Appendix L

CALCULUS

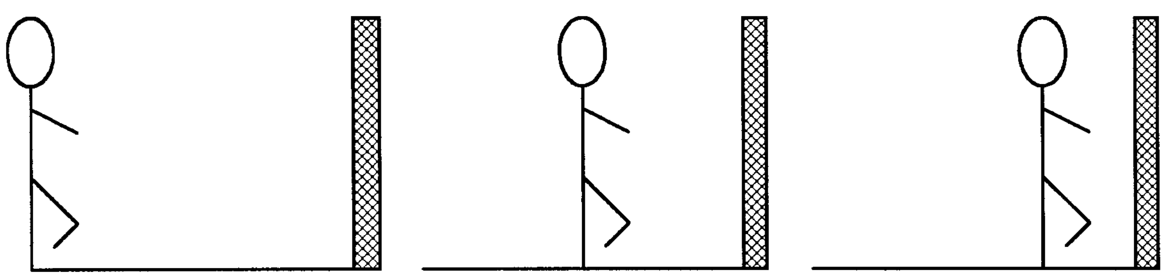
Calculus is sometimes called "*the* calculus." We don't say "*the* geometry," "*the* algebra," or "*the* trigonometry," so calculus seems to be elevated above the other branches of mathematics. What makes calculus so special?

Calculus *is* special because it is so powerful. In science and engineering, it is perhaps *the* most important branch of mathematics because so many of our physical laws can be expressed only in the language of calculus.

One problem with calculus is that students are intimidated by it. They hear about friends who may have made good engineers but could not pass calculus and had to change majors. We will not tell you that calculus is easy because it is a challenging subject. However, the difficulty with calculus is in the details, not the central ideas, for these are rather easily understood. It is our intent to highlight the central ideas and leave the details to your calculus course.

L.1 Limits

One of the central ideas of calculus is that of *limits*. A limit can be easily understood by considering the following example. Suppose you stand facing a wall and someone tells you to walk toward the wall, but cover only half the distance. While standing at your new position, he again asks you to walk toward the wall, but only cover half the distance.



This request is repeated over and over. Although he never told you to actually walk into the wall, for all practical purposes, you have. Therefore, the *limit* in this case is the wall.

L.2 The Small Numbers of Calculus

Calculus is the branch of mathematics that deals with exceedingly small numbers. For example, imagine we play a little game. We think of the smallest number we can, and you try to beat us by thinking of a smaller number. We open by saying "$x = 0.0000000000001$ is a very small number." You say "that's nothing, what about $x = 0.000000000000000000000000001$?" We say "Oh yeah, try to beat this

$$x = 0.000000000000000000000000000000000000000000000000000000000000000000000000001"$$

We could play this game forever and keep finding smaller and smaller numbers. Provided we never quite reach the *limit* of zero, we have identified the small numbers that are the subject of calculus. The way calculus denotes this very small x that approaches zero, but isn't quite zero, is with *differential x*, or dx. Similarly, a very small y that approaches zero, but isn't quite zero, is *differential y*, or dy. Mathematically, these ideas would be expressed as follows

$$dx = \lim_{x \to 0} x \qquad\qquad dy = \lim_{y \to 0} y$$

Because it is impossible to write down this number that approaches zero, but isn't quite zero, it has an aura of mystery. Although you should treat this very small number with proper respect, it still is just a number. If fact, it can even have dimensions or units attached to it, just like an ordinary number.

Let's go back to our game. After playing it for 2 years, we finally realize the futility of trying to think of the smallest number in the universe. We decide to have a truce and agree that 10 zeros is sufficient to describe this very small number. (We realize that we would actually need an infinite number of zeros to write this number, but this would take an infinite number of pages, which would require an infinite number of trees, which would cause an infinite number of environmentalists to be upset with us.) With this ground rule, we can do mathematical operations on our little numbers. For example, the equation

$$dy = 3 \cdot dx \tag{L-1}$$

has meaning. If we let dx be 0.00000000001, then dy can be calculated

$$dy = 3(0.00000000001) = 0.00000000003 \tag{L-2}$$

Similarly, we can divide our little numbers

$$\frac{dy}{dx} = \frac{0.00000000003}{0.00000000001} = 3 \tag{L-3}$$

Now suppose we renegotiate our truce and say that 15 zeros are required to represent our small numbers, but we still agree that dy is 3 times larger than dx. Then we can still divide our small numbers **and get the same answer**.

$$\frac{dy}{dx} = \frac{0.0000000000000003}{0.0000000000000001} = 3 \tag{L-4}$$

We are allowed to add (subtract) ratios of small numbers. For example, if we agree that the small number dz is 5 times larger than dx, and dy is three times larger than dx, then we can write the equation

$$a = \frac{dz}{dx} + \frac{dy}{dx} \tag{L-5}$$

and still get the same answer regardless of the number of zeros we use to represent our small numbers

$$a = \frac{0.00000000005}{0.00000000001} + \frac{0.00000000003}{0.00000000001} = 5 + 3 = 8 \qquad \text{(10-zero scale)} \tag{L-6}$$

$$a = \frac{0.0000000000000005}{0.0000000000000001} + \frac{0.0000000000000003}{0.0000000000000001} = 5 + 3 = 8 \qquad \text{(15-zero scale)} \qquad \text{(L-7)}$$

We can multiply both sides of Equation L-5 by dx and get

$$a\ dx = dz + dy \qquad \text{(L-8)}$$

This equation also gives consistent results at the 10-zero and 15-zero scales:

$$8(0.00000000001) = 0.00000000005 + 0.00000000003 \qquad \text{(L-9)}$$

$$8(0.00000000000000001) = 0.0000000000000005 + 0.0000000000000003 \qquad \text{(L-10)}$$

Now consider the following equation

$$a = dz + dy \qquad \text{(L-11)}$$

If we evaluate this equation at both scales

$$a = 0.00000000005 + 0.00000000003 = 0.00000000008 \qquad \text{(L-12)}$$

$$a = 0.000000000000005 + 0.0000000000000003 = 0.0000000000000008 \qquad \text{(L-13)}$$

we do not get consistent results. Therefore, this type of equation, which gets inconsistent results at the 10-zero and 15-zero scales, must be avoided.

In summary, we have seen that we can treat our small numbers like ordinary numbers, provided we exercise some care.

L.3 Branches of Calculus

Calculus is divided into two main branches. *Differential calculus* emphasizes the operations of subtraction and division whereas *integral calculus* emphasizes addition and multiplication. Differential calculus applies the mathematical operation of *differentiation* and integral calculus applies the operation of *integration*. As you will see, these operations are the antioperations of each other, just as addition is the antioperation of subtraction and subtraction is the antioperation of addition.

L.4 Differential Calculus

Differential calculus is based on the concept of *slope*. For linear equations, we understand slope to be "rise over run." As shown in Figure L.1, the slope of the linear equation

$$y = 2x + 1 \qquad \text{(L-14)}$$

is 2 and is everywhere the same.

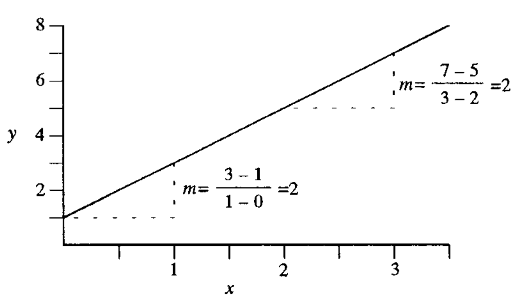


Figure L.1 The equation $y = 2x + 1$.

The slope is easy to determine for a linear equation, but what of nonlinear equations such as

$$y = x^2 \tag{L-15}$$

Figure L.2 is a plot of this function. It shows that as a region of the curve is sufficiently enlarged, the curve approximates a straight line, so the rise-over-run meaning of slope is retained. It must be pointed out, however, that the slope is *not* everywhere the same as with linear equations. The slope depends on where you measure it. An alternative method for finding the slope of a curve is shown in Figure L.3. A straight line is drawn tangent to the curve. The slope of the straight line is also the slope of the curve at the point of tangency.

Figure L.4 shows the function evaluated at two points, P and Q. P is kept fixed at (1,1) and Q varies. The slope m of the line connecting the two points is calculated as

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{L-16}$$

Table L.1 shows that as Q becomes closer to P (i.e., as Δx becomes smaller), the slope approaches a constant value of 2. In fact, as Δx becomes infinitesimally small (i.e., Δx becomes dx), the slope becomes exactly 2. In mathematical terms, this is stated as

$$\lim_{\Delta x \to 0} m = 2 \tag{L-17}$$

which says that in the limit, as Δx becomes zero, the slope becomes 2.

So far, we have determined the slope at a particular point. What if we wished to generalize the analysis? The slope is still defined the same way:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{L-18}$$

We can now substitute the equation $y = x^2$ and do some algebraic simplifications:

$$m = \frac{x_2^2 - x_1^2}{x_2 - x_1} = \frac{x_2^2 - x_1^2}{\Delta x} = \frac{(x_1 + \Delta x)^2 - x_1^2}{\Delta x} \tag{L-19}$$

$$m = \frac{(x_1^2 + 2x_1\Delta x + \Delta x^2) - x_1^2}{\Delta x} = \frac{\Delta x(2x_1 + \Delta x)}{\Delta x} \tag{L-20}$$

$$m = 2x_1 + \Delta x \tag{L-21}$$

Notice that this final equation agrees with the numerical results presented in Table L.1. If we were to take the *limit* as Δx goes to zero, then the Δx becomes a dx:

$$\lim_{\Delta x \to 0} m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 2x + dx \tag{L-22}$$

Because the dx is negligible compared to $2x$, then this equation simplifies to

$$\frac{dy}{dx} = 2x \tag{L-23}$$

In words, this equation says "the *first derivative* of $y = x^2$ with respect to x is $2x$." Other notations that are commonly encountered are

