Chapter 3

The Karnaugh Map

The algebraic methods developed in Chapter 2 allow us, in theory, to simplify any function. However, there are a number of problems with that approach. There is no formal method, such as first apply Property 10, then P14, etc. The approach is totally heuristic, depending heavily on experience. After manipulating a function, we often cannot be sure whether or not it is a minimum. We may not always find the minimum, even though it appears that there is nothing else to do. Furthermore, it gets rather difficult to do algebraic simplification with more than four or five variables. Finally, it is easy to make copying mistakes as we rewrite the equations.

In this chapter we will examine an approach that is easier to implement, the *Karnaugh map* (sometimes referred to as a K-map). This is a graphical approach to finding suitable product terms for use in sum of product expressions. (The product terms that are "suitable" for use in minimum sum of products expressions are referred to as *prime implicants*. We will define that term shortly.) The map is useful for problems of up to six variables and is particularly straightforward for most problems of three or four variables. Although there is no guarantee of finding a minimum solution, the methods we will develop nearly always produce a minimum. We will adapt the approach (with no difficulty) to finding minimum product of sums expressions, to problems with don't cares, and to multiple output problems.

We introduced the Karnaugh map in Section 2.6. In this chapter, we will develop techniques to find minimum sum of product expressions using the map. We will start with three- and four-variable maps and will include five- and six-variable maps later.

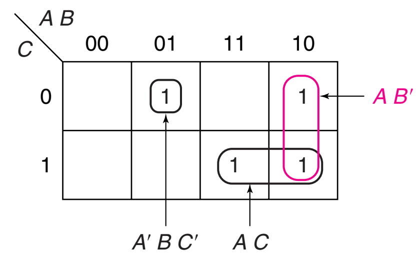
We can plot any function on the map. Either, we know the minterms, and use that form of the map (as we did earlier), or we put the function in sum of products form and plot each of the product terms.

EXAMPLE 3.1

Map

$$F = AB' + AC + A'BC'$$

The map for F is shown below, with each of the product terms circled. Each of the two-literal terms corresponds to two squares on the map (since one of the variables is missing). The AB' term is in the 10 column. The AC term is in the $C = 1$ row and in the 11 and 10 columns (with a common 1 in the A position). Finally, the minterm $A'BC'$ corresponds to one square, in the 01 ($A'B$) column and in the $C = 0$ row.

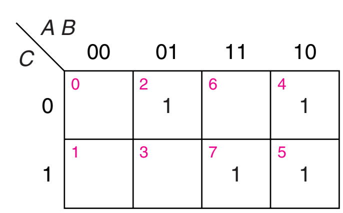


We could have obtained the same map by first expanding F to minterm form algebraically, that is,

$$\begin{aligned} F &= AB'(C' + C) + AC(B' + B) + A'BC' \\ &= AB'C' + AB'C + AB'C + ABC + A'BC' \\ &= m_4 + m_5 + m_5 + m_7 + m_2 \\ &= m_2 + m_4 + m_5 + m_7 \end{aligned}$$

(removing duplicates and reordering)

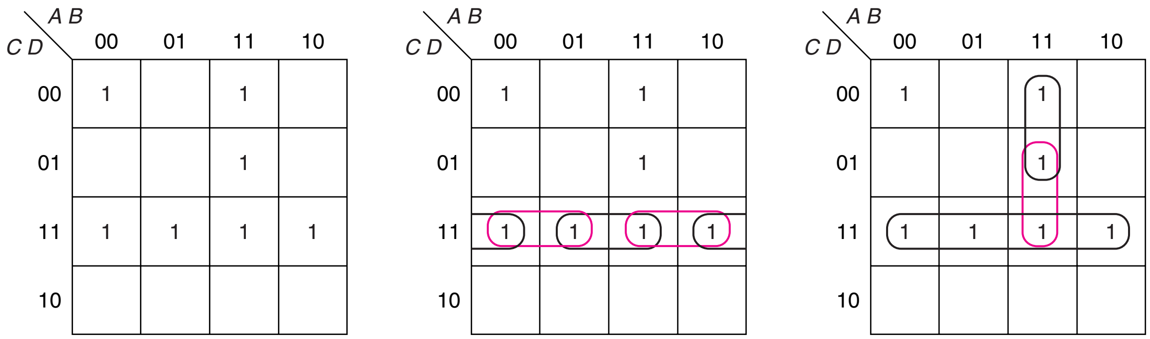
We can then use the numeric map and produce the same result.



