic functions, such as the hyperbolic sine and cosine.

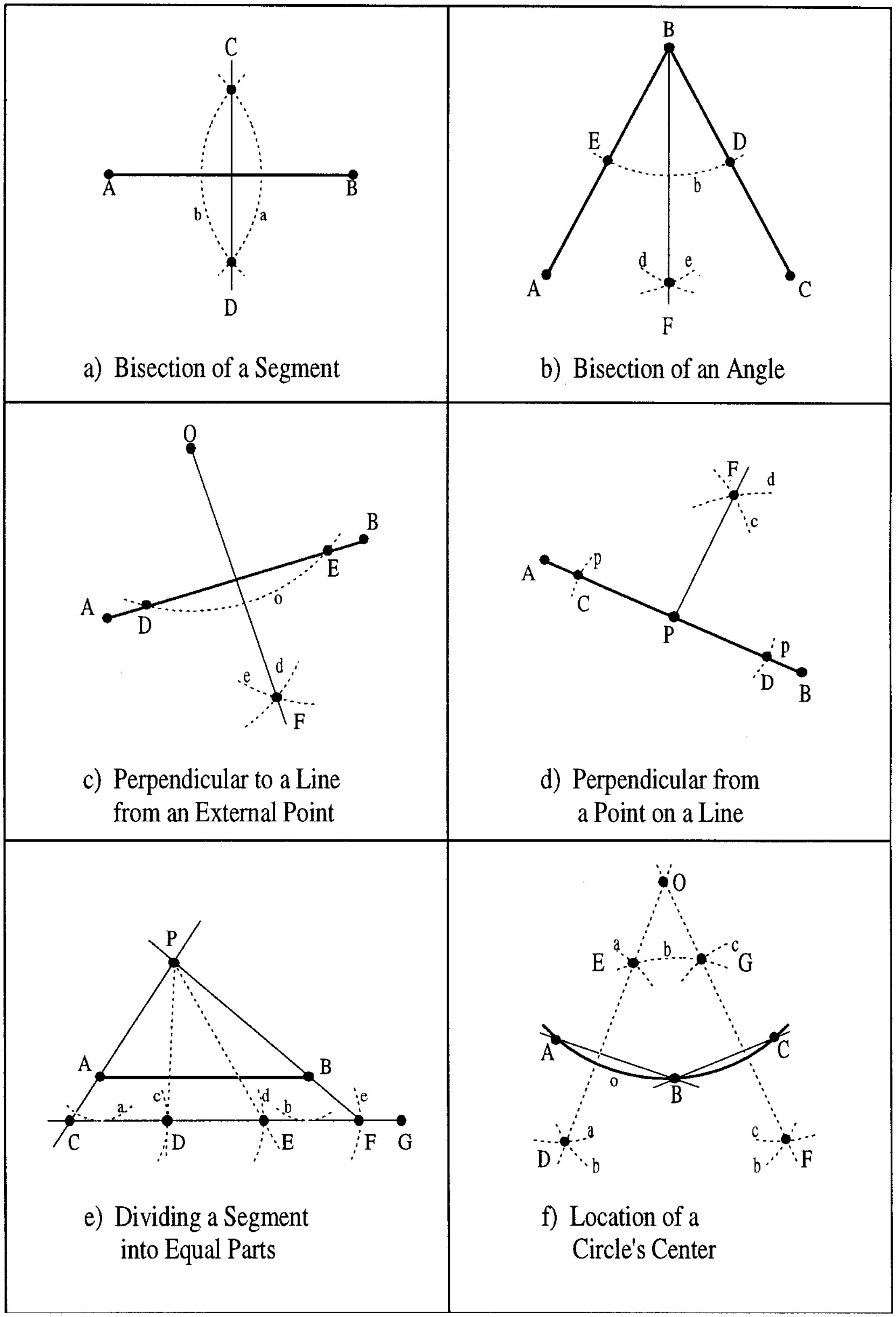


Figure I.16 Constructions.

Scalars are quantities that have a magnitude, but no directionality. In contrast, vectors have both magnitude and directionality. When multiplied by a positive scalar, a vector changes its magnitude, but not its direction. Vectors may be added (subtracted) by resolving them into their component vectors and adding (subtracting) each of the components. Alternatively, trigonometric relationships (e.g., law of sines) may be used to add (subtract) vectors.