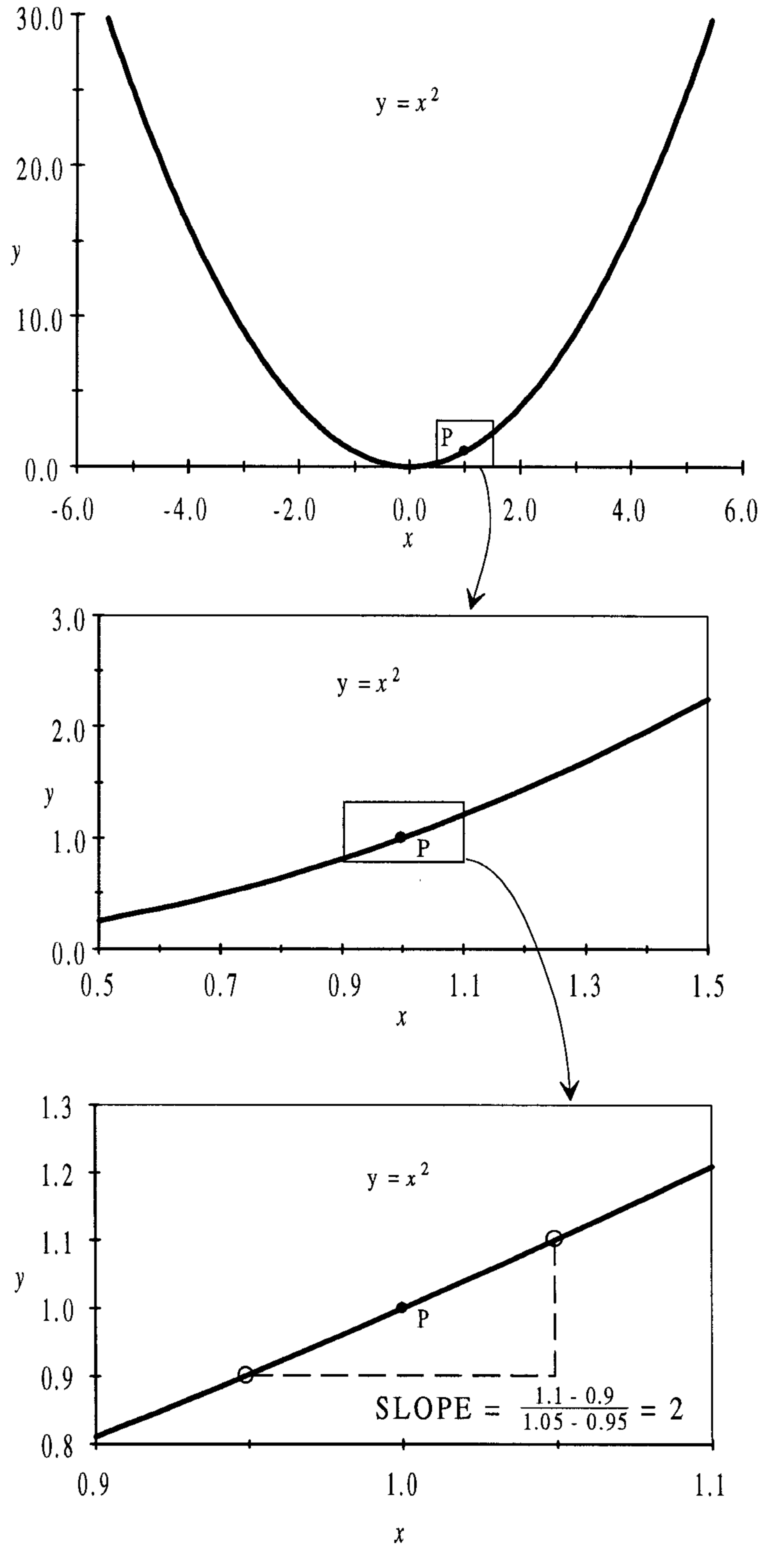


Figure L.2 Enlarged views of the function $y = x^2$.

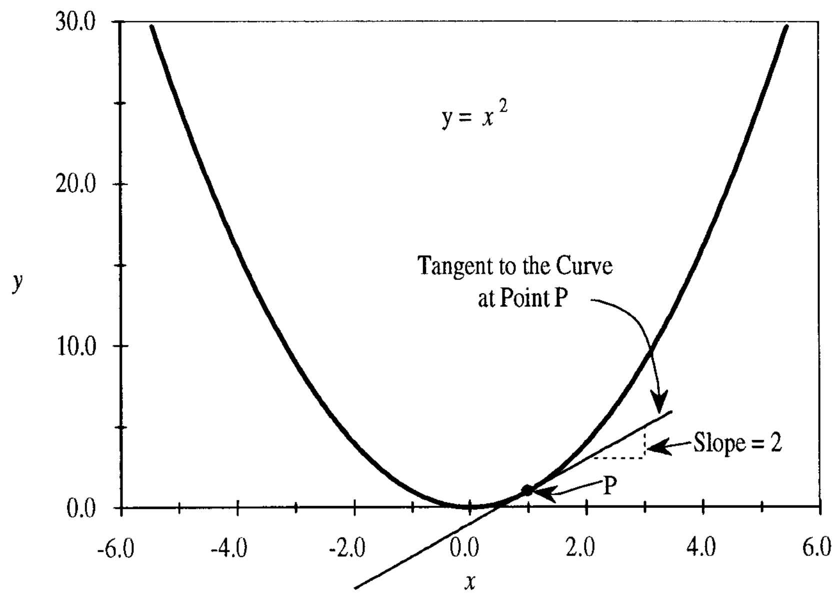


Figure L.3 Tangent to the function $y = x^2$.

Table L.1 Slope as Δx gets smaller					
	P		Q		
Δx	x_1	y_1	x_2	$y_2 = x^2$	slope= $\frac{\Delta y}{\Delta x}$
2	1	1	3	9	4
1	1	1	2	4	3
0.5	1	1	1.5	2.25	2.5
0.1	1	1	1.1	1.21	2.1
0.01	1	1	1.01	1.0201	2.01
0.001	1	1	1.001	1.002	2.001
0.0001	1	1	1.0001	1.0002000 1	2.0001
⇓	⇓	⇓			⇓
dx	1	1			2.00000

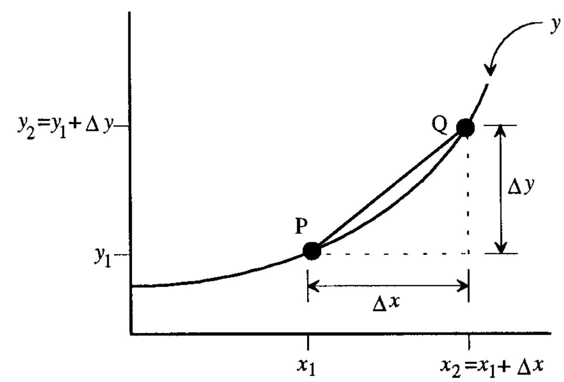


Figure L.4 The equation $y = x^2$.

$$\frac{d}{dx}(y) = 2x \qquad y' = 2x \qquad \dot{y} = 2x \qquad \text{(L-24)}$$

The first notation emphasizes the idea that differentiation *operates* on the original function $y = x^2$ to produce the new function $2x$. The y' and $\dot{y}$ notation are short-hand methods for indicating the first derivative.

The dimensions or units of the first derivative would be the same as those of y/x. For example, if y has units of meters and x has units of seconds, then $\frac{dy}{dx}$ has units of meters per second (m/s). The short-hand notation does not properly account for the dimensions or units associated with the variables y and x, whereas the other two notations do.

The *second derivative* repeats the derivative operation on our new function $2x$. This can be easily done by defining our new function as z

$$z = \frac{dy}{dx} = 2x \tag{L-25}$$

The derivative of z is

$$\frac{d}{dx}(z) = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{z_2 - z_1}{x_2 - x_1}\right) = \lim_{\Delta x \to 0} \left(\frac{2x_2 - 2x_1}{x_2 - x_1}\right) = \lim_{\Delta x \to 0} \left(\frac{2(x_1 + \Delta x) + 2x_1}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{2x_1 + 2\Delta x - 2x_1}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{2\Delta x}{\Delta x}\right) = 2\frac{dx}{dx} = 2 \tag{L-26}$$

The derivative is 2, just as we would expect because z is a linear equation.

The notations for the second derivative that operates on our original equation $y = x^2$ are

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2 \qquad \frac{d^2y}{dx^2} = 2 \qquad y'' = 2 \qquad \ddot{y} = 2 \tag{L-27}$$

The first notation shows the derivative operator $\frac{d}{dx}$ operating on the function $\frac{dy}{dx}$. The second notation is as though we multiplied the symbols of the first notation [the denominator is best visualized as $(dx)^2$]. The last two notations are shorthand; every time the derivative operation is repeated, another prime or dot is added.

The dimensions or units of the second derivative are the same as y/x^2. For example, if y had units of meters and x had units of seconds, then $\frac{d^2y}{dx^2}$ has units of meters per second squared (m/s^2). The two shorthand notations do not properly account for the dimensions or units associated with y and x, whereas the other two do.

The following equations summarize what we have done:

$$y = x^2 \qquad \text{(Original function)} \tag{L-28}$$

$$\frac{dy}{dx} = 2x \qquad \text{(First derivative)} \tag{L-29}$$

$$\frac{d^2y}{dx^2} = 2 \qquad \text{(Second derivative)} \tag{L-30}$$

L.4.1 Derivatives of Arbitrary Functions

So far, we have taken the derivative of a specific equation. Now, let's generalize to an arbitrary function $f(x)$:

$$y = f(x) \tag{L-31}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(x + dx) - f(x)}{dx} \tag{L-32}$$

L.4.2 Derivatives of Power Equation

The definition of the derivative may be applied to the power equation

$$y = ax^n \tag{L-33}$$

which is a generalization of the function $y = x^2$ that we have already studied. This function may be evaluated at x and $(x + dx)$ according to Equation L-32

$$\frac{dy}{dx} = \frac{a(x+dx)^n - ax}{dx} \tag{L-34}$$

By applying the *binomial formula* (see Appendix J), the first term in the numerator can be expanded

$$\frac{dy}{dx} = \frac{a\left(x^n + nx^{n-1}dx + \frac{n(n-1)}{2!}x^{n-2}(dx)^2 + \cdot\cdot\cdot + (dx)^n\right) - ax^n}{dx} \tag{L-35}$$

Multiplying the terms by a gives

$$\frac{dy}{dx} = \frac{ax^n + anx^{n-1}dx + \frac{an(n-1)}{2!}x^{n-2}(dx)^2 + \cdot\cdot\cdot + a(dx)^n - ax^n}{dx} \tag{L-36}$$

The first and last terms in the numerator cancel leaving

$$\frac{dy}{dx} = \frac{anx^{n-1}dx}{dx} + \frac{an(n-1)x^{n-2}}{2!}\frac{(dx)^2}{dx} + \cdot\cdot\cdot + \frac{a(dx)^n}{dx} \tag{L-37}$$

$$\frac{dy}{dx} = anx^{n-1} + \frac{an(n-1)x^{n-2}}{2!}dx + \cdot\cdot\cdot + a(dx)^{n-1} \tag{L-38}$$

The terms that contain dx (an infinitesimally small number) are negligible compared to the first term, so this simplifies to

$$\frac{dy}{dx} = anx^{n-1} \tag{L-39}$$

See for yourself that this equation is consistent with the equation $y = x^2$ that we first studied.

