Appendix J

POLYNOMIALS AND COMPLEX NUMBERS

Polynomials are given by the following equation:

$$y = \sum_{i=1}^{n} a_i x^{i-1} = a_1 x^0 + a_2 x^1 + a_3 x^2 + \cdots + a_n x^{n-1} = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1} \tag{J-1}$$

Although the notation is very cumbersome, it is worth the effort to become comfortable with polynomials because they occur frequently in engineering. For example, the equation that describes the position of an object moving under the influence of gravity is a polynomial

$$x = x_0 + v_0 t + \tfrac{1}{2} a_0 t^2 \tag{J-2}$$

where x is the position at any time t, x_0 is the position at time zero, v_0 is the velocity at time zero, and a_0 is the acceleration. Although you will become more familiar with this equation in other chapters, the essential point is that this very important equation has the form of a polynomial.

The equation for a straight line

$$y = mx + b \tag{J-3}$$

is a very short polynomial with only two terms. In our discussion of least squares linear regression, we showed you techniques for finding the constants m and b that fit straight-line data. Similar techniques (that are beyond the scope of this book) exist for finding the polynomial constants that fit curved data.

Complex numbers have the form

$$x = a + bi \tag{J-4}$$

which looks very much like a polynomial, except that i is not a variable, but the constant $\sqrt{-1}$. Observant readers should find this odd, because it is impossible to multiply two numbers together to obtain negative one; therefore, complex numbers are sometimes called "imaginary numbers." Although it would seem that an "imaginary" number would have no place in the "real" world of engineering, they are surprisingly useful. For example, they are often used in mechanical and electrical engineering to describe angles, a topic you will explore in your advanced mathematics courses.

J.1 Polynomials

Polynomials have the following standard form:

$$y = P_n(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1 \tag{J-5}$$

where n is the "degree" and n-1 is the "order." Alternatively, polynomials may be expressed in their factored form:

$$y = P_n(x) = x(\cdots(x(a_n x + a_{n-1}) + \cdots a_2) + a_1 \tag{J-6}$$

An example of a fourth-order polynomial in its standard and factored form follows:

$$y = 7x^4 + 3x^3 + 6x^2 + 2x + 4$$

$$y = x(7x^3 + 3x^2 + 6x + 2) + 4$$

$$y = x(x(7x^2 + 3x + 6) + 2) + 4$$

$$y = x(x(x(7x + 3) + 6) + 2) + 4$$

Evaluating a polynomial is easily performed by substituting the x value into the equation. For example, the above polynomial is evaluated at $x = -1$

$$y = 7(-1)^4 + 3(-1)^3 + 6(-1)^2 + 2(-1) + 4 = 12$$

Because the exponents are all integers, the exponential terms are evaluated by multiplying the x values the specified number of times. With all the exponents, this calculation has a total of 13 multiplications. Because multiplication is a very time consuming operation for a computer, it is desirable to reduce the number of multiplications to a minimum. This can be accomplished by evaluating the factored form of this equation.

$$y = (-1)((-1)((-1)(7(-1) + 3) + 6) + 2) + 4 = 12$$

The factored form has only four multiplications, so this would be a substantially faster computer calculation.

J.1.1 Polynomial Addition/Subtraction

Polynomials may be added (or subtracted) by adding (subtracting) the coefficients of identical terms. The sum of the following two polynomials of unequal degree (i.e., $n > m$)

$$y_1 = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1 \tag{J-7}$$

$$y_2 = b_m x^{m-1} + b_{m-1}x^{m-2} + \cdots + b_2 x + b_1 \tag{J-8}$$

is

$$y_1 + y_2 = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + (a_m + b_m)x^{m-1} + (a_{m-1} + b_{m-1})x^{m-2} + \cdots + (a_2 + b_2)x + (a_1 + b_1) \tag{J-9}$$

For example, the sum of the following two equations

$$y_1 = 3x^4 + 7x^3 + 5x^2 + 2x + 1$$

$$y_2 = 2x^2 + 3x + 7$$

is

$$y_1 + y_2 = 3x^4 + 7x^3 + 7x^2 + 5x + 8$$

J.1.2 Polynomial Multiplication

Polynomials are multiplied by taking each term in the first polynomial and multiplying it times each term in the second polynomial. For example,

$$(4x + 7)(3x^2 + 2x + 1)$$

$$4x(3x^2 + 2x + 1) + 7(3x^2 + 2x + 1)$$

$$12x^3 + 8x^2 + 4x + 21x^2 + 14x + 7$$

$$12x^3 + 8x^2 + 21x^2 + 4x + 14x + 7$$

$$12x^3 + 29x^2 + 18x + 7$$

J.1.3 Factoring

Factoring is a process of decomposing a polynomial into other polynomials which, when multiplied, give the original polynomial. Table J.1 shows some common factors.