Constructions are techniques for performing various drafting functions, such as bisecting angles or lines, using only a compass and straight edge.

Further Readings

M. S. Spiegel, *Mathematical Handbook of Formulas and Tables,* Schaum's Outline Series in Mathematics, McGraw-Hill, New York, 1968.

Handbook of Mathematical Formulas, Tables, Graphs, Functions, Transforms, Research and Education Association, New York, 1980.

Problems

I.1 Convert the following degree-minute-second angles into decimal degrees and radians:

a. 23° 45' 34"
b. 254° 35' 16"
c. 60° 45' 43"
d. 405° 84' 63"

I.2 Convert the following radian angles into decimal degree angles and degree-minute-second angles:

a. 2.456 radians
b. 0.234π radians
c. 0.234 radians
d. 8π radians

I.3 Calculate the swept circumferential length of a circle with at 1-cm radius for the following angles:

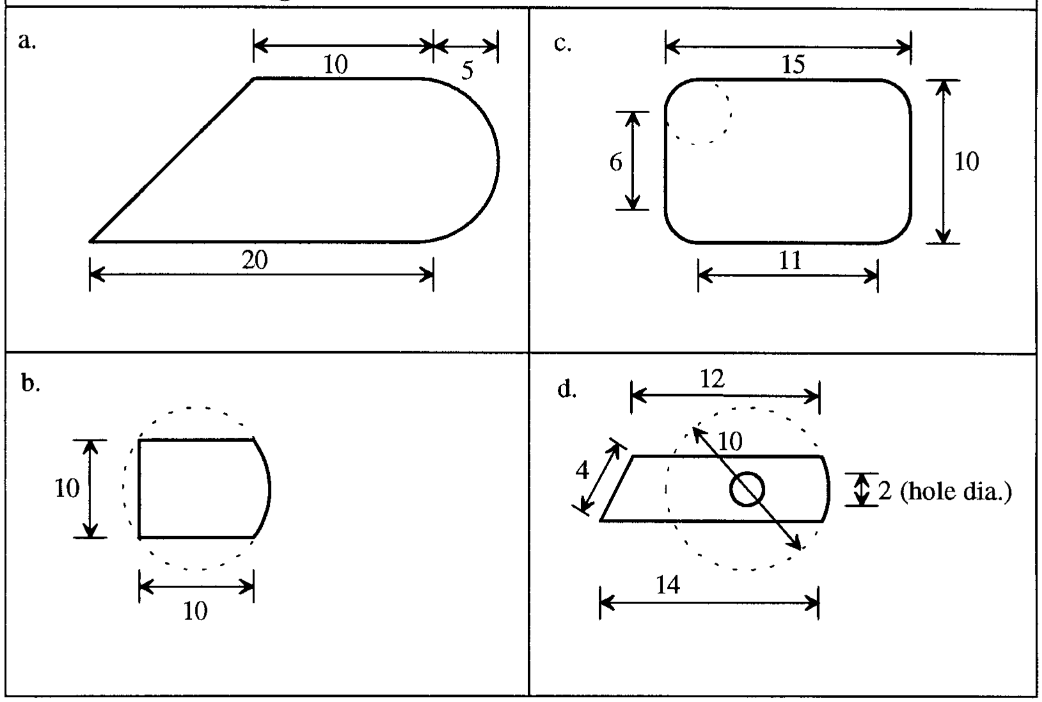
a. 3.45 radians
b. 2π radians
c. 36.45°
d. 78° 34' 45"

I.4 A radio antenna is 100 ft high. To keep it from blowing over in a strong wind, it is stabilized by three cables. The cables are secured to the antenna 75 ft above the ground and are secured to the ground 50 ft from the antenna base. You are the engineer responsible for telling the purchasing agent how much cable to order. You decide that an extra 5% is needed for end fittings and to accommodate any sagging due to gravity. How much cable should you tell the purchasing agent to order? What is the cable angle at the ground and at the antenna?

I.5 The cable in Problem I.4 is composed of 15 braided steel wires, each of which is 0.25 inch in diameter. Estimate the total mass of the delivered cable. (*Note*: The density of steel is 489 lb_m/ft^3.)

I.6 A new laser cutter can be programmed by computer to cut any desired shape in metal plate. The laser can cut at a rate of 2 inches per minute. How long will it take to cut out the patterns in Figure I.17? (*Notes*: All dimensions are given in inches. The dotted lines are not part of the plate nor do they indicate a hidden part. They indicate imaginary figures (e.g., circles) of which the solid line is a segment.)