L.4.3 Derivative of Exponential Equations

The derivative of the exponential equation

$$y = b^x \tag{L-40}$$

proceeds in a similar manner. This function may be evaluated at x and $(x + dx)$ according to Equation L-32.

$$\frac{d}{dx}(b^x) = \frac{b^{x+dx} - b^x}{dx} = \frac{b^x b^{dx} - b^x}{dx} = \left(\frac{b^{dx} - 1}{dx}\right)b^x = kb^x \tag{L-41}$$

Here we have a very interesting result. When we take the derivative of b^x, we get the same function back times a constant. The problem is that the term in the brackets is very hard to evaluate. We know that dx is a very small number, so we can write it as

$$\frac{b^{\sim 0} - 1}{\sim 0} = \frac{(\sim 1) - 1}{\sim 0} = \frac{\sim 0}{\sim 0} = k \tag{L-42}$$

We are taking the ratio of two small numbers which makes it difficult to calculate. However, by applying some logic, we can evaluate it. We know that if the base b were 1, the value of k would be exactly zero.

$$\frac{1^{\sim 0} - 1}{\sim 0} = \frac{1 - 1}{\sim 0} = \frac{0}{\sim 0} = 0 \tag{L-43}$$

If b were a very large number (like ∞), then k would be a very large number (like ∞)

$$\frac{(\infty)^{-0}-1}{\sim 0}=\frac{\infty-1}{\sim 0}=\frac{\infty}{\sim 0}=\infty \tag{L-44}$$

Therefore, we know that k has a range from 0 to ∞. There must exist a base such that $k = 1$; let's call it base e. (Although you already know the numerical value for this base is the Euler number, pretend that you don't so we can have the satisfaction of showing you where e came from.) Therefore, if we take the derivative of e^x, we get e^x back!

$$\frac{d}{dx}(e^x)=e^x \tag{L-45}$$

The sidebar "Euler's Formula for e" shows a method to obtain the numerical value for e by using Euler's formula. Later, we will show you how to obtain a numerical value for e by using a power series. (Hopefully, you are getting the idea that power series are pretty important because they arise again and again.)

L.4.4 Derivative of Trigonometric Functions (Advanced Topic)

We can also take the derivative of trigonometric functions, such as sin θ. Figure L.5 shows two right triangles inscribed in a circle; one has an angle θ and the other has an angle (θ + dθ). A small triangle is also formed by these two larger triangles. This small triangle has two sides of dx and dy, and a hypotenuse of $rd\theta$. The hypotenuse is actually an arc from the large circle, but at this small scale, the arc is essentially a straight line. The length of the arc is $r\,d\theta$, provided the angle is measured in radians (recall that a radian is the pure number by which the radius is multiplied to obtain the swept circumference).

From the definition of sine, we can write the following two expressions

$$\sin\theta=\frac{y}{r} \tag{L-46}$$

$$\sin(\theta+d\theta)=\frac{y+dy}{r} \tag{L-47}$$

for the two large right triangles. The derivative of sine can be obtained from Equation L-32

$$\frac{d}{d\theta}(\sin\theta)=\frac{\sin(\theta+d\theta)-\sin\theta}{d\theta} \tag{L-48}$$

$$=\frac{\frac{y+dy}{r}-\frac{y}{r}}{d\theta}=\frac{y+dy-y}{rd\theta}=\frac{dy}{rd\theta} \tag{L-49}$$

By looking at the small triangle, we see that $\frac{dy}{r\,d\theta}$ is the definition of cos θ.

$$\frac{d}{d\theta}(\sin\theta)=\cos\theta \tag{L-50}$$

We can perform a similar analysis to find the derivative of cos θ. From the definition of cosine, we can write the following two equations that apply to the two large right triangles in Figure L.5.

$$\cos\theta=\frac{x}{r} \tag{L-51}$$

$$\cos(\theta+d\theta)=\frac{x-dx}{r} \tag{L-52}$$

Euler's Formula for e

In Equation L-41, we defined k as follows

$$k \equiv \frac{b^{dx} - 1}{dx}$$

Furthermore, we defined e as the base for which $k = 1$

$$1 = \frac{e^{dx} - 1}{dx}$$

This can be rearranged as follows

$$dx = e^{dx} - 1$$

$$1 + dx = e^{dx}$$

$$e = (1 + dx)^{1/dx}$$

Recall that dx is defined as follows

$$dx \equiv \lim_{x \to 0} x$$

This allows us to calculate the numerical value of e

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

If we define $n \equiv 1/x$, then we have an alternate method to calculate e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Even though it was discovered well before his birth, this formula for e is sometimes called "Euler's formula" in honor of Swiss mathematician Leonhard Euler (1707 – 1783).

The following table shows the numerical value for e calculated by these two formulas:

x	n	e
1	1	2.0000000000
0.1	10	2.5937424601
0.01	100	2.7048138294
0.001	1,000	2.7169239322
0.0001	10,000	2.7181459268
0.00001	100,000	2.7182682371
⇓	⇓	
0.000000	∞	2.7182818284

From Equation L-32, we can find the derivative of cosine

$$\frac{d}{d\theta}(\cos\theta) = \frac{\cos(\theta + d\theta) - \cos\theta}{d\theta} \tag{L-53}$$

$$= \frac{\frac{x - dx}{r} - \frac{x}{r}}{d\theta} = \frac{x - dx - x}{r\,d\theta} = -\frac{dx}{r\,d\theta} \tag{L-54}$$

It's a Matter of Interest

When money is borrowed, it is necessary to pay a fee for the use of the money; this fee is called *interest.* The amount of money that is borrowed is called the *principal.* Normally, interest is expressed as a percentage of the principal.

Suppose you borrow \$1000 for 2 years at 10% interest per year. At the end of the first year, you owe not only the principal, but you also owe \$100 interest, so your loan is really \$1100. At the end of the second year, you will repay the \$1100 plus the \$110 interest. This process of refiguring the amount of the loan is called *compounding.* In this case, the loan was compounded each year. In general, the amount to be repaid can be calculated by the following compound-interest formula:

$$S = P(1+i)^t \qquad \text{(Compound interest, 1 compounding per year)}$$

where S is the amount repaid, P is the principal, t is the time (in years), and i is the annual interest rate expressed as a fraction (in this case, 0.10).

Instead of being compounded once a year, the loan could be compounded every month. In this case, the amount to be repaid can be calculated by

$$S = P\left(1+\frac{i}{m}\right)^{mt} \qquad \text{(Compound interest, } m \text{ compoundings per year)}$$

where m is the number of compoundings each year (in the case of monthly compounding, $m = 12$).

Rather than compound every month, we could envision a process whereby the compounding occurs continuously. In this case, there would be an infinite number of compoundings each year

$$S = \lim_{m\to\infty} P\left(1+\frac{i}{m}\right)^{mt} \qquad \text{(Compound interest, infinite compoundings per year)}$$

By substituting the following definition

$$\frac{1}{n} \equiv \frac{i}{m}$$

we obtain

$$S = \lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{it}$$

Prior to 1618, mathematicians involved with calculating interests noted that the term in the square bracket reaches a limiting value of 2.71828 · · ·, a number which, in 1727, Leonhard Euler (1707 – 1783) christened e

$$S = Pe^{it} \qquad \text{(Continuous compound interest)}$$

By looking at the small triangle, we see that $\frac{dx}{r\,d\theta}$ is the definition of $\sin\theta$. Therefore, the derivative of cosine is

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta \qquad \text{(L-55)}$$

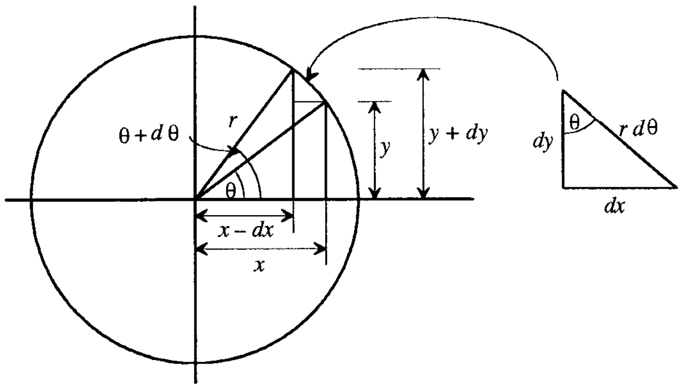


Figure L.5 Derivative of sine and cosine.

L.4.5 Summary of Derivatives

Table L.2 summarizes the derivatives we have just derived. The table also includes some other important derivatives. If you wish to see how they were derived, please consult a calculus text.

Derivative 2 in Table L.2 is worth remembering. It says "the derivative of a constant times a function is the constant times the derivative of the function." Also, Derivative 3 is very important. It says "the derivative of a sum is the sum of the derivatives."