We are now ready to define some terminology related to the Karnaugh map. An *implicant* of a function is a product term that can be used in a sum of products expression for that function, that is, the function is 1 whenever the implicant is 1 (and maybe other times, as well). From the point of view of the map, an implicant is a rectangle of 1, 2, 4, 8, . . . (any power of 2) 1's. That rectangle may not include any 0's. All minterms are implicants.

Consider the function, F, of Map 3.1. The second map shows the first four groups of 2; the third map shows the other groups of 2 and the group of 4.

Map 3.1 A function to illustrate definitions.



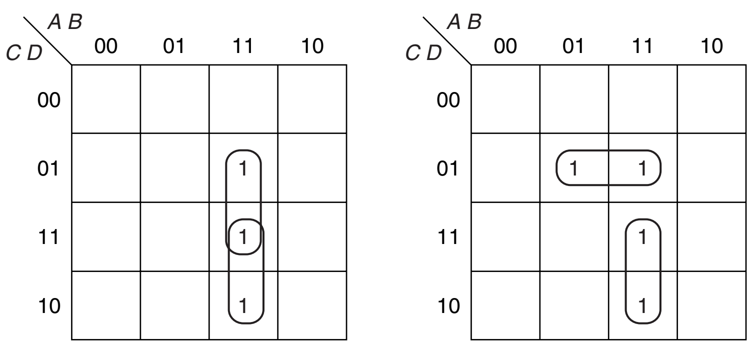
The implicants of F are

Minterms	*Groups of 2*	*Groups of 4*
$A'B'C'D'$	$A'CD$	CD
$A'B'CD$	BCD	
$A'BCD$	ACD	
$ABC'D'$	$B'CD$	
$ABC'D$	ABC'	
$ABCD$	ABD	
$AB'CD$		

Any sum of products of expression for F must be a sum of implicants. Indeed, we must choose enough implicants such that each of the 1's of F are included in at least one of these implicants. Such a sum of products expression is sometimes referred to as a *cover* of F and we sometimes say that an implicant *covers* certain minterms (for example, ACD covers m_{11} and m_{15}).

Implicants must be rectangular in shape and the number of 1's in the rectangle must be a power of 2. Thus, neither of the functions whose maps are shown in Example 3.2 are covered by a single implicant, but rather by the sum of two implicants each (in their simplest form).

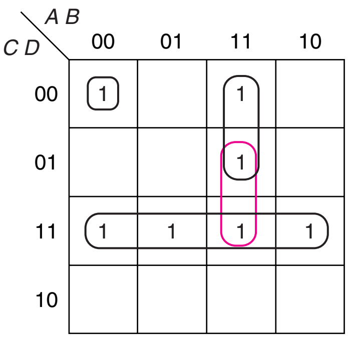
EXAMPLE 3.2



G H

G consists of three minterms, $ABC'D$, $ABCD$, and $ABCD'$, in the shape of a rectangle. It can be reduced no further than is shown on the map, namely, to $ABC + ABD$, since it is a group of three 1's, not two or four. Similarly, H has the same three minterms plus $A'BC'D$; it is a group of four, but not in the shape of a rectangle. The minimum expression is, as shown on the map, $BC'D + ABC$. (Note that ABD is also an implicant of G, but it includes 1's that are already included in the other terms.)

Map 3.2 Prime implicants.



