SECTION 3.3

MULTIPLICATION

State office buildings at the Empire State Plaza, Albany, New York

PROBLEM OPENER

Lee has written a two-digit number in which the units digit is her favorite digit. When she subtracts the tens digit from the units digit, she gets 3. When she multiplies the original two-digit number by 21, she gets a three-digit number whose hundreds digit is her favorite digit and whose tens and units digits are the same as those in her original two-digit number. What is her favorite digit?

The skyscraper in the center of the preceding photo is called the Tower Building. There is an innovative window-washing machine mounted on top of this building. The machine lowers a cage on a vertical track so that each column of 40 windows can be washed. After one vertical column of windows has been washed, the machine moves to the next column. The rectangular face visible in the photograph has 36 columns of windows. The total number of windows is $40 + 40 + 40 + \ldots + 40$, a sum in which 40 occurs 36 times. This sum equals the product 36×40, or 1440. We are led to different expressions for the sum and product by considering the rows of windows across the floors. There are 36 windows in each floor on this face of the building and 40 floors. Therefore, the number of windows is $36 + 36 + 36 + \ldots + 36$, a sum in which 36 occurs 40 times. This sum is equal to 40×36, which is also 1440. For sums such as these in which one number is repeated, multiplication is a convenient method for doing addition.

Historically, multiplication was developed to replace certain special cases of addition, namely, the cases of *several equal addends.* For this reason we usually see **multiplication** of whole numbers explained and defined as **repeated addition.**

MULTIPLICATION OF WHOLE NUMBERS For any whole numbers r and s, the **product** of r and s is the sum with s occurring r times. This is written as

$$r \times s = \underbrace{s + s + s + \ldots + s}_{r \text{ times}}$$

The numbers r and s are called **factors.**

One way of representing multiplication of whole numbers is with a **rectangular array** of objects, such as the rows and columns of windows at the beginning of this section. Figure 3.11 shows the close relationship between the use of *repeated addition* and *rectangular arrays* for illustrating products. Part (a) of the figure shows squares in 4 groups of 7 to illustrate 7 + 7 + 7 + 7, and part (b) shows the squares pushed together to form a 4 × 7 rectangle.

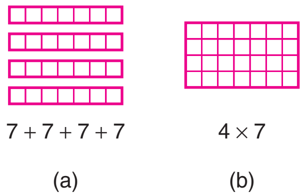


Figure 3.11

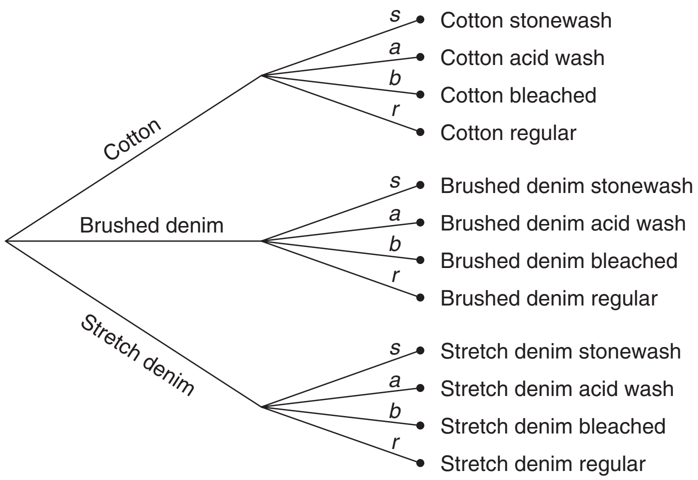
In general, $r \times s$ is the number of objects in an $r \times s$ rectangular array.

Another way of viewing multiplication is with a figure called a **tree diagram.** Constructing a tree diagram is a counting technique that is useful for certain types of multiplication problems.

Example A

A catalog shows jeans available in cotton, brushed denim, or stretch denim and in stonewash (s), acid wash (a), bleached (b), or regular color (r). How many types of jeans are available?

Solution A tree diagram for this problem is shown below. The tree begins with 3 branches, each labeled with one of the types of material. Each of these branches leads to 4 more branches, which correspond to the colors. The tree has 3 × 4 = 12 endpoints, one for each of the 12 different types of jeans.



Models for Multiplication Algorithms

Research provides evidence that students will rely on their own computational strategies (Cobb et al. 1991). Such inventions contribute to their mathematical development (Gravemeijer 1994; Steffe 1994).

Standards 2000, p. 86

Physical models for multiplication can generate an understanding of multiplication and suggest or motivate procedures and rules for computing. There are many suitable models for illustrating multiplication. Base-ten pieces are used in the following examples.

Figure 3.12 illustrates 3 × 145, using base-ten pieces. First 145 is represented as shown in (a). Then the base-ten pieces for 145 are tripled. The result is 3 flats, 12 longs, and 15 units, as shown in (b). Finally, the pieces are regrouped: 10 units are replaced by 1 long, leaving 5 units; and 10 longs are replaced by 1 flat, leaving 3 longs. The result is 4 flats, 3 longs, and 5 units, as shown in (c).

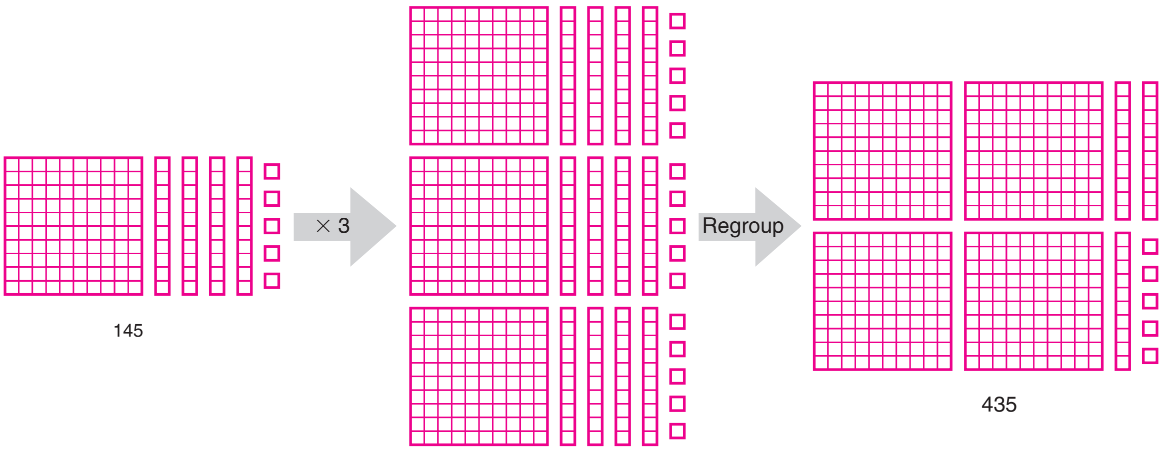


Figure 3.12 (a) (b) (c)

Base-ten pieces can be used to illustrate the pencil-and-paper algorithm for computing. Consider the product 3 × 145 shown in Figure 3.12. First a 5, indicating the remaining 5 units in part (c), is recorded in the units column, and the 10 units that have been regrouped are recorded by writing 1 in the tens column (see below). Then 3 is written in the tens column for the remaining 3 longs, and 1 is recorded in the hundreds column for the 10 longs that have been regrouped.