Table L.2 Summary of derivatives and integrals (reference)		
	$e = 2.718281828459045\cdots$	$u = f_1(x)$
	a and n = constant	$v = f_2(x)$
	x = variable	C = Constant of integration
	Derivatives	**Integrals**
1	$\frac{d}{dx}(x) = 1$	$\int dx = x + C$
2	$\frac{d}{dx}(au) = a\frac{du}{dx}$	$\int au\, dx = a\int u\, dx$
3	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$	$\int (u+v)\, dx = \int u\, dx + \int v\, dx$
4	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
5	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$
6	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
7	$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
8	$\frac{d}{dx}(a^x) = (\ln a)a^x$	$\int a^x dx = \left(\frac{1}{\ln a}\right)a^x + C$
9	$\frac{d}{dx}(\sin x) = \cos x$	$\int \sin x\, dx = -\cos x + C$
10	$\frac{d}{dx}(\cos x) = -\sin x$	$\int \cos x\, dx = \sin x + C$
11	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \tan x\, dx = \ln\lvert\sec x\rvert + C$

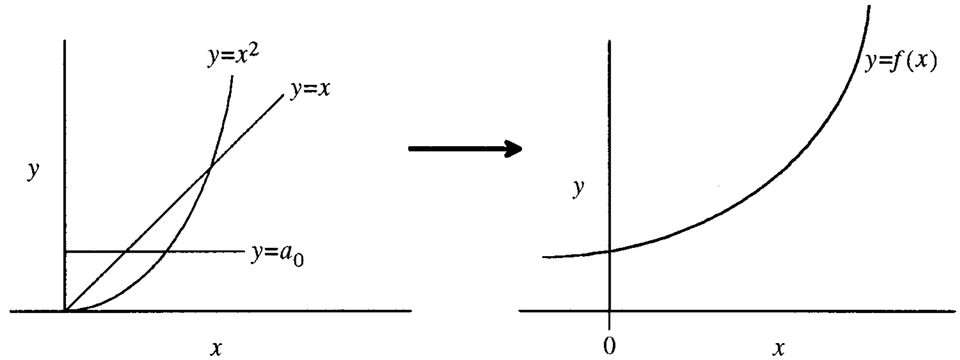


Figure L.6 Power series approximation to $y = f(x)$.

L.5 Power Series (Advanced Topic)

A *power series* starts from the premise that many arbitrary functions $y=f(x)$ can be represented by adding terms containing increasing powers of x

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n \tag{L-56}$$

(see Figure L.6). Each term must be scaled by an appropriate constant, a_n; the challenge is to find the appropriate constants.

To illustrate the idea of a power series with a concrete example, let's suppose we wished to express the polynomial

$$y = f(x) = 10 + 7x + 2x^2 + 4x^3 \tag{L-57}$$

as a power series. It is obvious by looking at Equations L-56 and L-57 that the appropriate constants are $a_0 = 10$, $a_1 = 7$, $a_2 = 2$, and $a_3 = 4$. However, we will treat this example equation as we would any other equation and demonstrate that the power series can accurately describe it.

If the power series accurately represents the proposed arbitrary function, then it must be true that the derivatives of the arbitrary function and the derivatives of the power series are identical

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n \qquad y = f(x) = 10 + 7x + 2x^2 + 4x^3 \tag{L-58}$$

$$y' = f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} \qquad y' = f'(x) = 7 + 4x + 12x^2 \tag{L-59}$$

$$y'' = f''(x) = 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2} \qquad y'' = f''(x) = 4 + 24x \tag{L-60}$$

$$y''' = f'''(x) = 6a_3 + \cdots + n(n-1)(n-2)a_nx^{n-3} \qquad y''' = f'''(x) = 24 \tag{L-61}$$

$$\Downarrow$$

$$y^{(n)} = f^{(n)}(x) = n!a_n \tag{L-62}$$

If we evaluate these equations at $x = 0$, the values of the constants are easily calculated

$$f(0) = a_0 \qquad\qquad a_0 = f(0) = 10$$

$$f'(0) = a_1 \qquad\qquad a_1 = f'(0) = 7$$

$$f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2} \qquad\qquad a_2 = \frac{f''(0)}{2} = \frac{4}{2} = 2$$

$$f'''(0) = 6a_3 \Rightarrow a_3 = \frac{f'''(0)}{6} \qquad\qquad a_3 = \frac{f'''(0)}{6} = \frac{24}{6} = 4$$

$$\Downarrow$$

$$f^{(n)}(0) = n!a_n \Rightarrow a_n = \frac{f^{(n)}(0)}{n!}$$

By substituting the constants for our example into Equation L-56, we get

$$y = f(x) = 10 + 7x + 2x^2 + 4x^3 \qquad \text{(L-57)}$$

which is just what we expected. The general equation for a power series representation of $y = f(x)$ is

$$y = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdot\cdot\cdot + \frac{f^{(n)}(0)}{n!}x^n \qquad \text{(L-63)}$$

$$y = f(x) = 10 + 7x + 2x^2 + 4x^3 \qquad \text{(L-57)}$$

The implications of this equation are very significant. It says that if an arbitrary function can be differentiated and evaluated at $x = 0$, then this function can be evaluated at any x!

Throughout our mathematics discussion, you have been introduced to *transcendental functions*, such as e^x and $\sin x$. With power functions, now we can actually evaluate these transcendental functions. For example, we have surmised that there exists a base e such that

$$\frac{d}{dx}(e^x) = e^x \qquad \text{(L-45)}$$

Because we get back the same function we started with, it is very easy to take the derivative multiple times and evaluate all of them at $x = 0$

$$f(0) = e^0 = 1$$

$$f'(0) = e^0 = 1$$

$$f''(0) = e^0 = 1$$

$$\Downarrow$$

$$f^{(n)}(0) = e^0 = 1$$

These evaluations of the derivative may be substituted into Equation L-63 so the transcendental function e^x can be determined

$$y = f(x) = e^x = 1 + 1x + \frac{1}{2!}x^2 + \cdot\cdot\cdot + \frac{1}{n!}x^n \qquad \text{(L-64)}$$

As more terms are used, the power series more accurately represents the function e^x.

To determine the value of the base e, we merely have to evaluate this function at $x = 1$:

$$e = e^1 = 1 + 1(1) + \frac{1}{2!}(1)^2 + \cdots + \frac{1}{n!}(1)^n = 2.718281828459045 \cdots \qquad \text{(L-65)}$$

We can perform a similar analysis to evaluate the transcendental function sin x simply by knowing that the derivative of sin x is cos x and the derivative of cos x is –sin x. We must also know that sin (0) is 0 and cos (0) is 1.

$$f(0) = \sin(0) = 0$$

$$f'(0) = \cos(0) = 1$$

$$f''(0) = -\sin(0) = 0$$

$$f'''(0) = -\cos(0) = -1$$

$$f''''(0) = \sin(0) = 0$$

etc.

Therefore the series representation of sin x may be determined by substituting these expressions into Equation L-63:

$$y = f(x) = \sin x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{(L-66)}$$

Table L.3 shows the power series for many functions commonly encountered in engineering.

Table L.3 Power series representation of transcendental functions (reference)	
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$-\infty < x < \infty$
$a^x = e^{x \ln a} = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \cdots$	$-\infty < x < \infty$
$\ln x = 2\left[\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right]$	$x > 0$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$-\infty < x < \infty$
$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!}$	$\lvert x \rvert < \frac{\pi}{2}$
$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{2^{2n}B_n x^{2n-1}}{(2n)!}$	$0 < \lvert x \rvert < \pi$
$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{E_n x^{2n}}{(2n)!}$	$\lvert x \rvert < \frac{\pi}{2}$
$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15,120} + \cdots + \frac{2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!}$	$0 < \lvert x \rvert < \pi$
$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots$	$\lvert x \rvert < 1$

Table L.3 (Continued)	
$\cos^{-1}x=\frac{\pi}{2}-\left(x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+\cdots\right)$	$\lvert x\rvert<1$
$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\lvert x\rvert<1$
$\tan^{-1}x=\pm\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\cdots$	$+$ if $x\geq 1$, $-$ if $x\leq -1$
$\cot^{-1}x=\frac{\pi}{2}-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)$	$\lvert x\rvert<1$
$\cot^{-1}x=p\pi+\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\cdots$	$p=0$ if $x>1$, $p=1$ if $x<-1$
$\sec^{-1}x=\frac{\pi}{2}-\left(\frac{1}{x}+\frac{1}{2\cdot 3x^3}+\frac{1\cdot 3}{2\cdot 4\cdot 5x^5}+\cdots\right)$	$\lvert x\rvert>1$
$\csc^{-1}x=\frac{1}{x}+\frac{1}{2\cdot 3x^3}+\frac{1\cdot 3}{2\cdot 4\cdot 5x^5}+\cdots$	$\lvert x\rvert>1$
$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$	$-\infty<x<\infty$
$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$	$-\infty<x<\infty$
$\tanh x=x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\cdots+\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!}$	$\lvert x\rvert<\frac{\pi}{2}$
$\coth x=\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}+\cdots+\frac{(-1)^{n-1}2^{2n}B_n x^{2n-1}}{(2n)!}$	$0<\lvert x\rvert<\pi$
$\mathrm{sech}x=1-\frac{x^2}{2}+\frac{5x^4}{24}-\frac{61x^6}{720}+\cdots+\frac{(-1)^n E_n x^{2n}}{(2n)!}$	$\lvert x\rvert<\frac{\pi}{2}$
$\mathrm{csch}x=\frac{1}{x}-\frac{x}{6}+\frac{7x^3}{360}-\frac{31x^5}{15,120}+\cdots+\frac{(-1)^n 2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!}$	$0<\lvert x\rvert<\pi$
$\sinh^{-1}x=x-\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}-\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots$	$\lvert x\rvert<1$
$\sinh^{-1}x=\pm\left(\ln\lvert 2x\rvert+\frac{1}{2\cdot 2x^2}-\frac{1\cdot 3}{2\cdot 4\cdot 4x^4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6}-\cdots\right)$	$+$ if $x\geq 1$, $-$ if $x\leq -1$
$\cosh^{-1}x=\left[\ln(2x)-\left(\frac{1}{2\cdot 2x^2}+\frac{1\cdot 3}{2\cdot 4\cdot 4x^4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6}+\cdots\right)\right]$	if $\cosh^{-1}x>0, x\geq 1$
$\cosh^{-1}x=-\left[\ln(2x)-\left(\frac{1}{2\cdot 2x^2}+\frac{1\cdot 3}{2\cdot 4\cdot 4x^4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6}+\cdots\right)\right]$	if $\cosh^{-1}x<0, x\geq 1$
$\tanh^{-1}x=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$	$\lvert x\rvert<1$
$\coth^{-1}x=\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}+\cdots$	$\lvert x\rvert>1$
$B_n=\frac{(2n)!}{2^{2n-1}\pi^{2n}}\left(1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots\right)$ (Bernoulli numbers)	
$E_n=\frac{2^{2n+2}(2n)!}{\pi^{2n+1}}\left(1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}+\cdots\right)$ (Euler numbers)	

L.6 Integral Calculus

The central idea of *integral calculus* is to determine the area under a curve. For example, we would use integral calculus to find the area under the function

$$y = kx^2 \qquad \text{(L-67)}$$

between $x = 1$ and $x = 2$. Most equations in engineering have units associated with the variables. In this example, we will assume x and y each have units of ft, and that k has units of ft^{-1}. To keep the mathematics simple, we will assume that $k = 1\ ft^{-1}$ so that Equation L-67 may be written as

$$y = x^2 \qquad \text{(L-68)}$$

There are many approaches to finding the area under this equation:

L.6.1 Method 1 (Counting Boxes)

This method involves plotting the function on graph paper and counting the number of boxes under the curve. In Figure L.7, there are

$$6(1) + 0.75 + 0.33 + 0.04 + 0.95 + 0.68 + 0.39 + 0.11 = 9.25 \text{ boxes.}$$

It is necessary to calibrate the box area to the function. For example, in Figure L.7, four boxes have an area of 1 ft^2. Therefore, the total area under the curve is estimated to be

$$A = 9.25 \text{ boxes} \times \frac{1\ \text{ft}^2}{4 \text{ boxes}} = 2.31\ \text{ft}^2$$

L.6.2 Method 2 (Experimental Measurement)

This method experimentally measures the area under the curve. One approach is to cut a piece of tracing paper with unknown area A_1 to fit under the curve (see Figure L.8). This is then weighed on an analytical balance to obtain weight W_1. To calibrate the weight, another piece of tracing paper of identical thickness to the first is cut to have a known area A_2 and weighed to obtain weight W_2. The unknown area A_1 is obtained by setting up the following proportion

$$\frac{A_1}{A_2} = \frac{W_1}{W_2} \qquad \text{(L-69)}$$

This may be solved explicitly for A_1 and the weight measurements substituted

$$A_1 = \frac{W_1}{W_2} A_2 = \frac{0.125\ \text{g}}{0.053\ \text{g}}\left(1\ \text{ft}^2\right) = 2.36\ \text{ft}^2$$

Because paper adsorbs moisture from the air, this method can be improved by using plastic film rather than paper. Another experimental approach is to use a *planimeter*, which is a mechanical device for measuring area by tracing the surface. There are also computer planimeters in which the object with the unknown area is placed on a digitizing tablet and traced with a wand.

L.6.3 Method 3 (Average Height)

This method approximates the area under the curve with a single rectangle of height y_{ave} and width $(b - a)$ (see Figure L.9). The area is calculated according to the formula

$$A = y_{\text{ave}} \Delta x = y_{\text{ave}}(b - a) \qquad \text{(L-70)}$$

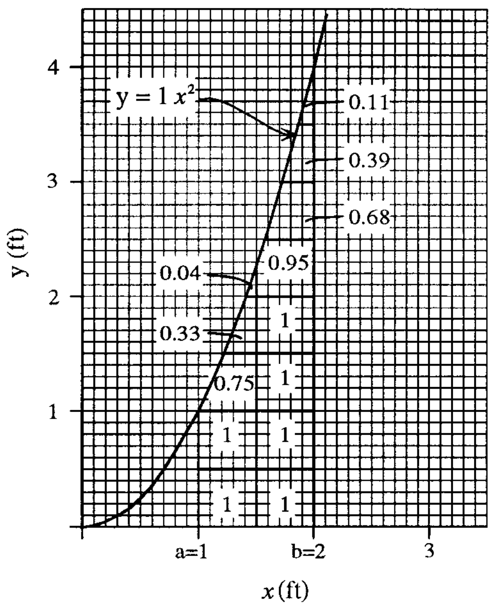


Figure L.7 Method 1 (counting boxes).

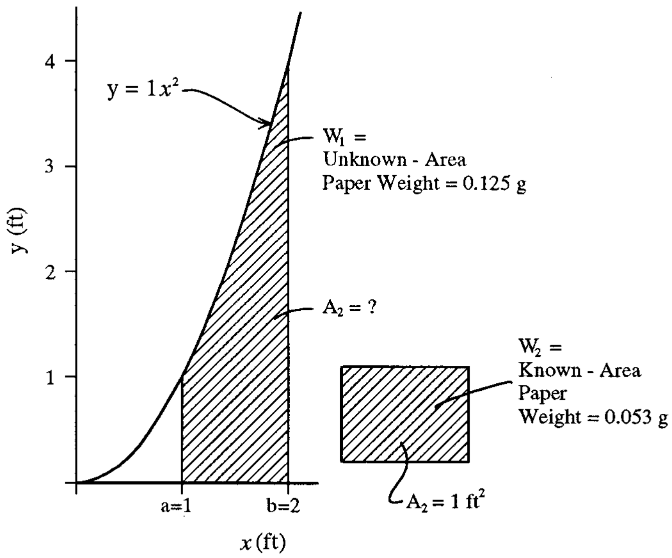


Figure L.8 Method 2 (experimental measurement).

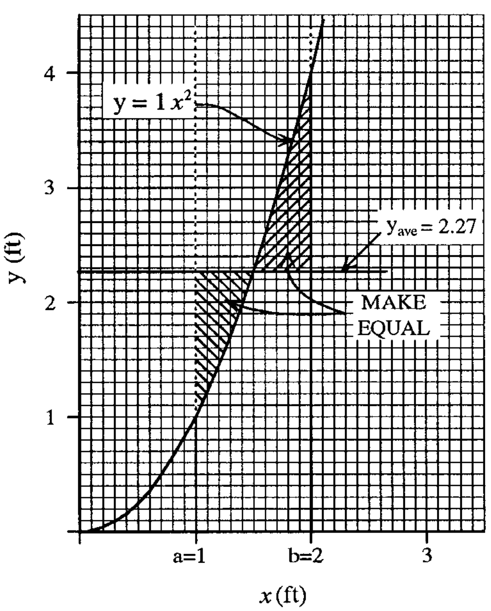


Figure L.9 Method 3 (average height).

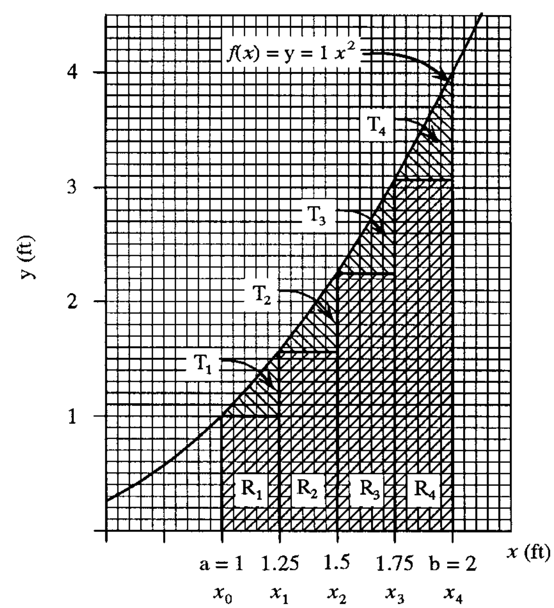


Figure L.10 Method 4 (trapezoidal rule).

The difficulty is to find the appropriate average height. This can be accomplished by drawing a horizontal line such that the two areas between the curve and the horizontal line are equal. By eye, the appropriate average height is 2.27 ft, so the area under the curve is

$$A = (2.27 \text{ ft})(2 \text{ ft} - 1 \text{ ft}) = 2.27 \text{ ft}^2$$

L.6.4 Method 4 (Trapezoidal Rule)

This method approximates the area under the curve with a series of rectangles and triangles (see Figure L.10). The interval from a to b is divided into n equal divisions, so the width of the rectangles Δx can be calculated according to

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = 0.25 \tag{L-71}$$

The total area under the curve is the sum of the rectangle area R and the triangle area T.

$$A = [R_1 + T_1] + [R_2 + T_2] + [R_3 + T_3] + [R_4 + T_4] \tag{L-72}$$

The rectangle and triangle areas are determined as follows

$$A = \left[f(x_0)\Delta x + \frac{f(x_1) - f(x_0)}{2}\Delta x\right] + \left[f(x_1)\Delta x + \frac{f(x_2) - f(x_1)}{2}\Delta x\right] + \left[f(x_2)\Delta x + \frac{f(x_3) - f(x_2)}{2}\Delta x\right]$$

$$+\left[f(x_3)\Delta x + \frac{f(x_4) - f(x_3)}{2}\Delta x\right] \tag{L-73}$$

which simplifies to

$$A = \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + f(x_3) + \frac{f(x_4)}{2}\right]\Delta x \tag{L-74}$$

Numerical values for this example may be substituted

$$A = \left[\frac{(1)^2}{2} + (1.25)^2 + (1.5)^2 + (1.75)^2 + \frac{(2)^2}{2}\right](0.25)$$

$$= [0.5 + 1.5625 + 2.25 + 3.06 + 2](0.25) = 2.343125 \text{ ft}^2$$

The more general statement of the trapezoidal rule is

$$A = \left[\frac{f(x_0)}{2} + \sum_{i=1}^{n-1} f(x_i) + \frac{f(x_n)}{2}\right]\Delta x \tag{L-75}$$

where n is the number of segments into which the function is divided. Program L.1 shows an example of the trapezoidal rule.

L.6.5 Method 5 (Simpson's Rule)

This method improves upon the trapezoidal rule by approximating each segment of the function with a parabola rather than a straight line. Whereas the trapezoidal rule connected the endpoints of each segment with a straight line

$$y = mx + b \tag{L-76}$$

Simpson's Rule connects the endpoints of each segment with a parabola

$$y = nx^2 + mx + b \tag{L-77}$$

Because parabolas are curved, they can more closely approximate the function. According to Simpson's rule, the area under a function can be estimated by

$$A = \frac{\Delta x}{3}\left\{f(x_0) + 4\left[\sum_{\text{odd } i=1}^{n-1} f(x_i)\right] + 2\left[\sum_{\text{even } i=2}^{n-2} f(x_i)\right] + f(x_n)\right\} \quad \text{(L-78)}$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{(L-79)}$$

and n must be even. We can apply this rule to determine the area under the function we have been studying, i.e., $y = x^2$.

$$\Delta x = \frac{2-1}{4} = 0.25$$

$$A = \frac{0.25}{3}[(1)^2 + 4((1.25)^2 + (1.75)^2) + 2((1.5)^2) + (2)^2] = 2.3333333$$

Program L.2 shows an example of Simpson's rule.