A *prime implicant* is an implicant that (from the point of view of the map) is not fully contained in any one other implicant. For example, it is a rectangle of two 1's that is not part of a single rectangle of four 1's. On Map 3.2, all of the prime implicants of F are circled. They are $A'B'C'D'$, ABC', ABD, and CD. Note that the only minterm that is not part of a larger group is m_0 and that the other four implicants that are groups of two 1's are all part of the group of four.

From an algebraic point of view, a prime implicant is an implicant such that if any literal is removed from that term, it is no longer an implicant. From that viewpoint, $A'B'C'D'$ is a prime implicant because $B'C'D'$, $A'C'D'$, $A'B'D'$, and $A'B'C'$ are not implicants (that is, if we remove any literal from that term, we get a term that is 1 for some input combinations for which the function is to be 0). However, ACD is not a prime implicant since when we remove A, leaving CD, we still have an implicant. (Surely, the graphical approach of determining which implicants are prime implicants is easier than the algebraic method of attempting to delete literals.)

The purpose of the map is to help us find minimum sum of products expressions (where we defined minimum as being minimum number of product terms (implicants) and among those with the same number of implicants, the ones with the fewest number of literals. However, the only product terms that we need consider are prime implicants. Why? Say we found an implicant that was not a prime implicant. Then, it must be contained in some larger implicant, a prime implicant, one that covers more 1's. But that larger implicant (say four 1's rather than two) has fewer literals. That alone makes a solution using the term that is not a prime implicant not a minimum. (For example, CD has two literals, whereas, ACD has three.) Furthermore, that larger implicant covers more 1's, which often will mean that we need fewer terms.

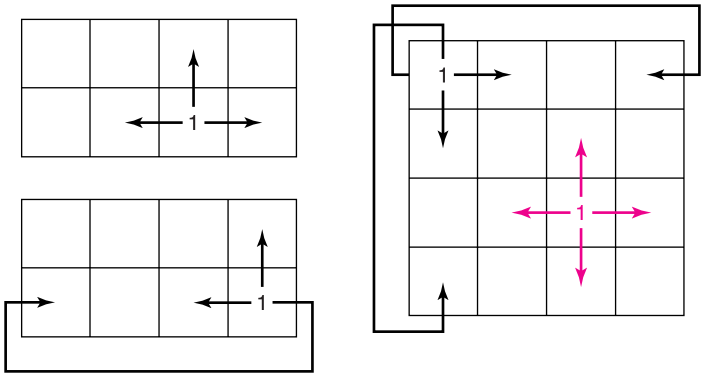
An *essential prime implicant* is a prime implicant that includes at least one 1 that is not included in any other prime implicant. (If we were to circle all of the prime implicants of a function, the essential prime implicants are those that circle at least one 1 that no other prime implicant circles.) In the example of Map 3.2, $A'B'C'D'$, ABC', and CD are essential prime implicants; ABD is not. The term *essential* is derived from the idea that we must use that prime implicant in any minimum sum of products expression. A word of caution is in order. There will often be a prime implicant that is used in a minimum solution (even in all minimum solutions when more than one equally good solution exists) that is not "essential." That happens when each of the 1's covered by this prime implicant could be covered in other ways. We will see examples of that in Section 3.1.

3.1 MINIMUM SUM OF PRODUCT EXPRESSIONS USING THE KARNAUGH MAP

In this section, we will describe two methods for finding minimum sum of products expressions using the Karnaugh map. Although these methods involve some heuristics, we can all but guarantee that they will lead to a minimum sum of products expression (or more than one when multiple solutions exist) for three- and four-variable problems. (They also work for five- and six-variable maps, but our visualization in three dimensions is more limited. We will discuss this in detail in Section 3.5.)

In the process of finding prime implicants, we will be considering each of the 1's on the map starting with the most *isolated* 1's. By isolated, we mean that there are few (or no) adjacent squares with a 1 in it. In an **n**-variable map, each square has **n** adjacent squares. Examples for three- and four-variable maps are shown in Map 3.3.