Flats	Longs	Units
1	1	
1	4	5
	×	3
4	3	5

The next example illustrates how multiplication by 10 can be carried out with base-ten pieces. Multiplying by 10 is especially convenient because 10 units can be placed together to form 1 long, 10 longs to form 1 flat, and 10 flats to form 1 long-flat (row of flats).

To multiply 34 and 10, we replace each base-ten piece for 34 by the base-ten piece for the next higher power of 10 (Figure 3.13). We begin with 3 longs and 4 units and end with 3 flats, 4 longs, and 0 units. This illustrates the familiar fact that the product of any whole number and 10 can be computed by placing a zero at the right end of the numeral for the whole number.

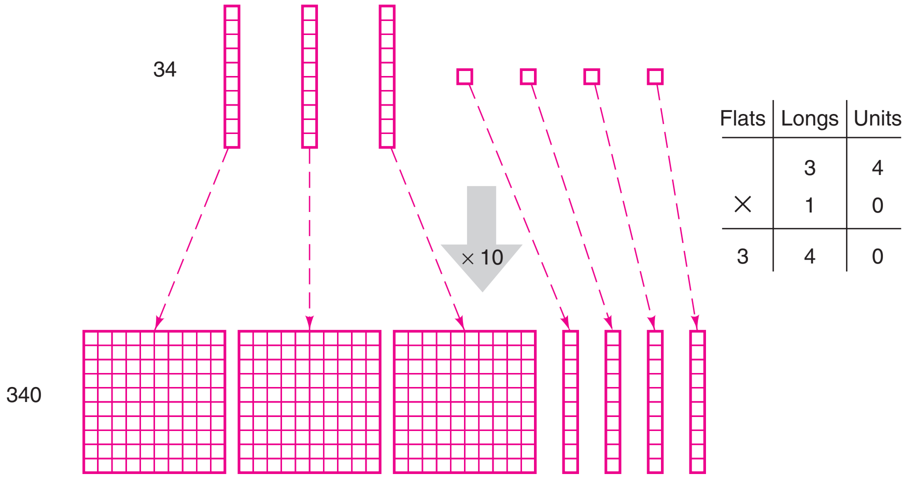


Figure 3.13

Computing the product of two numbers by repeated addition of base-ten pieces becomes impractical as the size of the numbers increases. For example, computing 18×23 requires representing 23 with base-ten pieces 18 times. For products involving two-digit numbers, rectangular arrays are more convenient.

To compute 18×23, we can draw a rectangle with dimensions 18 by 23 on grid paper (Figure 3.14). The product is the number of small squares in the rectangular array. This number can be easily determined by counting groups of 100 flats and strips of 10 longs. The total number of small squares is 414. Notice how the array in Figure 3.14 can be viewed as 18 horizontal rows of 23, once again showing the connection between the repeated-addition and rectangular-array views of multiplication.

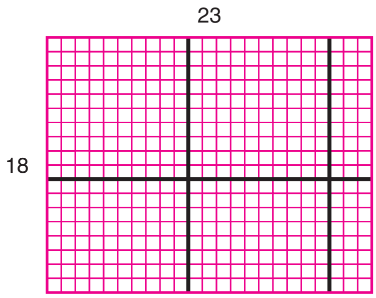


Figure 3.14

The pencil-and-paper algorithm for multiplication computes **partial products.** When a two-digit number is multiplied by a two-digit number, there are four partial products.

The product 13×17 is illustrated in Figure 3.15. The four regions of the grid formed by the heavy lines represent the four partial products. Sometimes it is instructive to draw arrows from each partial product to the corresponding region on the grid.

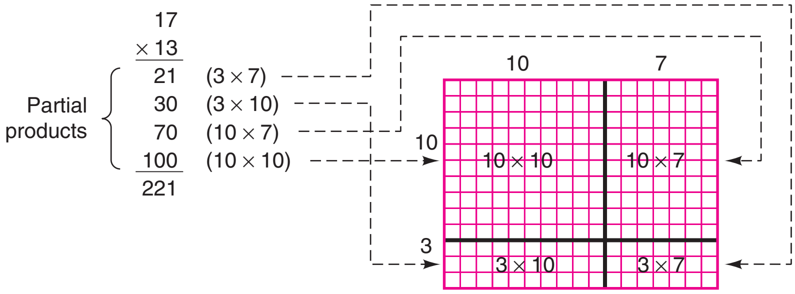


Figure 3.15

HISTORICAL HIGHLIGHT

$\rightarrow 1 \times 52 = 52$
$\rightarrow 2 \times 52 = 104$
$4 \times 52 = 208$
$\rightarrow 8 \times 52 = 416$

$$\begin{array}{r} 52 \\ 104 \\ +416 \\ \hline 572 \end{array}$$

One of the earliest methods of multiplication is found in the Rhind Papyrus. This ancient scroll (ca. 1650 B.C.), more than 5 meters in length, was written to instruct Egyptian scribes in computing with whole numbers and fractions. Beginning with the words "Complete and thorough study of all things, insights into all that exists, knowledge of all secrets . . . ," it indicates the Egyptians' awe of mathematics. Although most of its 85 problems have a practical origin, there are some of a theoretical nature. The Egyptians' algorithm for multiplication was a succession of doubling operations, followed by addition. To compute 11 × 52, they would repeatedly double 52, then add *one* 52, *two* 52s, and *eight* 52s to get *eleven* 52s.

Number Properties

Four properties for addition of whole numbers were stated in Section 3.2. Four corresponding properties for multiplication of whole numbers are stated below, along with one additional property that relates the operations of addition and multiplication.

CLOSURE PROPERTY FOR MULTIPLICATION This property states that the product of any two whole numbers is also a whole number.

For any two whole numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a} \times \boldsymbol{b} \text{ is a unique whole number}$$

IDENTITY PROPERTY FOR MULTIPLICATION The number 1 is called an **identity for multiplication** because when multiplied by another number, it leaves the identity of the number unchanged. For example,

$$1 \times 14 = 14 \qquad 34 \times 1 = 34 \qquad 1 \times 0 = 0$$

The number 1 is unique in that it is the only number that is an identity for multiplication.

For any whole number b,

$$1 \times b = b \times 1 = b$$

and 1 is a unique identity for multiplication.

COMMUTATIVE PROPERTY FOR MULTIPLICATION This number property says that in any product of two numbers, the numbers may be interchanged (commuted) without affecting the product. This property is called the **commutative property for multiplication.** For example,

$$347 \times 26 = 26 \times 347$$

For any whole numbers a and b,

$$a \times b = b \times a$$

The commutative property is illustrated in Figure 3.16, which shows two different views of the same rectangular array. Part (a) represents 7×5, and part (b) represents 5×7. Since part (b) is obtained by rotating part (a), both figures have the same number of small squares, so 7×5 is equal to 5×7.

Using area models, properties of operations such as commutativity of multiplication become more apparent.