Program L.1 Trapezoidal program.

```
      program trapezoidal

*     This program uses the trapezoidal rule to approximate the area under the
*     curve f(x) = x**2 on the interval from x = 1 to 2. The function f is
*     written as a statement function. n is the number of intervals, 60, used
*     in the computation. a and b are the limits of integration.

      implicit none

      integer n, i
      real area, a, b, f, x, xi, delta_x

      f(x)=x**2

      a=1.0
      b=2.0
      n=60

      delta_x=(b-a)/real(n)
      area=0.0

      do i=1, n-1
        xi=a+(real(i)*delta_x)
        area=area+f(xi)
      end do

      area=(f(a)/2.0+area+f(b)/2.0)*delta_x

      write(*,*)'The area under f is approximately equal to: ',area

      end
```

Program L.2 Simpson's rule.

```
      program simpson

*     This program uses Simpson's rule to approximate the area under the
*     curve f(x) = x**2 on the interval from x = 1 to 2. The function f is
*     written as a statement function. n is the number of intervals, 60, used
*     in the computation. a and b are the limits of integration.

      implicit none

      integer n, i

      real f, x, a, b, delta_x, y(0:99), area, even, odd

      f(x)=x**2

      a=1.0
      b=2.0
      n=60
      area=0.0

      delta_x=(b-a)/real(n)

      do i=0,n
        x=a+delta_x*real(i)
        y(i)=f(x)
      end do

      odd=0.0
      do i=1,n-1,2
        odd=odd+y(i)
      end do

      even=0.0
      do i=2,n-2,2
        even=even+y(i)
      end do

      area=(delta_x/3.0)*(f(a)+4.0*odd+2.0*even+f(b))

      write(*,*)'The area under f is approximately equal to: ',area

      end
```

L.6.6 Method 6 (Average-Height Rectangles)

This method divides the area into rectangles (see Figure L.11). The area under the curve is approximately the sum of all the rectangle areas:

$$A = R_1 + R_2 + R_3 + R_4 \tag{L-80}$$

The width of each rectangle is given by Equation L-79. The height of each rectangle is equal to the average value of the function evaluated at the endpoints of each segment. Therefore, the area A is

$$A = \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \frac{f(x_2)+f(x_3)}{2}\Delta x + \frac{f(x_3)+f(x_4)}{2}\Delta x \tag{L-81}$$

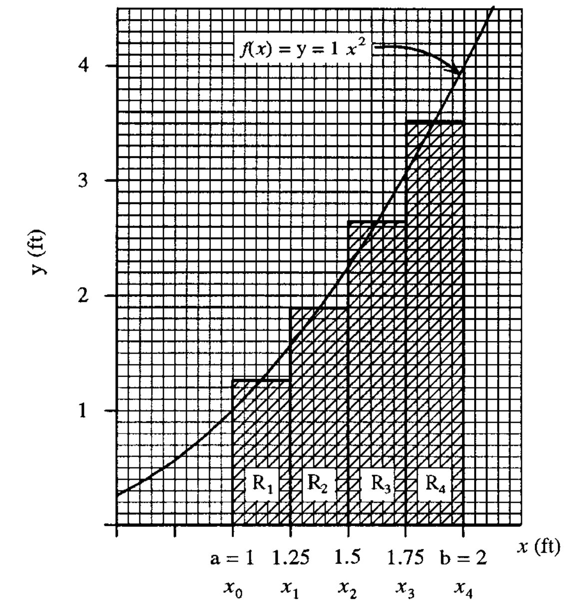


Figure L.11 Method 6 (average-height rectangles).

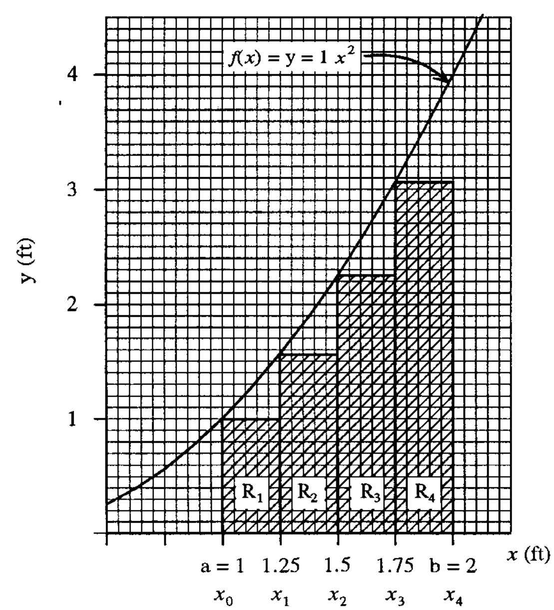


Figure L.12 Method 7 (coarse inscribed rectangles).

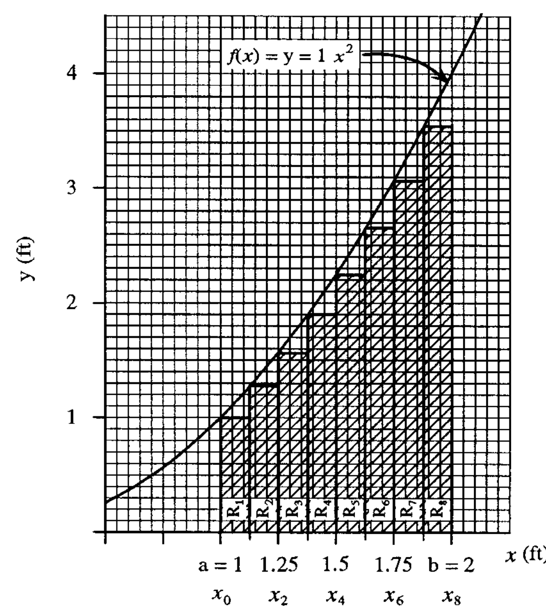


Figure L.13 Method 7 (fine inscribed triangles).

which may be simplified to

$$A = \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + f(x_3) + \frac{f(x_4)}{2}\right]\Delta x \qquad \text{(L-82)}$$

which is the same as the trapezoidal rule.

L.6.7 Method 7 (Inscribed Rectangles)

This method approximates the area with inscribed rectangles (see Figure L.12). The total area is approximately the sum of all the rectangle areas.

$$A = R_1 + R_2 + R_3 + R_4 \tag{L-83}$$

The width of each rectangle is given by Equation L-79. The height of each rectangle is equal to the function evaluated at the lower bound of the segment. (*Note*: If the function had a negative slope, the height would be evaluated at the upper bound of the segment.)

$$A = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \tag{L-84}$$

which simplifies to

$$A = [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\Delta x = \left[\sum_{i=0}^{3} f(x_i)\right]\Delta x = \left[\sum_{i=0}^{3} x_i^2\right]\Delta x \tag{L-85}$$

Our function $y = x^2$ may now be evaluated as

$$A = [(1)^2 + (1.25)^2 + (1.5)^2 + (1.75)^2](0.25) = 1.97$$

This answer is not very close to the actual value because there is so much area not covered by the rectangles.

If we were to decrease the width of the rectangles and increase their number, we would cover more area under the curve (see Figure L.13). Now the area under the curve is

$$A = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 \tag{L-86}$$

The width of each rectangle is

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{8} = 0.125 \tag{L-87}$$

and the height is the function evaluated at the lower bound of the segment

$$A = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x \tag{L-88}$$

which simplifies to

$$A = [f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)]\Delta x = \left[\sum_{i=0}^{7} f(x_i)\right]\Delta x = \left[\sum_{i=0}^{7} x_i^2\right]\Delta x \tag{L-89}$$

Our function $y = x^2$ may now be evaluated.

$$A = [(1)^2 + (1.125)^2 + (1.25)^2 + (1.375)^2 + (1.5)^2 + (1.625)^2 + (1.75)^2 + (1.875)^2](0.125) = 2.148$$

We can see that by decreasing the width of Δx, we have more closely approximated the area under the curve.

L.6.8 Method 8 (Integration)

This method carries the process of decreasing Δx to an infinite degree, that is, Δx becomes dx. When this is done, the *area under the curve becomes exact.* A special notation is required for this process as shown below:

$$A_a^b = \lim_{\Delta x \to 0,\, n \to \infty} \left[\sum_{i=0}^{n-1} f(x_i)\right]\Delta x = \int_a^b f(x)\, dx \tag{L-90}$$

In words, this says "the area under the function $f(x)$ from a to b (i.e., A_a^b) is the function integrated from the lower bound a to the upper bound b with respect to x." The integral sign $\int$ looks like the letter s for sum, which is very appropriate because integration is actually a summation process.

It's wonderful to have this fancy notation for finding the area under a curve, but unless it can be evaluated, it's useless. Fortunately, we can accomplish this by considering Figure L.14. We observe that a vertical line with length $f(b) - f(a)$ is the sum of the "rises" of each slope; that is,

$$f(b) - f(a) = \Delta y_1 + \Delta y_2 + \Delta y_3 + \Delta y_4 \tag{L-91}$$

We can divide both sides by Δx:

$$\frac{f(b) - f(a)}{\Delta x} = \frac{\Delta y_1}{\Delta x} + \frac{\Delta y_2}{\Delta x} + \frac{\Delta y_3}{\Delta x} + \frac{\Delta y_4}{\Delta x} = \sum_{i=1}^{4} \frac{\Delta y_i}{\Delta x} \tag{L-92}$$

If we multiply both sides by Δx, take the limit of this summation as Δx goes to zero and n goes to infinity, then we have

$$f(b) - f(a) = \lim_{\Delta x \to 0,\, n \to \infty} \left[\sum_{i=1}^{n} \frac{\Delta y_i}{\Delta x} \right] \Delta x = \int_a^b \left(\frac{dy}{dx} \right) dx = \int_a^b f'(x)\ dx \tag{L-93}$$

This is an amazing result! It says in order to integrate the function

$$\int_a^b f'(x)\ dx \tag{L-94}$$

it is necessary to find a complementary function $f(x)$, where $f(x)$ and $f'(x)$ are related as follows:

$$f'(x) = \frac{d}{dx}(f(x)) \tag{L-95}$$

Once this complementary function is identified, the integral can be evaluated by simply evaluating this complementary function at the upper and lower bounds.