Figure I.17 Patterns for Problems I.6 and I.7.



I.7 The patterns described in Problem I.6 are cut from 0.25-inch thick steel plate. Calculate the mass of each plate after it is cut. (*Note*: The density of steel is 0.282 lb_m/in^3.)

I.8 The objects in Figure I.18 are cast from aluminum. Calculate their mass. (*Note*: All dimensions are given in inches. The density of aluminum is 0.0955 lb_m/in^3.)

I.9 Calculate the total exterior surface area of the objects described in Problem I.8.

I.10 The castings described in Problem I.8 are placed on a table to cool. The table is well insulated, so no heat is lost from surfaces in contact with the table. However, heat is lost by *all* surfaces exposed to the air. Calculate the heat loss (in Btu/min) if the object loses 5 Btu/min from each square inch exposed to the air.

I.11 An "ultralight" is a very small plane generally constructed from a kit. They are small enough to be put in a trailer and pulled by an automobile. An ultralight enthusiast puts his plane in his trailer and drives from his home to a country field where he can take off. At noon, he leaves from his home and drives due north on a highway going 55 miles per hour for 15 minutes. He then heads due east on a dirt road going 20 miles per hour for 30 minutes. He arrives at the country field and spends 1 hour assembling his plane. He then takes off and heads straight for home going 45 miles per hour. As he flies over his home, he "flaps his wings" to show off to his kids. What time in the afternoon does he arrive at this home?

I.12 Add the vectors **A** and **B** in Figure I.19 to find the resultant vector **C**. Report the resultant vector as the sum of the unit vectors **i** and **j** (i.e., **C**= x **i** + y **j**). Calculate the magnitude of the resultant vector. Determine the angle between the resultant vector and the x axis.

Figure I.18 Objects for Problems I.8, I.9, and I.10.