Standards 2000, p. 152

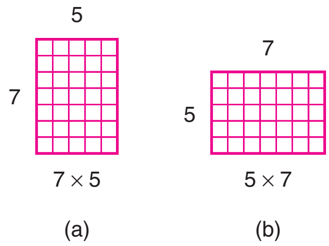


Figure 3.16

As the multiplication table in Figure 3.17 shows, the commutative property for multiplication approximately cuts in half the number of basic multiplication facts that must be memorized. Each product in the shaded part of the table corresponds to an equal product in the unshaded part of the table.

Example B

Since $3 \times 7 = 21$, we know by the commutative property for multiplication that $7 \times 3 = 21$. What do you notice about the location of each product in the shaded part of the table relative to the location of the corresponding equal product in the unshaded part of the table?

Solution If the shaded part of the table is folded onto the unshaded part, each product in the shaded part will coincide with an equal product in the unshaded part. In other words, the table is symmetric about the diagonal from upper left to lower right.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Figure 3.17

Notice that the numbers in the rows of the multiplication table in Figure 3.17 form arithmetic sequences, for example, 2, 4, 6, 8, . . . and 3, 6, 9, 12. . . . One reason that children learn to count by 2s, 3s, and 5s is to acquire background for learning basic multiplication facts. The keystrokes in the next two diagrams will produce a sequence which increases by 2s and a sequence which decreases by 5s on most calculators.

KEYSTROKES	VIEW SCREEN
2	2
+ 2 =	4
+ 2 =	6
+ 2 =	8

KEYSTROKES	VIEW SCREEN
35	35
− 5 =	30
− 5 =	25
− 5 =	20

ASSOCIATIVE PROPERTY FOR MULTIPLICATION In any product of three numbers, the middle number may be associated with and multiplied by either of the two end numbers. This property is called the **associative property for multiplication.** For example,

$$\underbrace{6 \times (7 \times 4)} = \underbrace{(6 \times 7) \times 4}$$

Associative property
for multiplication

For any whole numbers ***a*, *b*,** and ***c*,**

$$\boldsymbol{a \times (b \times c) = (a \times b) \times c}$$

Figure 3.18 illustrates the associative property for multiplication. Part (a) represents 3×4, and (b) shows 5 of the 3×4 rectangles. The number of small squares in (b) is $5 \times (3 \times 4)$. Part (c) is obtained by subdividing the rectangle (b) into 4 copies of a 3×5 rectangle. The number of small squares in (c) is $4 \times (3 \times 5)$, which, by the commutative property for multiplication, equals $(5 \times 3) \times 4$. Since the numbers of small squares in (b) and (c) are equal, $5 \times (3 \times 4) = (5 \times 3) \times 4$.

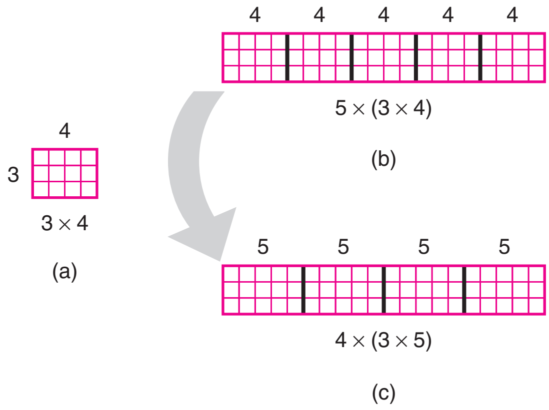


Figure 3.18

The commutative and associative properties are often used to obtain convenient combinations of numbers for mental calculations, as in the next example.

Example C

Try computing $25 \times 46 \times 4$ in your head before reading further.

Solution The easy way to do this is by rearranging the numbers so that 25×4 is computed first and then 46×100. The following equations show how the commutative and associative properties permit this rearrangement.

Associative property
for multiplication

$$(25 \times 46) \times 4 = (46 \times 25) \times 4 = 46 \times (25 \times 4)$$

Commutative property
for multiplication

DISTRIBUTIVE PROPERTY When multiplying a sum of two numbers by a third number, we can add the two numbers and then multiply by the third number, or we can multiply each number of the sum by the third number and then add the two products.

For example, to compute $35 \times (10 + 2)$, we can compute 35×12, or we can add 35×10 to 35×2. This property is called the **distributive property for multiplication over addition.**

$$35 \times 12 = 35 \times (10 + 2) = (35 \times 10) + (35 \times 2)$$

Distributive property

For any whole numbers a, b, and c,

$$a \times (b + c) = a \times b + a \times c$$

One use of the distributive property is in learning the basic multiplication facts. Elementary school children are often taught the "doubles" ($2 + 2 = 4$, $3 + 3 = 6$, $4 + 4 = 8$, etc.) because these number facts together with the distributive property can be used to obtain other multiplication facts.

Example D How can $7 \times 7 = 49$ and the distributive property be used to compute 7×8?

Solution

$$7 \times 8 = \underbrace{7 \times (7 + 1)} = \underbrace{49 + 7} = 56$$

Distributive property

The distributive property can be illustrated by using rectangular arrays, as in Figure 3.19. The dimensions of the array in (a) are 6 by $(3 + 4)$, and the array contains 42 small squares. Part (b) shows the same squares separated into two rectangular arrays with dimensions 6 by 3 and 6 by 4. Since the number of squares in both figures is the same, $6 \times (3 + 4) = (6 \times 3) + (6 \times 4)$.

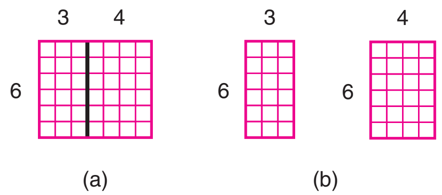


Figure 3.19

The distributive property also holds for multiplication over subtraction.

Example E Show that the two sides of the following equation are equal.

$$6 \times (20 - 8) = (6 \times 20) - (6 \times 8)$$

Solution $6 \times (20 - 8) = 6 \times 12 = 72$ and $(6 \times 20) - (6 \times 8) = 120 - 48 = 72$

Mental Calculations

In the following paragraphs, three methods are discussed for performing mental calculations of products. These methods parallel those used for performing mental calculations of sums and differences.

COMPATIBLE NUMBERS We saw in Example C that the commutative and associative properties permit the rearrangement of numbers in products. Such rearrangements can often enable computations with compatible numbers.

Example F

Find a more convenient arrangement that will yield compatible numbers, and compute the following products mentally.

1. $5 \times 346 \times 2$
2. $2 \times 25 \times 79 \times 2$

Solution **1.** $5 \times 2 \times 346 = 10 \times 346 = 3460$ **2.** $2 \times 2 \times 25 \times 79 = 100 \times 79 = 7900$

SUBSTITUTIONS In certain situations the distributive property is useful for facilitating mental calculations. For example, to compute 21×103, first replace 103 by $100 + 3$ and then compute 21×100 and 21×3 in your head. Try it.

$$21 \times 103 = \underbrace{21 \times (100 + 3)} = \underbrace{2100 + 63} = 2163$$

Distributive property

Occasionally it is convenient to replace a number by the difference of two numbers and to use the fact that multiplication distributes over subtraction. Rather than compute 45×98, we can compute 45×100 and subtract 45×2.

$$45 \times 98 = \underbrace{45 \times (100 - 2)} = \underbrace{4500 - 90} = 4410$$

Distributive property

Example G

Find a convenient substitution, and compute the following products mentally.