For example, we have been evaluating the area under the function $y = x^2$ from $x = 1$ to $x = 2$. That is to say,

$$f'(x) = x^2 \tag{L-96}$$

The challenge is to find a complementary function $f(x)$ such that

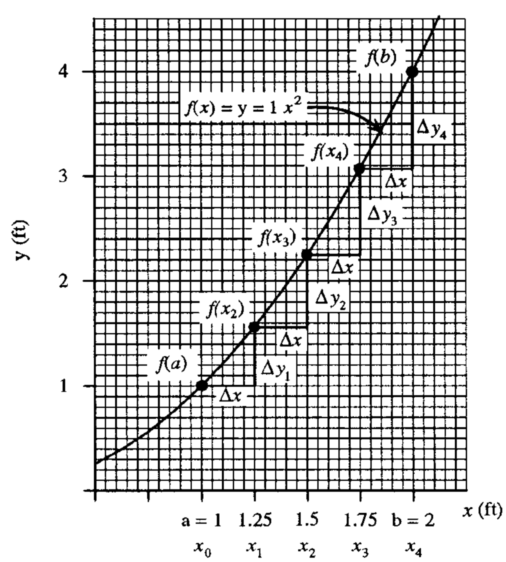


Figure L.14 The vertical line of length $f(b) - f(a)$ is the sum of the rises Δy_1, Δy_2, Δy_3, and Δy_4.

$$\frac{d}{dx}(f(x)) = f'(x) = x^2 \quad \text{(L-97)}$$

Fortunately this problem is simple enough that $f(x)$ can be found by inspection

$$f(x) = \frac{1}{3}x^3 + C \quad \text{(L-98)}$$

where C is called the *constant of integration.* We can check if we got the right function by differentiating it:

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + C\right) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) + \frac{d}{dx}(C) = \frac{1}{3}\frac{d}{dx}(x^3) + 0 = \frac{1}{3}(3x^2) = x^2 \quad \text{(L-99)}$$

Because we have verified that this is the right complementary function, now we can integrate our equation $y = x^2$ from $x = 1$ to $x = 2$:

$$\int_1^2 x^2 dx = \left[\frac{1}{3}x^3 + C\right]_1^2 = \left[\left(\frac{1}{3}(2)^3 + C\right) - \left(\frac{1}{3}(1)^3 + C\right)\right] = 2.33333333 \quad \text{(L-100)}$$

which is the exact area under the curve. Notice that the constant of integration canceled, which is always the case for *definite integrals* (i.e., where the limits are specified). For *indefinite integrals* (i.e., where the limits are not specified), the constant of integration must be reported:

$$\int x^2 dx = \frac{1}{3}x^3 + C \quad \text{(L-101)}$$

L.6.9 Comparisons

Table L.4 compares the various methods we employed to find the area under the curve. It is interesting to note that Simpson's rule gave the exact area. This is expected because Simpson's rule approximates the curve as a parabola and $y = x^2$ is a parabola.

The difficult part of the integration process is finding the complementary function that, when differentiated, gives the function under which the area is being determined. Table L.2 lists a few common functions. More extensive tables can be found in calculus texts and the *CRC Handbook of Chemistry and Physics.* A very extensive listing is *Table of Integrals, Series, and Products* by I. S. Gradshteyn and I. M. Ryzhik (Academic Press).

Table L.4 Comparison of integration methods

	Method	Area (ft^2)	Error (%)
1	Graph paper	2.31	0.86
2	Weigh tracing paper	2.36	1.14
3	Average height	2.27	2.71
4	Trapezoidal rule ($n = 4$)	2.34313	0.42
5	Simpson's rule ($n = 4$)	2.33333	0.00
6	Average-height rectangles ($n = 4$)	2.34313	0.42
7a	Inscribed rectangle ($n = 4$)	1.97	15.57
7b	Inscribed rectangle ($n = 8$)	2.148	7.94
8	Integration	2.33333	0.00

L.7 Calculus Summary

We can summarize our brief tour of calculus by examining the following equations:

$$f(x) = \frac{1}{3}x^3$$

$$\int x^2 dx \text{ (integrate)} \Uparrow \; \Downarrow \text{(differentiate)} \frac{d}{dx}\left(\frac{1}{3}x^3\right)$$

$$f'(x) = x^2$$

$$\int 2x\, dx \text{ (integrate)} \Uparrow \Downarrow \text{ (differentiate)} \frac{d}{dx}(x^2)$$

$$f''(x) = 2x$$

$$\int 2\, dx \text{ (integrate)} \Uparrow \; \Downarrow \text{ (differentiate)} \frac{d}{dx}(2x)$$

$$f'''(x) = 2$$

We can easily see that integration is the antioperation of differentiation, and that differentiation is the antioperation of integration. It should be noted that in the above equations, the constant of integration was always set to zero to keep the equations simple.

These concepts of integration and differentiation are central to physics; in fact, Newton had to invent calculus before he could describe the physics of motion. The following sets of equations are used to describe motion:

$$x = \frac{1}{2}a_0 t^2 + v_0 t + x_0 \text{ (position)} \tag{L-102}$$

$$\int (a_0 t + v_0)\, dt \text{ (integrate)} \Uparrow \; \Downarrow \text{ (differentiate)} \frac{d}{dt}\left(\frac{1}{2}a_0 t^2 + v_0 t + x_0\right)$$

$$v = \frac{dx}{dt} = a_0 t + v_0 \text{ (velocity)} \tag{L-103}$$

$$\int (a_0)\, dt \text{ (integrate)} \Uparrow \; \Downarrow \text{ (differentiate)} \frac{d}{dt}(a_0 t + v_0)$$

$$a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = a_0 \text{ (acceleration)} \tag{L-104}$$

where a_0 is the constant acceleration, v_0 is the initial velocity at time zero, and x_0 is the initial position at time zero. These are all constants; in fact, v_0 and x_0 may be viewed as the constants of integration. For objects that are free falling under the influence of gravity with negligible air drag, the acceleration is 9.8 m/s^2, which is commonly given the symbol g:

$$g = 9.8 \text{ m/s}^2 \tag{L-105}$$

Newton, Co-Discoverer of Calculus
Some people consider Sir Isaac Newton (1642 – 1727) to be the greatest intellect who ever lived. He was a sickly child who did not distinguish himself in school and failed at his first occupation, farming. His uncle recognized Isaac's innate brightness and sent him to Cambridge where he graduated in 1665 without distinction. Some early experiments with prisms made him famous and he became a distinguished professor at Cambridge University in 1669. His teaching responsibilities required that he give only eight (rather poor) lectures each year; the rest of his time was devoted to thought. In 1689, he was elected to the British Parliament. For many years, he never made a speech. Then, breaking his silence, he rose to speak. Quiet passed through the House to hear the words of wisdom from this famous scholar. He merely wanted to request that a window be closed to stop a draft. In addition to his many great accomplishments in optics, mechanics of motion, and astronomy, Newton discovered calculus (a field he called the "method of fluxions"). Without this essential tool, it would have been impossible to describe motion. In Germany, calculus was independently discovered by Liebniz; the claims that Liebniz stole it from Newton were unfounded. Although Newton shares the credit for discovering calculus, the calculus notations invented by Liebniz are considered more elegant and have been adopted by mathematicians the world over. Nonetheless, out of patriotic pride, it took English mathematicians many years to adopt the Liebniz notations.
I. Asimov, *Asimov's Biographical Encyclopedia of Science and Technology*, 2nd ed., Doubleday, Garden City, New York, 1982.

Example L.1

Problem Statement: For fun, an engineer buys a police radar gun used to check the speed of motorists. He points this radar gun at a highway that passes by the 50-story office building where he works. Whenever he gets bored, he points the radar gun at the nearby traffic and enjoys seeing traffic slow down to 55 mph. One day, while amusing himself in this manner, he notices a cannonball falling out of the sky (see Figure L.15). He points the radar gun at the falling cannonball and measures the velocity to be 100 m/s. A few seconds later, he measures the velocity to be 150 m/s. How much time elapsed between the two measurements?

Solution:

$$v_1 = v_0 + g\,t_1 \Rightarrow t_1 = \frac{v_1 - v_0}{g}$$

$$v_2 = v_0 + g\,t_2 \Rightarrow t_2 = \frac{v_2 - v_0}{g}$$

$$t_2 - t_1 = \frac{v_2 - v_0}{g} - \frac{v_1 - v_0}{g} = \frac{v_2 - v_1}{g} = \frac{150\text{ m/s} - 100\text{ m/s}}{9.8\text{ m/s}^2} = 5.1\text{ s}$$

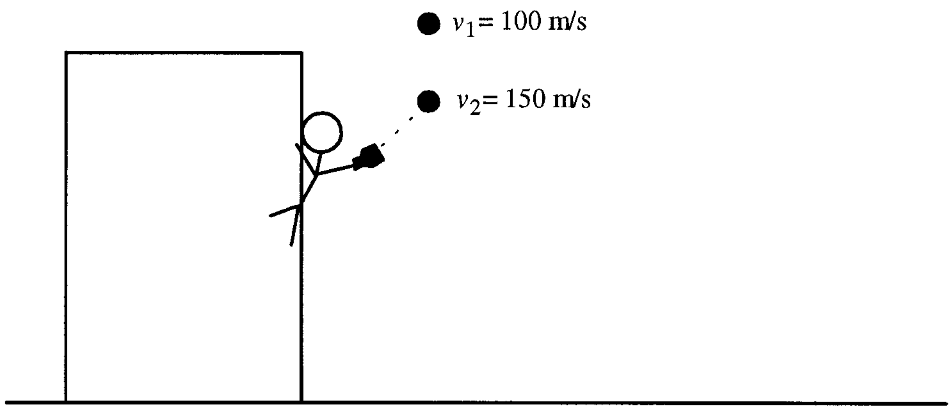


Figure L.15 Cannonball falling out of the sky.

L.8 Averages

An *average* is a single number that characterizes the most typical value in a data set. For *discrete data* (i.e., a set of distinct numbers), there are three kinds of averages defined as follows:

$$A = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right) \qquad \text{(Arithmetic mean)} \qquad \text{(L-106)}$$

$$G = (a_1 a_2 a_3 \cdots a_n)^{1/n} = \left(\prod_{i=1}^{n} a_i\right)^{1/n} \quad (a_i > 0) \qquad \text{(Geometric mean)} \qquad \text{(L-107)}$$

$$H = \left[\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right)\right]^{-1} \quad (a_i > 0) \qquad \text{(Harmonic mean)} \qquad \text{(L-108)}$$

where n is the number of discrete data points. For any discrete data set, $A > G > H$.

Example L.2

***Problem Statement*:** What is the arithmetic mean, geometric mean, and harmonic mean for the following discrete data set: 2, 3.4, 5, 8, 5.6, and 3?