Map 3.3 Adjacencies on three- and four-variable maps.



Map Method 1

1. Find all essential prime implicants. Circle them on the map and mark the minterm(s) that make them essential with an asterisk (*). Do this by examining each 1 on the map that has not already been circled. It is usually quickest to start with the most isolated

1's, that is, those that have the fewest adjacent squares with 1's in them.

2. Find *enough* other prime implicants to cover the function. Do this using two criteria:
   a. Choose a prime implicant that covers as many new 1's (that is, those not already covered by a chosen prime implicant).
   b. Avoid leaving isolated uncovered 1's.

It is often obvious what "enough" is. For example, if there are five uncovered 1's and no prime implicants cover more than two of them, then we need at least three more terms. Sometimes, three may not be sufficient, but it usually is.

We will now look at a number of examples to demonstrate this method. First, we will look at the example used to illustrate the definitions.

EXAMPLE 3.3

As noted, m_0 has no adjacent 1's; therefore, it ($A'B'C'D'$) is a prime implicant. Indeed, it is an essential prime implicant, since no other prime implicant covers this 1. (That is always the case when minterms are prime implicants.) The next place that we look is m_{12}, since it has only one adjacent 1. Those 1's are covered by prime implicant ABC'. Indeed, no other prime implicant covers m_{12}, and thus ABC' is essential. (Whenever we have a 1 with only one adjacent 1, that group of two is an essential prime implicant.) At this point, the map has become

CD \ AB	00	01	11	10
00	1*		1*	
01			1	
11	1	1	1	1
10				

and

$$F = A'B'C'D' + ABC' + \cdots$$

Each of the 1's that have not yet been covered are part of the group of four, CD. Each has two adjacent squares with 1's that are part of that group. That will always be the case for a group of four. (Some squares, such as m_{15} may have more than two adjacent 1's.) *CD* is essential because no other prime implicant covers m_3, m_7, or m_{11}. However, once that group is circled, as shown below, we have covered the function:

CD \ AB	00	01	11	10
00	1*		1*	
01			1	
11	1*	1*	1	1*
10				

resulting in

$$F = A'B'C'D + ABC' + CD$$

In this example, once we have found the essential prime implicants, we are done; all of the 1's have been covered by one (or more) of the essential prime implicants. We do not need step 2. There may be other prime implicants that were not used (such as *ABD* in this example).

Another function that is covered using only essential prime implicants is shown in Example 3.4.

EXAMPLE 3.4

We start looking at the most isolated 1, m_{11}. It is covered only by the group of two shown, *wyz*. The other essential prime implicant is $y'z'$, because of m_0, m_8, or m_{12}. None of these are covered by any other prime implicant; each makes that prime implicant essential. The second map shows these two terms circled.

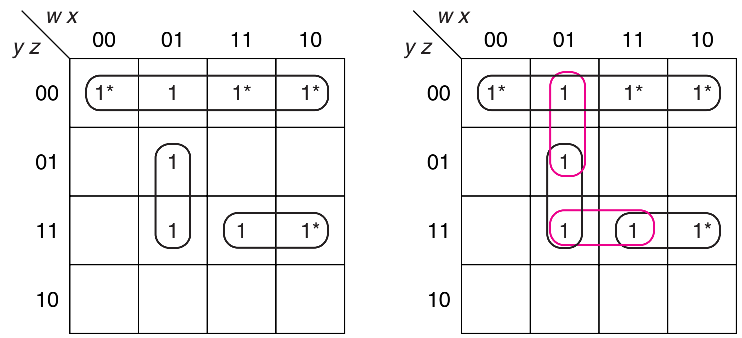
yz \ wx	00	01	11	10
00	1	1	1	1
01		1		
11		1	1	1
10				

yz \ wx	00	01	11	10
00	1*	1	1*	1*
01		1		
11		1	1	1*
10				

That leaves two 1's uncovered. Each of these can be covered by two different prime implicants; but the only way to cover them both with one term is shown on the first map below.