1. 25×99
2. 42×11
3. 34×102

Solution **1.** $25 \times (100 - 1) = 2500 - 25 = 2475$ **2.** $42 \times (10 + 1) = 420 + 42 = 462$ **3.** $34 \times (100 + 2) = 3400 + 68 = 3468$

Other relationships can be seen by decomposing and composing area models. For example, a model for 20×6 can be split in half and the halves rearranged to form a 10×12 rectangle, showing the equivalence of 10×12 and 20×6.

Standards 2000, p. 152

EQUAL PRODUCTS This method of performing mental calculations is similar to the *equal differences* method used for subtraction. It is based on the fact that the product of two numbers is unchanged when one of the numbers is divided by a given number and the other number is multiplied by the same number. For example, the product 12×52 can be replaced by 6×104 by dividing 12 by 2 and multiplying 52 by 2. At this point we can mentally calculate 6×104 to be 624. Or we can continue the process of dividing and multiplying by 2, replacing 6×104 by 3×208, which can also be mentally calculated.

Figure 3.20 illustrates why one number in a product can be halved and the other doubled without changing the product. The rectangular array in part (a) of the figure represents 22×16. If this rectangle is cut in half, the two pieces can be used to form an 11×32 rectangle, as in (b). Notice that 11 is half of 22 and 32 is twice 16. Since the rearrangement has not changed the number of small squares in the two rectangles, the products 22×16 and 11×32 are equal.

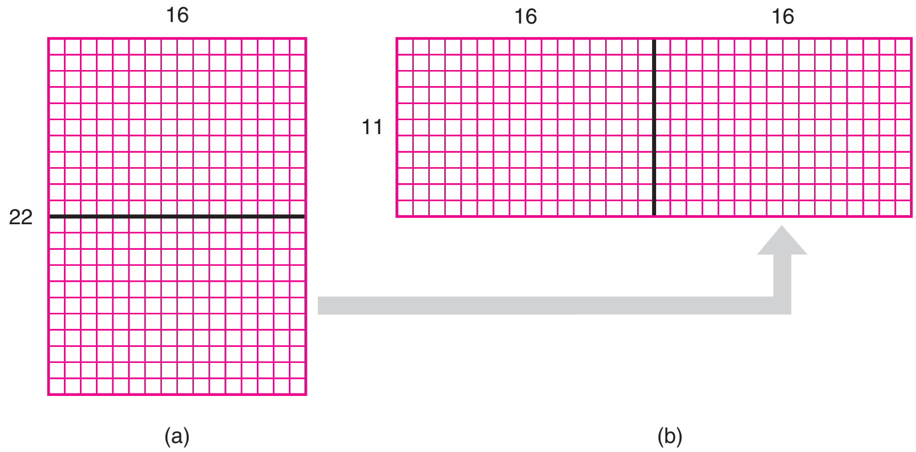


Figure 3.20

The equal-products method can also be justified by using number properties. The following equations show that $22 \times 16 = 11 \times 32$. Notice that multiplying by $\frac{1}{2}$ and 2 is the same as multiplying by 1. This is a special case of the inverse property for multiplication, which is discussed in Section 5.3.

$$\begin{aligned} 22 \times 16 &= 22 \times 1 \times 16 && \text{identity property for multiplication} \\ &= 22 \times \left(\frac{1}{2} \times 2\right) \times 16 && \text{inverse property for multiplication} \\ &= \left(22 \times \frac{1}{2}\right) \times (2 \times 16) && \text{associative property for multiplication} \\ &= 11 \times 32 \end{aligned}$$

Example H

Use the method of equal products to perform the following calculations mentally.

1. 14×4
2. 28×25
3. 15×35

Solution **1.** $14 \times 4 = 7 \times 8 = 56$ **2.** $28 \times 25 = 14 \times 50 = 7 \times 100 = 700$ **3.** $15 \times 35 = 5 \times 105 = 525$

Estimation of Products

The importance of estimation is noted in NCTM's K–4 Standard, *Estimation:*

Instruction should emphasize the development of an estimation mindset. Children should come to know what is meant by an estimate, when it is appropriate to estimate, and how close an estimate is required in a given situation. If children are encouraged to estimate, they will accept estimation as a legitimate part of mathematics.*

**Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 1989), p. 115.

The techniques of *rounding,* using *compatible numbers,* and *front-end estimation* are used in the following examples.

ROUNDING Products can be estimated by rounding one or both numbers. Computing products by rounding is somewhat more risky than computing sums by rounding, because any error due to rounding becomes multiplied. For example, if we compute 47 × 28 by rounding 47 to 50 and 28 to 30, the estimated product 50 × 30 = 1500 is greater than the actual product. This may be acceptable if we want an estimate greater than the actual product. For a closer estimate, we can round 47 to 45 and 28 to 30. In this case the estimate is 45 × 30 = 1350.

Example I

Use rounding to estimate these products. Make any adjustments you feel might be needed.

1. 28 × 63
2. 81 × 57
3. 194 × 26

Solution Following is one estimate for each product. You may find others. **1.** 28 × 63 ≈ 30 × 60 = 1800. Notice that since 63 is greater than 28, increasing 28 by 2 has more of an effect on the estimate than decreasing 63 to 60 (see Figure 3.21). So the estimate of 1800 is greater than the actual answer. **2.** 81 ×57 ≈ 80 × 60 = 4800 **3.** 194 × 26 ≈ 200 × 25 = 5000

When students leave grade 5, . . . they should be able to solve many problems mentally, to estimate a reasonable result for a problem, . . . and to compute fluently with multidigit whole numbers.

Standards 2000, p. 149

Figure 3.21 shows the effect of estimating 28 × 63 by rounding to 30 × 60. Rectangular arrays for both 28 × 63 and 30 × 60 can be seen on the grid. The gray region shows the increase from rounding 28 to 30, and the red region shows the decrease from rounding 63 to 60. Since the gray region (2 × 60 = 120) is larger than the red region (3 × 28 = 84), we are adding more than we are removing, and so the estimate is greater than the actual product.

Figure 3.21

COMPATIBLE NUMBERS Using compatible numbers becomes a powerful tool for estimating products when it is combined with techniques for performing mental calculations. For example, to estimate $4 \times 237 \times 26$, we might replace 26 by 25 and use a different ordering of the numbers.

$$4 \times 237 \times 26 \approx 4 \times 25 \times 237 = 100 \times 237 = 23{,}700$$

Example J

Use compatible numbers and mental calculations to estimate these products.

1. $2 \times 117 \times 49$
2. $34 \times 46 \times 3$

Solution **1.** $2 \times 117 \times 49 \approx 2 \times 117 \times 50 = 100 \times 117 = 11{,}700$ **2.** $34 \times 46 \times 3 = (3 \times 34) \times 46 \approx 100 \times 46 = 4600$

FRONT-END ESTIMATION This technique is similar to that used for computing sums. The leading digit of each number is used to obtain an estimated product. To estimate 43×72, the product of the leading digits of the numbers is $4 \times 7 = 28$, so the estimated product is 2800.

$$43 \times 72 \approx 40 \times 70 = 2800$$

Similarly, front-end estimation can be used for estimating the products of numbers whose leading digits have different place values.

$$61 \times 874 \approx 60 \times 800 = 48{,}000$$

Example K

Use front-end estimation to estimate these products.