***Solution*:**

$$A = \frac{2 + 3.4 + 5 + 8 + 5.6 + 3}{6} = 4.50$$

$$G = [(2)(3.4)(5)(8)(5.6)(3)]^{1/6} = 4.07$$

$$H = \left[\frac{1}{6}\left(\frac{1}{2} + \frac{1}{3.4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{5.6} + \frac{1}{3}\right)\right]^{-1} = 3.68$$

For *continuous data* (e.g., car speed that changes continuously with time), there are an infinite number of data points, so n would be infinity. The definitions for the arithmetic, geometric, and harmonic means are meaningless when $n = \infty$. Therefore we must invoke another concept for the meaning of "average."

Method 3 for determining the area under a curve used the concept of an average value for y (i.e., y_{ave}) (see Figure L.9). Recall that we were studying the *continuous* function $y = f(x) = x^2$ and were trying to determine the area under the curve from $x = 1$ to $x = 2$. We said that

$$A = y_{ave}(b - a) \qquad \text{(L-70)}$$

We also know that the area under the curve can be determined exactly by the expression

$$A = \int_a^b x^2 dx \qquad \text{(L-109)}$$

By equating these two expressions for area A and solving for y_{ave} we get

$$y_{ave} = \frac{\int_a^b x^2 dx}{(b - a)} \qquad \text{(L-110)}$$

We can generalize this for an arbitrary function $f(x)$ as

$$f(x)_{ave} = \frac{\int_a^b f(x) dx}{(b - a)} \qquad \text{(L-111)}$$

L.9 Multiple Integrals (Advanced Topic)

Multiple integrals are useful for deriving the area and volume of geometric figures. Two integrations are required to determine the area of a geometric figure and three integrations are needed to find the volume of geometric figures.

$$A = \int\int f(x, y)\, dx\, dy \qquad \text{(L-112)}$$

$$V = \int\int\int f(x, y, z)\, dx\, dy\, dz \qquad \text{(L-113)}$$

The multiple integration for x is performed while holding all the other variables constant.

For example, suppose we wished to find the area of a rectangle (see Figure L.16). The area of a little piece dA is

$$dA = dx \cdot dy \qquad \text{(L-114)}$$

The total area can be found by taking the double integral

$$A = \int dA = \int_0^B \int_0^A dx\, dy = \int_0^B \left\{ \int_0^A dx \right\} dy = \int_0^B \left\{ [x]_0^A \right\} dy = \int_0^B \{[A - 0]\}\, dy = \int_0^B A\, dy$$

$$= A \int_0^B dy = A[y]_0^B = A[B - 0] = A \cdot B \qquad \text{(L-115)}$$

Of course you already knew the answer, but it illustrates all the steps required to perform a double integration.

Let's perform another double integration that isn't so obvious. Have you ever wondered how the equation for the area of a circle was derived? Although we all learned it in grade school, it was taught as a formula to memorize without understanding.

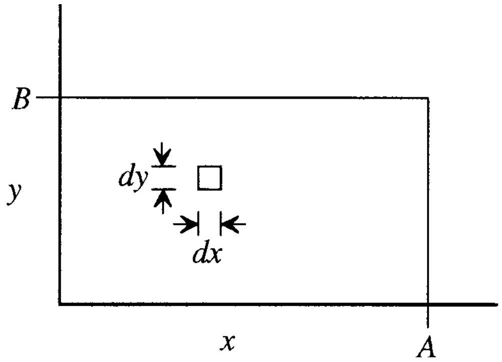


Figure L.16 Area of a rectangle.

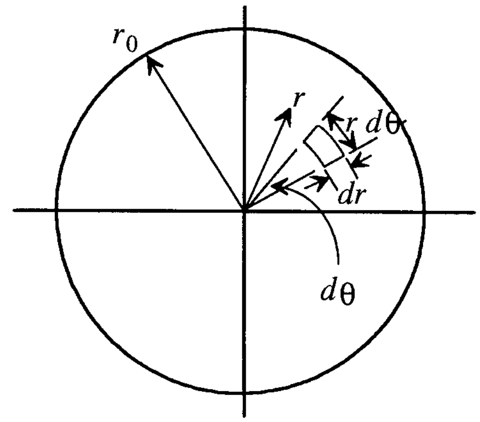


Figure L.17 Area of a circle.

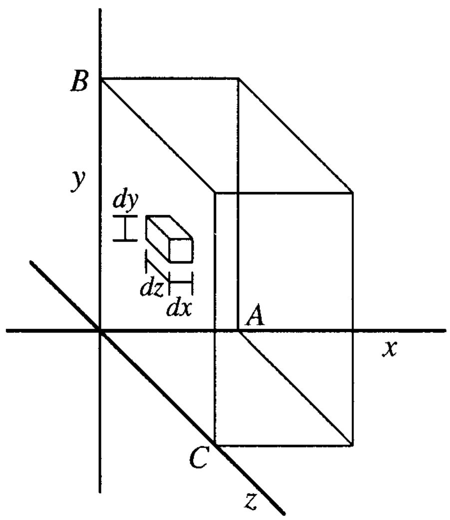


Figure L.18 Volume of a rectangular solid.

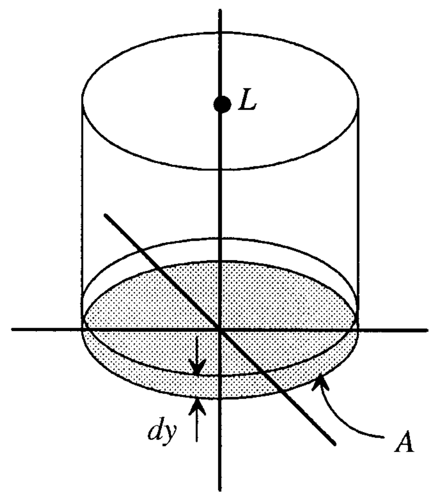


Figure L.19 Volume of a cylinder.

Figure L.17 shows that the area of the small "square" is

$$dA = (r\,d\theta)(dr) \tag{L-116}$$

where one side is a differential radius wide and the other side is the swept length of a circle segment (i.e., $r\,d\theta$ provided θ is measured in radians). The radius will be integrated from 0 to r_0 and the angle will be integrated from 0 to 2π (i.e., a full 360°). The area is

$$A = \int dA = \int_0^{r_0}\int_0^{2\pi}(r\,d\theta)(dr) = \int_0^{r_0}\int_0^{2\pi}(d\theta)(r\,dr) = \int_0^{r_0}\{[\theta]_0^{2\pi}\}r\,dr = \int_0^{r_0}\{[2\pi - 0]\}r\,dr$$

$$= \int_0^{r_0} 2\pi\,r\,dr = 2\pi\int_0^{r_0} r\,dr = 2\pi\left[\frac{r^2}{2}\right]_0^{r_0} = 2\pi\left[\frac{r_0^2}{2} - \frac{0^2}{2}\right] = 2\pi\left[\frac{r_0^2}{2}\right] = \pi r_0^2 \tag{L-117}$$

In the previous derivation, we treated r as a variable and r_0 as the radius of that particular circle. If we generalize and allow the circle radius to be any r, then the equation may be written as

$$A = \pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2 \tag{L-118}$$

where d is the circle diameter. A *very* common error is to use the diameter for r which makes the area wrong by a factor of 4. The reader is advised to commit the diameter form of the equation to memory.

A triple integration is performed to determine the volume of a geometric shape. For example, consider the rectangular solid shown in Figure L.18. The differential volume is

$$dV = dx\, dy\, dz \tag{L-119}$$

The total volume is determined by

$$V = \int dV = \int_0^C \int_0^B \int_0^A dx\, dy\, dz = \int_0^C \int_0^B [x]_0^A dy\, dz = \int_0^C \int_0^B [A-0]\, dy\, dz = \int_0^C \int_0^B A\, dy\, dz$$

$$V = A \int_0^C \int_0^B dy\, dz = A \int_0^C [y]_0^B dz = A \int_0^C [B-0]\, dz = A \int_0^C B\, dz = AB \int_0^C dz$$

$$V = AB[z]_0^C = AB[C-0] = ABC \tag{L-120}$$

Again, you knew this would be the answer, but you can see all the integration steps required to arrive at the final result.

A triple integral may be viewed as an area integrated over a length. Figure L.19 shows a cylinder. The differential volume is

$$dV = A\, dy \tag{L-121}$$

The total volume is

$$V = \int dV = \int_0^L A\, dy = A \int_0^L dy = A[y]_0^L = A[L-0] = AL = \pi r^2 L \tag{L-122}$$

where πr^2 was substituted for the area A. It should be noted that A could be removed to the left of the integral sign *only* because A is constant in the y direction. If the geometrical shape were a cone, then A could not be removed from the integration process because the cross-sectional area varies in the y direction. For a cone, it would be necessary to find a relationship for the cross-sectional area as a function of y and include this function in the integration.

Here, we have shown some simple examples of area and volume integrals. Previously, we showed formulas for the area and volume of many geometric figures. From the brief discussion presented here, you will have an appreciation for the steps required to derive these formulas.

L.10 Summary

Because so many physical laws describe the manner in which a quantity **changes**, calculus is a primary mathematical tool for engineers. A key idea in calculus is limits. For example, the differential of x (i.e., dx) is the limit as x becomes arbitrarily close to zero, but is not quite zero.

There are two main branches of calculus: differential calculus and integral calculus. Differential calculus is used to find the slope of a mathematical function and integral calculus is used to find the area under a mathematical function. Differentiation is the antioperation of integration, and vice versa. For simple mathematical functions, differentiation and integration may be performed analytically; i.e., an equation can be derived that gives the differential or integral. However, for complex mathematical relationships, it is often necessary to use numerical methods for finding the derivative or integral (e.g., trapezoidal rule or Simpson's rule).

Throughout our tour of mathematics, we have found it necessary to use power series to approximate various transcendental functions such as exponents, logarithms, and trigonometric functions. In this chapter, we see that to obtain a power series approximation to a transcendental function, we must be able to repeatedly differentiate the function and evaluate it and its derivatives at zero.

Three types of means are used to analyze discrete data: arithmetic, geometric, and harmonic. For continuous data, the average is found by integrating the data from the lower limit to the upper limit and dividing by the difference between the upper and lower limits.

Multiple integrals are particularly useful for analyzing geometric shapes. For example, the geometry chapter presented formulas for the area and volume of various shapes; these formulas were derived using multiple integrals. A multiple integral is performed by holding all variables constant except the variable that is being integrated.

Further Readings

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