Thus, the minimum sum of product solution is

$$f = y'z' + wyz + w'xz$$



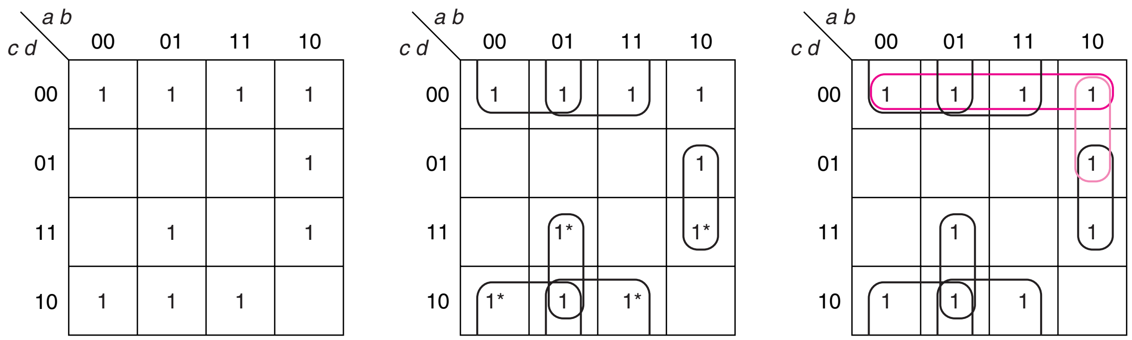
The other two prime implicants are $w'xy'$ and xyz, circled in green on the second map. They are redundant, however, since they cover no new 1's. Even though $w'xz$ must be used in a minimum solution, it does not meet the definition of an essential prime implicant; each of the 1's covered by it can be covered by other prime implicants.

Sometimes, after selecting all of the essential prime implicants, there are two choices for covering the remaining 1's, but only one of these produces a minimum solution, as in Example 3.5.

EXAMPLE 3.5

$$f(a, b, c, d) = \Sigma m(0, 2, 4, 6, 7, 8, 9, 11, 12, 14)$$

The first map shows the function and the second shows all essential prime implicants circled. In each case, one of the 1's (as indicated with an asterisk, *) can be covered by only that prime implicant. (That is obvious from the last map, where the remaining two prime implicants are circled.)



Only one 1 (m_8) is not covered by an essential prime implicant. It can be covered in two ways, by a group of four (in green) and a group of two (light green). Clearly, the group of four provides a solution with one less literal, namely,

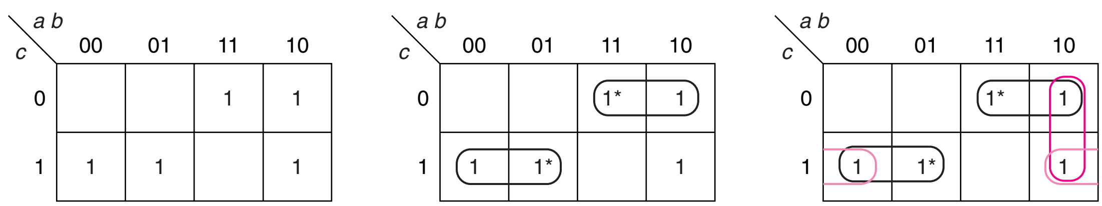
$$f = a'd' + bd' + a'bc + ab'd + c'd'$$

When asking whether a 1 makes a group of four an essential prime implicant on a four-variable map, we need find only two adjacent 0's. If there are fewer than two adjacent 0's, this 1 must be either in a group of eight or part of two or more smaller groups. Note that in Example 3.5, m_2 and m_{14} have two adjacent 0's, and thus each makes a prime implicant essential. In contrast, m_0, m_4, m_8, and m_{12} each have only one adjacent 0 and are each covered by two or three prime implicants. For a 1 to make a group of two essential, it must have three adjacent 0's. That is true for m_7 and m_{11}, but not for m_8 or m_9, each of which can be covered by two prime implicants.