1. 64×23 2. 68×87 3. 237×76 4. $30{,}328 \times 419$

Solution **1.** $64 \times 23 \approx 60 \times 20 = 1200$ **2.** $68 \times 87 \approx 60 \times 80 = 4800$ **3.** $237 \times 76 \approx 200 \times 70 = 14{,}000$ **4.** $30{,}328 \times 419 \approx 30{,}000 \times 400 = 12{,}000{,}000$

Order of Operations

Special care must be taken on some calculators when multiplication is combined with addition or subtraction. The numbers and operations will not always produce the correct answer if they are entered into the calculator in the order in which they appear.

Example L

Compute $3 + 4 \times 5$ by entering the numbers into your calculator as they appear from left to right.

Solution Some calculators will display 35, and others will display 23. The correct answer is 23 because multiplication should be performed before addition:

$$3 + 4 \times 5 = 3 + 20 = 23$$

To avoid confusion, mathematicians have developed the convention that when multiplication occurs with addition and/or subtraction, the multiplication should be performed first. This rule is called the **order of operations.**

Some calculators are programmed to follow the order of operations. On this type of calculator, any combination of products with sums and differences and without parentheses can be computed by entering the numbers and operations in the order in which they occur from left to right and then pressing [=]. If a calculator does not follow the order of operations, the products can be computed separately and recorded by hand or saved in the calculator's memory.

Example M

Use your calculator to evaluate $34 \times 19 + 82 \times 43$. Then check the reasonableness of your answer by using estimation and mental calculations.

Solution The exact answer is 4172. An estimate can be obtained as follows:

$$34 \times 19 + 82 \times 43 \approx 30 \times 20 + 80 \times 40 = 600 + 3200 = 3800$$

Notice that the estimation in Example M is 372 less than the actual product. However, it is useful in judging the reasonableness of the number obtained from the calculator: It indicates that the calculator answer is most likely correct. If $34 \times 19 + 82 \times 43$ is entered into a calculator as it appears from left to right and if the calculator is not programmed to follow the order of operations, then the incorrect result of 31,304 will be obtained, which is too large by approximately 27,000.

Problem-Solving Application

There is an easy method for mentally computing the products of certain two-digit numbers. A few of these products are shown here.

$25 \times 25 = 625$	$24 \times 26 = 624$	$71 \times 79 = 5609$
$37 \times 33 = 1221$	$35 \times 35 = 1225$	$75 \times 75 = 5625$

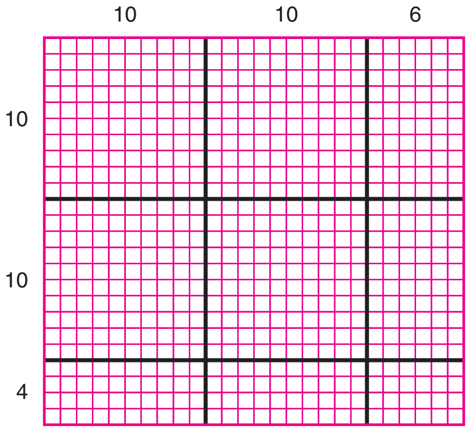
The solution to the following problem reveals the method of mental computation and uses *rectangular grids* to show why the method works.

PROBLEM

What is the method of mental calculation for computing the products of the two-digit numbers shown above, and why does this method work?

Understanding the Problem There are patterns in the digits in these products. One pattern is that the two numbers in each pair have the same first digit. Find another pattern. **Question 1:** What types of two-digit numbers are being used?

Devising a Plan Looking for patterns may help you find the types of numbers and the method of computing. Another approach is to represent some of these products on a grid. The following grid illustrates 24×26; the product is the number of small squares in the rectangle. To determine this number, we begin by counting large groups of squares. There are 6 hundreds. **Question 2:** Why is this grid especially convenient for counting the number of hundreds?



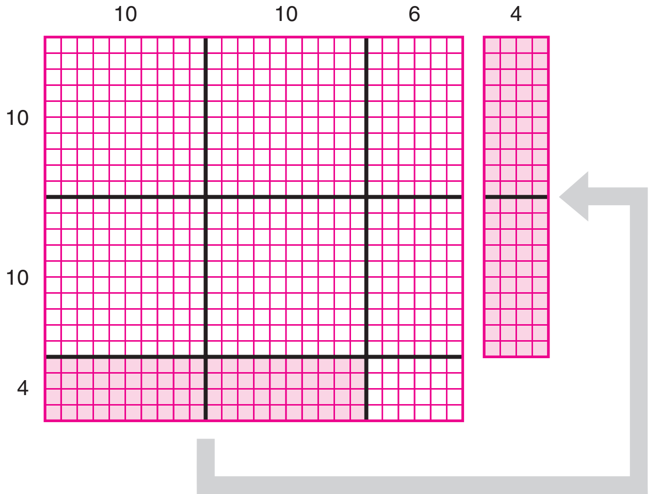
Carrying Out the Plan Sketch grids for one or more of the products being considered in this problem. For each grid it is easy to determine the number of hundreds. This is the key to solving the problem. **Question 3:** What is the solution to the original problem?

Looking Back Consider the following products of three-digit numbers:

$$103 \times 107 = 11{,}021 \qquad 124 \times 126 = 15{,}624$$

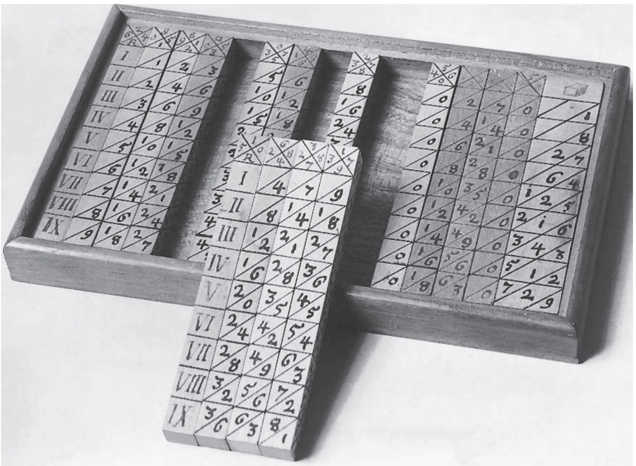
Question 4: Is there a similar method for mentally calculating the products of certain three-digit numbers?

Answers to Questions 1–4 **1.** In each pair of two-digit numbers, the tens digits are equal and the sum of the units digits is 10. **2.** The two blocks of 40 squares at the bottom of the grid can be paired with two blocks of 60 squares on the right side of the grid to form two more blocks of 100, as shown below. Then the large 20 × 30 grid represents 6 hundreds. The 4 × 6 grid in the lower right corner represents 4 × 6.