Table J.1 Factors (reference)	
$x^2 - y^2 =$	$(x - y)(x + y)$
$x^3 - y^3 =$	$(x - y)(x^2 + xy + y^2)$
$x^3 + y^3 =$	$(x + y)(x^2 - xy + y^2)$
$x^4 - y^4 =$	$(x - y)(x + y)(x^2 + y^2)$
$x^5 - y^5 =$	$(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$
$x^5 + y^5 =$	$(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$
$x^6 - y^6 =$	$(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
$x^4 + x^2y^2 + y^4 =$	$(x^2 + xy + y^2)(x^2 - xy + y^2)$
$x^4 + 4y^4 =$	$(x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$

J.1.4 Special Products

The special products shown in Table J.2 are calculated by the *binomial formula*

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n \qquad \text{(J-10)}$$

where n is an integer.

Table J.2 Special products (reference)	
$(x+y)^2 =$	$x^2+2xy+y^2$
$(x-y)^2 =$	$x^2-2xy+y^2$
$(x+y)^3 =$	$x^3+3x^2y+3xy^2+y^3$
$(x-y)^3 =$	$x^3-3x^2y+3xy^2-y^3$
$(x+y)^4 =$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$
$(x-y)^4 =$	$x^4-4x^3y+6x^2y^2-4xy^3+y^4$
$(x+y)^5 =$	$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$
$(x-y)^5 =$	$x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$
$(x+y)^6 =$	$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$
$(x-y)^6 =$	$x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$

J.2 Complex Numbers

A *complex number* x is composed of a *real part* a and an *imaginary part* b

$$x = a + bi \tag{J-11}$$

where

$$i = \sqrt{-1} \Rightarrow i^2 = -1 \tag{J-12}$$

Because it is impossible to multiply two numbers together to get -1, it is easy to see why b is called the *imaginary part.* Surprisingly, even though complex numbers have an imaginary part, they are often encountered in the solution of real engineering problems.

Table J.3 shows how common mathematical operations are performed on complex numbers

Table J.3 Mathematical operations with complex numbers (reference)		
Addition	$(a+bi)+(c+di) =$	$(a+c)+(b+d)i$
Subtraction	$(a+bi)-(c+di) =$	$(a-c)+(b-d)i$
Multiplication	$(a+bi)(c+di) =$	$(ac-bd)+(ad+bc)i$
Division	$\frac{a+bi}{c+di} =$	$\frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$

These relationships are derived by using ordinary algebraic rules and substituting -1 for i^2.

J.3 Summary

Polynomials have the general form

$$y = a_1 + a_2x + a_3x^2 + \cdot\cdot\cdot + a_nx^{n-1} \tag{J-1}$$

Polynomials may also be presented in their factored form, which requires fewer multiplications to evaluate. Various mathematical operations may be performed on polynomials, such as addition, subtraction, and multiplication. Factoring is a process of decomposing a polynomial into products of other polynomials. The binomial formula allows special products to be calculated.

Complex numbers have the form $x = a + b\,i$, where i is $\sqrt{-1}$. Because it is impossible to multiply two numbers together to obtain -1, b is called the imaginary part of the complex number. Various mathematical operations may be performed on complex numbers. Even though they have an imaginary part, complex numbers are surprising useful in the real world of engineering.

Further Readings

M. S. Spiegel, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline Series in Mathematics, McGraw-Hill, New York, 1968.

Handbook of Mathematical Formulas, Tables, Graphs, Functions, Transforms, Research and Education Association, New York, 1980.

Problems

J.1 Perform the following operations on polynomials:

a. $(3x^3+9x^2+5x+2)+(7x^3+8x^2+3x+9)$
b. $(5x^3+2x)+(5x^3+2x^2+9x+1)$
c. $(4x^4-3x+6)+(7x^2+5x+4)$
d. $(2x^3+4x^2+3x+5)-(8x^3+4x^2+7x+2)$
e. $(4x^3-9x^2-8x+5)-(-2x^3-4x^2+8x-3)$
f. $(4x+3)(3x^3+4x^2+2x+9)$
g. $(7x^2+4x+3)(8x^2+3x+4)$
h. $(8x^2+2x-1)(-3x^2+4x-2)$

J.2 Perform the following operations on complex numbers:

a. $(6+4i)+(3+3i)$
b. $(7-3i)-(2+4i)$
c. $(8-5i)(7+3i)$
d. $(8+3i)\div(2-i)$

J.3 Write a computer program that allows the user to input two complex numbers. The user then selects the operation to be performed on the two numbers (e.g., addition, subtraction, multiplication, division) and the computer prints the answer.

J.4 Write a computer program that calculates y given by the polynomial

$$y = 7x^7+8x^6+5x^5+9x^4-4x^3+7x^2-3x+1$$

The user may input any desired value for x. Write the program in two versions, one that calculates y directly by the given formula and another that evaluates it in the factor form. In each program, put the formula in a loop that repeats the calculation a large number of times (say 10,000) before printing the result. By comparing the length of time it takes to run each program, you can determine how efficient the factored form is relative to the standard form. Is the factored form 2 times faster, 3 times faster, or what?