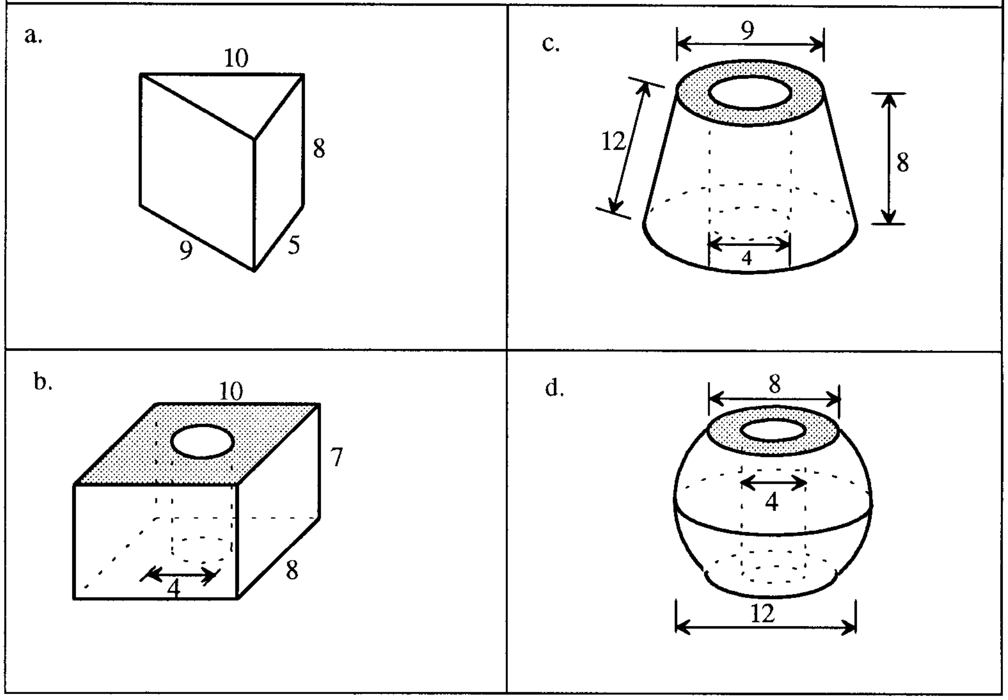
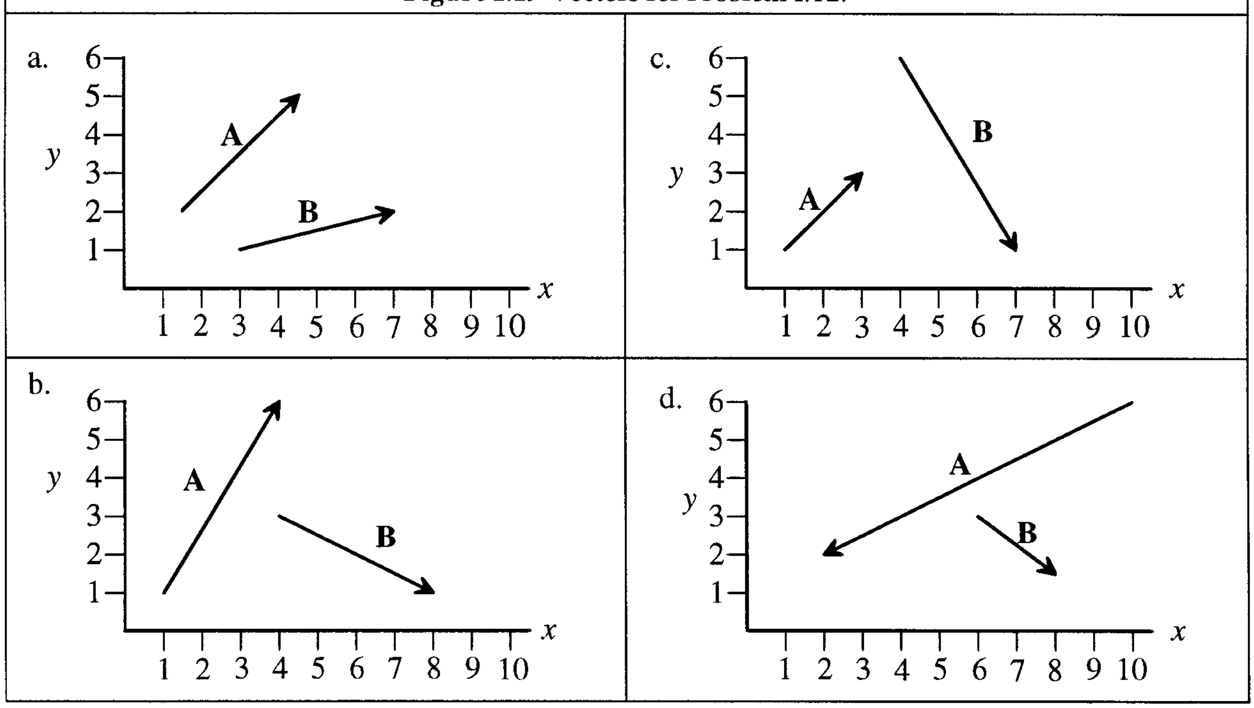


Figure I.19 Vectors for Problem I.12.



I.13 Write a computer program that calculates the resultant vector **C** by adding two vectors **A** and **B**. The input and resultant vectors will be specified using the unit vectors **i** and **j**. The program must also calculate the magnitude of the resultant vector and the angle between it and the x-axis.

I.14 Write a computer program that accomplishes the tasks described in the following problems:

a. Problem I.6
b. Problem I.7
c. Problem I.8
d. Problem I.9
e. Problem I.10

When you write the program, use variables in the formulas rather than the specific numbers given in the problem statement. Test your program with the numbers given in the problem statement, but also run the program with other inputs of your own choosing. Be aware that the inputs have restrictions (e.g., dimensions cannot be negative, hole diameters cannot be larger than the object itself). Before putting the numbers into the formulas, your program must verify that the numbers make sense. If they do not, an error message must be printed to the screen describing the nature of the problem. For example, if the user entered a negative number for one of the dimensions, the error message "Program Aborted, Negative Dimension Entered" would appear on the screen.

I.15 Write a computer function subprogram to calculate sine according to the power series Equation I.6. The program should stop when an additional term changes the value by less than 0.0001% (1 part in a million). Call this function subprogram from a main program. Compare the value obtained by your function subprogram to the one obtained by calling the intrinsic function for $\sin x$.

I.16 Write a computer function subprogram to calculate cosine according to the power series Equation I.7. The program should stop when an additional term changes the value by less than 0.0001% (1 part in a million). Call this function subprogram from a main program. Compare the value obtained by your function subprogram to the one obtained by calling the intrinsic function for $\cos x$.