3. The first two digits of the product are formed by multiplying the tens digit by the tens digit plus 1. The remaining digits of the product are obtained by multiplying the two units digits. **4.** Yes. For 124 × 126: 12 × 13 = 156 and 4 × 6 = 24, so 124 × 126 = 15,624.

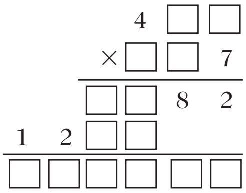
HISTORICAL HIGHLIGHT



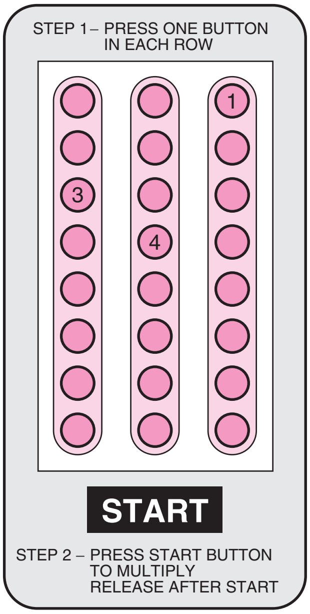
As late as the seventeenth century, multiplication of large numbers was a difficult task for all but professional clerks. To help people "do away with the difficulty and tediousness for calculations," Scottish mathematician John Napier (1550–1617) invented a method of using rods for performing multiplication. Napier's rods—or *bones* as they are sometimes called—contain multiplication facts for each digit. For example, the rod for the 4s has 4, 8, 12, 16, 20, 24, 28, 32, and 36. This photograph of a wooden set shows the fourth, seventh, and ninth rods placed together for computing products that have a factor of 479. For example, to compute 6×479 look at row VI of the three rods for 479. Adding the numbers along the diagonals of row VI results in the product of 2874.

PUZZLER

Supply the missing digits in this faded document puzzle.

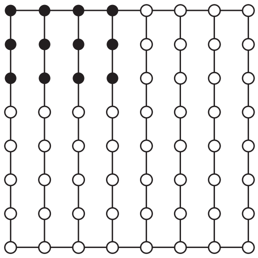


EXERCISES AND PROBLEMS 3.3



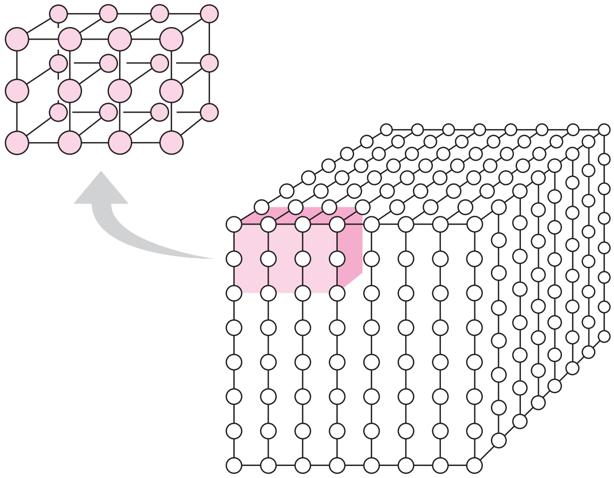
An exhibit illustrating multiplication at the California Museum of Science and Industry.

The children in the picture above are computing products of three numbers from 1 through 8. Each time three buttons are pressed on the switch box, the product is illustrated by lighted bulbs in the $8 \times 8 \times 8$ cube of bulbs. Buttons 3, 4, and 1 are for the product $3 \times 4 \times 1$. The 12 bulbs in the upper left corner of the cube will be lighted for this product, as shown in the following figure. Whenever the third number of the product is 1, the first two numbers determine a rectangular array of lighted bulbs on the front face of the cube (facing children).



$3 \times 4 \times 1$

The third number in a product illustrated by the cube of bulbs determines the number of times the array on the front face is repeated in the cube. The 24 bulbs in the upper left corner of the next figure will be lighted for $3 \times 4 \times 2$.



Describe the bulbs that will be lighted for the products in exercises 1 and 2.

1. **a.** $7 \times 3 \times 1$ **b.** $2 \times 8 \times 1$

2. **a.** $5 \times 4 \times 1$ **b.** $8 \times 8 \times 1$

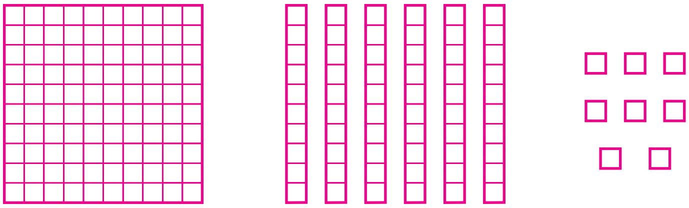
Describe the bulbs that will be lighted for the products in exercises 3 and 4.

3. **a.** $6 \times 4 \times 3$ **b.** $1 \times 8 \times 8$

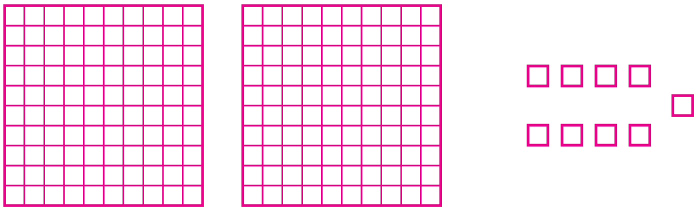
4. **a.** $2 \times 2 \times 2$ **b.** $8 \times 1 \times 8$

Sketch a new set of base pieces for each product in exercises 5 and 6, and then show regrouping.

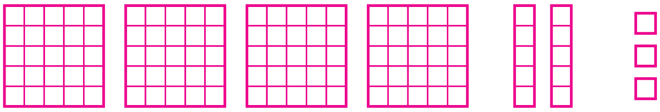
5. a. Multiply 168 by 3.



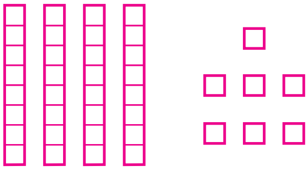
b. Multiply 209 by 4.



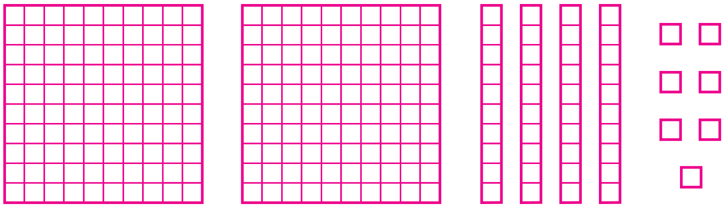
c. Multiply 423_{five} by 3.



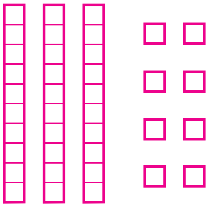
d. Multiply 47_{eight} by 5.



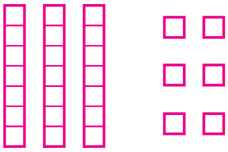
6. a. Multiply 247 by 2.



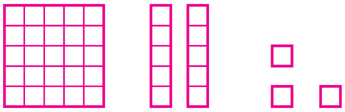
b. Multiply 38 by 5.