We will now consider some examples with multiple minimum solutions, starting with a three-variable function.

EXAMPLE 3.6

There are two essential prime implicants, as shown on the following maps:



After finding the two essential prime implicants, ac' and $a'c$, as shown on the center map, m_5 is still uncovered. As can be seen from the map on the right, there are two ways to cover that term, yielding two, equally good, minimum solutions:

$$f = ac' + a'c + ab'$$
$$= ac' + a'c + b'c$$

As an aside, we can show that these two solutions are mathematically equal. We can take the first expression and add to it the consensus of the last two terms, $a'c$ ¢ $ab' = b'c$, leaving

$$f = ac' + a'c + ab' + b'c$$

Notice that the consensus term is the third term of the second expression. We could do the same thing with the first and third terms of the second expression, ac' ¢ $b'c = ab'$ and add that to the second expression, obtaining

$$f = ac' + a'c + b'c + ab'$$

These two expressions are indeed the same set of terms in a different order.

EXAMPLE 3.7

$$g(w, x, y, z) = \Sigma m(2, 5, 6, 7, 9, 10, 11, 13, 15)$$

The function is mapped first, and the two essential prime implicants are shown on the second map, giving

$$g = xz + wz + \cdots$$

yz \ wx	00	01	11	10
00				
01		1	1	1
11		1	1	1
10	1	1		1

yz \ wx	00	01	11	10
00				
01		1*	1	1*
11		1	1	1
10	1	1		1

Although m_2 looks rather isolated, it can indeed be covered by $w'yz'$ (with m_6) or by $x'yz'$ (with m_{10}). After choosing the essential prime implicants, the remaining three 1's can each be covered by two different prime implicants. Since there are three 1's left to be covered (after choosing the essential prime implicants), and since all the remaining prime implicants are groups of two and thus have three literals, we need at least two more of these prime implicants. Indeed, there are three ways to cover the remaining 1's with two more prime implicants. Using the first criteria, we choose one of the prime implicants that covers two new 1's, $w'yz'$, as shown on the left map below.

yz \ wx	00	01	11	10
00				
01		1	1	1
11		1	1	1
10	1	1		1

yz \ wx	00	01	11	10
00				
01		1	1	1
11		1	1	1
10	1	1		1

yz \ wx	00	01	11	10
00				
01		1	1	1
11		1	1	1
10	1	1		1

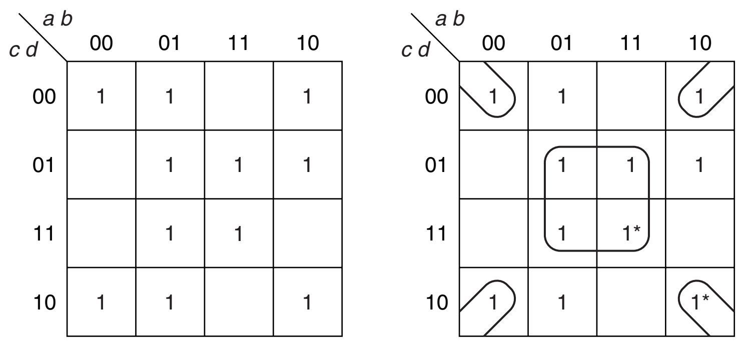
Then, only m_{10} remains and it can be covered either by $wx'y$ or by $x'yz'$, as shown on the center map. Similarly, we could have started with $x'yz'$, in which case we could use $w'xy$ to complete the cover, as on the third map. (We could also have chosen $w'yz'$, but that repeats one of the answers from before.) Thus, the three solutions are

$$g = xz + wz + w'yz' + wx'y$$
$$g = xz + wz + w'yz' + x'yz'$$
$$g = xz + wz + x'yz' + w'xy$$

All three minimum solutions require four terms and 10 literals.

At this point, it is worth stating the obvious. If there are multiple minimum solutions (as was true in this example), all such minimums have the same number of terms and the same number of literals. Any solution that has more terms or more literals is not minimum!