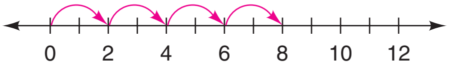
c. Multiply 36_{seven} by 5.



d. Multiply 123_{five} by 4.



Multiplication of whole numbers can be illustrated on the number line by a series of arrows. This number line shows 4×2.



Draw arrow diagrams for the products in exercises 7 and 8.

7. a. 3×4 **b.** 2×5
c. Use the number line to show that $3 \times 4 = 4 \times 3$.

8. a. 2×6 **b.** 6×2
c. Use the number line to show that $2 \times (3 + 2) = 2 \times 3 + 2 \times 2$.

Error analysis. Students who know their basic multiplication facts may still have trouble with the steps in the pencil-and-paper multiplication algorithm. Try to detect each type of error in exercises 9 and 10, and write an explanation.

9. a.
$$\begin{array}{r} {\scriptstyle 2} \\ 27 \\ \times\ 4 \\ \hline 48 \end{array}$$

b.
$$\begin{array}{r} {\scriptstyle 2} \\ 18 \\ \times\ 3 \\ \hline 34 \end{array}$$

10. a.
$$\begin{array}{r} {\scriptstyle 4} \\ 54 \\ \times 6 \\ \hline 342 \end{array}$$

b.
$$\begin{array}{r} {\scriptstyle 1} \\ 34 \\ \times 24 \\ \hline 76 \end{array}$$

In exercises 11 and 12, use base-ten grids to illustrate the partial products that occur when these products are computed with pencil and paper. Draw arrows from each partial product to its corresponding region on the grid. (Copy the base-ten grid from the inside cover or the website.)

11. **a.** $\begin{array}{r} 24 \\ \times\ 7 \\ \hline \end{array}$ **b.** $\begin{array}{r} 56 \\ \times\ 43 \\ \hline \end{array}$

12. **a.** $\begin{array}{r} 34 \\ \times\ 26 \\ \hline \end{array}$ **b.** $\begin{array}{r} 39 \\ \times\ 47 \\ \hline \end{array}$

Which number property is being used in each of the equalities in exercises 13 and 14?

13. **a.** $3 \times (2 \times 7 + 1) = 3 \times (7 \times 2 + 1)$
b. $18 + (43 \times 7) \times 9 = 18 + 43 \times (7 \times 9)$
c. $(12 + 17) \times (16 + 5)$
$= (12 + 17) \times 16 + (12 + 17) \times 5$

14. **a.** $(13 + 22) \times (7 + 5) = (13 + 22) \times (5 + 7)$
b. $(15 \times 2 + 9) + 3 = 15 \times 2 + (9 + 3)$
c. $59 + 41 \times 8 + 41 \times 26 = 59 + 41 \times (8 + 26)$

Determine whether each set in exercises 15 and 16 is closed for the given operation.

15. **a.** The set of odd whole numbers for multiplication
b. The set of whole numbers less than 100 for addition
c. The set of all whole numbers whose units digits are 6 for multiplication

16. **a.** The set of even whole numbers for multiplication
b. The set of whole numbers less than 1000 for multiplication
c. The set of whole numbers greater than 1000 for multiplication

In exercises 17 and 18, compute the exact products mentally, using *compatible numbers.* Explain your method.

17. **a.** $2 \times 83 \times 50$ **b.** $5 \times 3 \times 2 \times 7$

18. **a.** $4 \times 2 \times 25 \times 5$ **b.** $5 \times 17 \times 20$

In exercises 19 and 20, compute the exact products mentally, using substitution and the fact that multiplication distributes over addition. Show your use of the distributive property.

19. **a.** 25×12 **b.** 15×106

20. **a.** 18×11 **b.** 14×102

In exercises 21 and 22, compute the exact products mentally, using the fact that multiplication distributes over subtraction. Show your use of the distributive property.

21. **a.** 35×19 **b.** 30×99

22. **a.** 51×9 **b.** 40×98

In exercises 23 and 24, use the method of *equal products* to find numbers that are more convenient for making exact mental calculations. Show the new products that replace the original products.

23. **a.** 24×25 **b.** 35×60

24. **a.** 16×6 **b.** 36×5

In exercises 25 and 26, *round* the numbers and mentally estimate the products. Show the rounded numbers, and predict whether the estimated products are greater than or less than the actual products. Explain any adjustment you make to improve the estimates.

25. **a.** 22×17 **b.** 83×31

26. **a.** 71×56 **b.** 205×29

In exercises 27 and 28, use *compatible numbers* and mental calculations to estimate the products. Show your compatible-number replacements, and predict whether the estimated products are greater than or less than the actual products.

27. **a.** $4 \times 76 \times 24$ **b.** $3 \times 34 \times 162$

28. **a.** $5 \times 19 \times 74$ **b.** $2 \times 63 \times 2 \times 26$

In exercises 29 and 30, estimate the products, using *front-end estimation* and mental calculations. Show two estimates for each product, one using only the tens digits and one using combinations of the tens and units digits.

29. **a.** 36×58 **b.** 42×27

30. **a.** 62×83 **b.** 14×62

In exercises 31 and 32, *round* the given numbers and estimate each product. Then sketch a rectangular array for the actual product, and on the same figure sketch the rectangular array for the product of the rounded numbers. Shade the regions that show increases and/or decreases due to rounding. (Copy the base-ten grid from the inside cover or the website.)

31. **a.** 18×62 **b.** 43×29

32. **a.** 17×28 **b.** 53×31

In exercises 33 and 34, circle the operations in each expression that should be performed first. Estimate each expression mentally, and show your method of estimating. Use a calculator to obtain an exact answer, and compare this answer to your estimate.

33. **a.** $62 \times 45 + 14 \times 29$
b. $36 + 18 \times 40 + 15$

34. **a.** $114 \times 238 - 19 \times 605$
b. $73 - 50 + 17 \times 62$

In exercises 35 and 36, a geometric sequence is generated by beginning with the first number entered into the calculator and repeatedly carrying out the given keystrokes. Beginning with the first number entered, write each sequence which is produced by the given keystrokes.

35. **a.** Enter 5 and repeat the keystrokes [×] [3] [=] eight times.
b. Enter 20 and repeat the keystrokes [+] [5] [=] nine times.

36. **a.** Enter 91 and repeat the keystrokes [−] [2] [=] fourteen times
b. Enter 3 and repeat the keystrokes [×] [2] [=] six times.

An elementary school calculator with a constant function is convenient for generating a geometric sequence. For example, the sequence 2, 4, 8, 16, 32, . . . , will be produced by entering 1 [×] 2 and repeating pressing [=]. Write the first three terms of each sequence in exercises 37 and 38.

37.

	KEYSTROKES	VIEW SCREEN
a.	81 [×] 5 [=]	
	[=]	
	[=]	
b.	119 [×] 4 [=]	
	[=]	
	[=]	

38.

	KEYSTROKES	VIEW SCREEN
a.	17 [×] 3 [=]	
	[=]	
	[=]	
b.	142 [×] 6 [=]	
	[=]	
	[=]	

In exercises 39 and 40, estimate the second factor so that the product will fall within the given range. Check your answer with a calculator. Count the number of tries it takes you to land in the range.