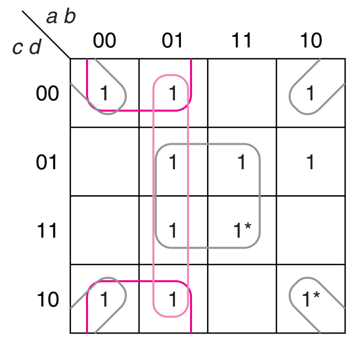
EXAMPLE 3.8

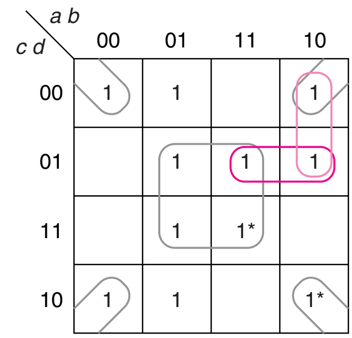


Once again there are two essential prime implicants, as shown on the right map. The most isolated 1's are m_{10} and m_{15}. Each has only two adjacent 1's. But all of the 1's in groups of four have at least two adjacent 1's; if there are only two, then that minterm will make the prime implicant essential. (Each of the other 1's in those groups of four has at least three adjacent 1's.) The essential prime implicants give us

$$f = b'd' + bd + \cdots$$

There are three 1's not covered by the essential prime implicants. There is no single term that will cover all of them. However, the two in the 01 column can be covered by either of two groups of four, as shown on the map on the left (one circled in green, the other in light green). And, there are two groups of two that cover m_9 (also one circled in green, the other in light green), shown on the map to the right.





We can choose one term from the first pair and (independently) one from the second pair. Thus, there are four solutions. We can write the solution as shown, where we take one term from within each bracket

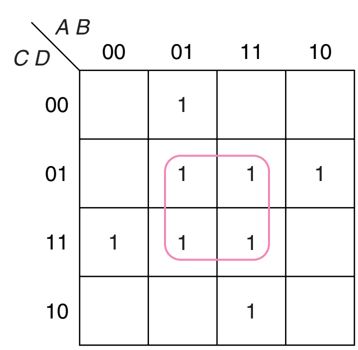
$$f = b'd' + bd + \begin{Bmatrix} a'd' \\ a'b \end{Bmatrix} + \begin{Bmatrix} ac'd \\ ab'c' \end{Bmatrix}$$

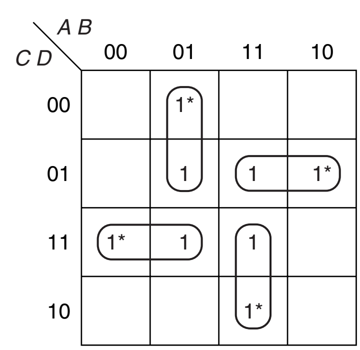
or we can write out all four expressions

$$\begin{aligned} f &= b'd' + bd + a'd' + ac'd \\ &= b'd' + bd + a'd' + ab'c' \\ &= b'd' + bd + a'b + ac'd \\ &= b'd' + bd + a'b + ab'c' \end{aligned}$$

EXAMPLE 3.9

This example is one we call "don't be greedy."