	Product	Range
Example	22 × _____	(900, 1000)
	22 × 40 = 880	Too small
	22 × 43 = 946	In the range in two tries

39. **a.** 32 × _____ (800, 850)
b. 95 × _____ (1650, 1750)

40. **a.** 103 × _____ (2800, 2900)
b. 6 × _____ (3500, 3600)

41. There are many patterns in the multiplication table (page 166) that can be useful in memorizing the basic multiplication facts.
a. What patterns can you see?
b. There are several patterns for products involving 9 as one of the numbers being multiplied. Find two of these patterns.

REASONING AND PROBLEM SOLVING

42. A student opened her math book and computed the sum of the numbers on two facing pages. Then she turned to the next page and computed the sum of the numbers on these two facing pages. Finally, she computed the product of the two sums, and her calculator displayed the number 62,997. What were the four page numbers?

43. Harry has $2500 in cash to pay for a secondhand car, or he can pay $500 down and $155 per month for 2 years. If he doesn't pay the full amount in cash, he knows he can make $150 by investing his money. How much will he lose if he uses the more expensive method of payment?

44. Kathy read 288 pages of her 603-page novel in 9 days. How many pages per day must she now read in order to complete the book and return it within the library's 14-day deadline?

45. A store carries five styles of backpacks in four different sizes. The customer also has a choice of two different kinds of material for three of the styles. If Vanessa is only interested in the two largest backpacks, how many different backpacks would she have to chose from?

46. **Featured Strategy: Making An Organized List** The five tags shown below are placed in a box and mixed. Three tags are then selected at a time. If a player's score is the product of the numbers, how many different scores are possible?

(3) (6) (2) (5) (1)

a. **Understanding the Problem** The problem asks for the number of different scores, so each score can be counted only once. The tags 6, 5, and 1 produce a score of 30. Find three other tags that produce a score of 30.

b. **Devising a Plan** One method of solving the problem is to form an organized list. If we begin the list with the number 3, there are six different possibilities for sets of three tags. List these six possibilities.

c. **Carrying Out the Plan** Continue to list the different sets of three tags and compute their products. How many different scores are there?

d. **Looking Back** A different type of organized list can be formed by considering the scores between 6 (the smallest score) and 90 (the greatest score). For example, 7, 8, and 9 can be quickly thrown out. Why?

Find patterns in problems 47 and 48 and determine if they continue to hold for the next few equations. If so, will they continue to hold for more equations? Show examples to support your conclusions.

47. $1 \times 9 + 2 = 11$
$12 \times 9 + 3 = 111$
$123 \times 9 + 4 = 1111$

48. $1 \times 99 = 99$
$2 \times 99 = 198$
$3 \times 99 = 297$

49. **a.** Select some two-digit numbers and multiply them by 99 and 999. Describe a few patterns and form some conjectures.
b. Test your conjectures on some other two-digit numbers. Predict whether your conjectures will continue to hold, and support your conclusions with examples.
c. Do your conjectures continue to hold for three-digit numbers times 99 and 999?

50. **a.** Select some two-digit numbers and multiply them by 11. Describe some patterns and form a conjecture.
b. Test your conjecture on some other two-digit numbers. Predict whether your conjectures will continue to hold, and support your conclusions with examples.
c. Does your conjecture hold for three-digit numbers times 11?

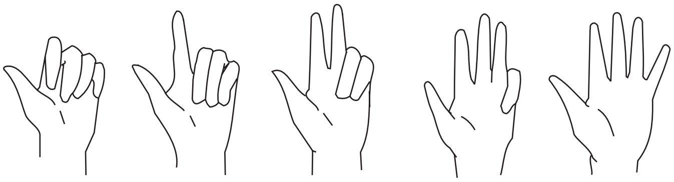
51. Samir has a combination lock with numbers from 1 to 25. This is the type of lock which requires three numbers to be opened: turn right for the first number, left for the second number, and right for the third number. Samir remembers the first two numbers, and they are not equal; but he can't remember which one is first and which is second. Also, he has forgotten the third number. What is the greatest number of different combinations that must be tried to open the lock?

52. When 6-year-old Melanie arrived home from school, she was the first to eat cookies from a freshly baked batch. When 8-year-old Felipe arrived home, he ate twice as many cookies as Melanie had eaten. When 9-year-old Hillary arrived home, she ate 3 fewer cookies than Felipe. When 12-year-old Nicholas arrived, he ate 3 times as many cookies as Hillary. Nicholas left 2 cookies, one for each of his parents. If Nicholas had eaten only 5 cookies, there would have been 3 cookies for each of his parents. How many cookies were in the original batch?

53. A mathematics education researcher is studying problem solving in small groups. One phase of the study involves pairing a third-grade girl with a third-grade boy. If the researcher wants between 70 and 80 different boy-girl combinations and there are 9 girls available for the study, how many boys are needed?

54. A restaurant owner has a luncheon special which consists of a cup of soup, half of a sandwich, and a beverage. She wants to advertise that a different combination of the three can be purchased 365 days of the year for $4.99 apiece. If she has 7 different kinds of soup and 6 different kinds of sandwiches, how many different kinds of beverages are needed to provide at least 365 different luncheons?

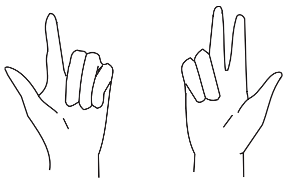
In problems 55 and 56, use the following system of finger positions to compute the products of numbers from 6 to 10. Here are the positions for the digits from 6 to 10.



6 7 8 9 10

The two numbers that are to be multiplied are each represented on a different hand. The sum of the raised fingers is the number of 10s, and the product of the closed fingers is the number of 1s.

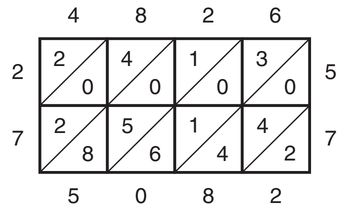
55. Explain how the position illustrated below shows that $7 \times 8 = 56$.



7 × 8

56. Describe the positions of the fingers for 7×6. Does the method work for this product?

One of the popular schemes used for multiplying in the fifteenth century was called the **lattice method.** The two numbers to be multiplied, 4826 and 57 in the example at the right, are written above and to the right of the lattice. The partial products are written in the cells. The sums of numbers along the diagonal cells, beginning at the lower right with 2, 4 + 4 + 0, etc., form the product 275,082.



Show how the lattice method can be used to compute the products in exercises 57 and 58.

57. 34×78

58. 306×923

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- Math Investigation 3.3 *Number Chains*

Section-Related: • Links • Writing/Discussion Problems • Bibliography

PUZZLER

One night three men registered at a hotel. They were charged \$30 for their room. The desk clerk later realized that she had overcharged them by \$5 and sent the refund up with the bellboy. The bellboy knew it would be difficult to split the \$5 three ways. Therefore, he kept a \$2 "tip" and gave the men only \$3. Each man had originally paid \$10 and was given back \$1. Thus the room cost each man \$9, and together they paid \$27. This total plus the \$2 tip is \$29. What happened to the other dollar?