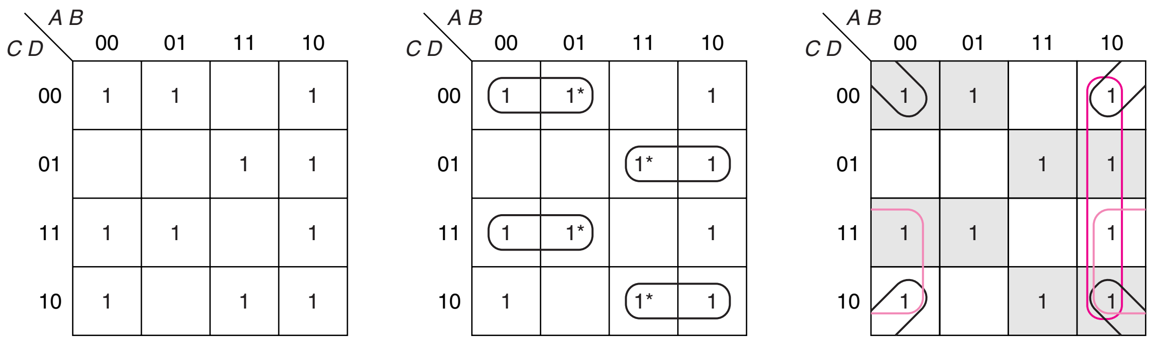




At first glance, one might want to take the only group of four (circled in light green). However, that term is not an essential prime implicant, as is obvious once we circle all of the essential prime implicants and find that the four 1's in the center are covered. Thus, the minimum solution is

$$G = A'BC' + A'CD + ABC + AC'D$$

EXAMPLE 3.10



The four essential prime implicants are shown on the second map, leaving three 1's to be covered:

$$F = A'C'D' + AC'D + A'CD + ACD' + \cdots$$

These squares are shaded on the third map. The three other prime implicants, all groups of four, are also shown on the third map. Each of these covers two of the remaining three 1's (no two the same). Thus any two of $B'D'$, AB', and $B'C$ can be used to complete the minimum sum of products expression. The resulting three equally good answers are

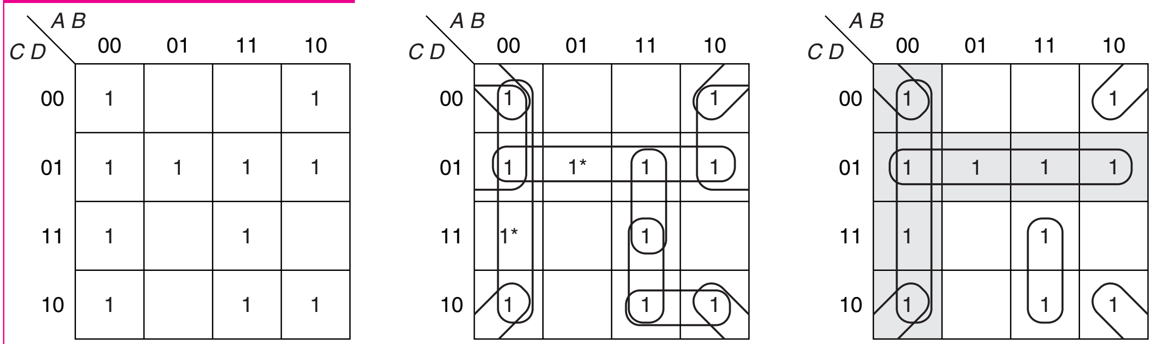
$$F = A'C'D' + AC'D + A'CD + ACD' + B'D' + AB'$$
$$F = A'C'D' + AC'D + A'CD + ACD' + B'D' + B'C$$
$$F = A'C'D' + AC'D + A'CD + ACD' + AB' + B'C$$

Before doing additional (more complex) examples, we will introduce a somewhat different method for finding minimum sum of products expressions.

Map Method 2

1. Circle all of the prime implicants.
2. Select all essential prime implicants; they are easily identified by finding 1's that have only been circled once.
3. Then choose enough of the other prime implicants (as in Method 1). Of course, these prime implicants have already been identified in step 1.

EXAMPLE 3.11



All of the prime implicants have been circled on the center map. Note that m_0 has been circled three times and that several minterms have been circled twice. However, m_3 and m_5 have only been circled once. Thus, the prime implicants that cover them, $A'B'$ and $C'D$ are essential. On the third map, we have shaded the part of the map covered by essential prime implicants to highlight what remains to be covered. There are four 1's, each of which can be covered in two different ways, and five prime implicants not used yet. No prime implicant covers more than two new 1's; thus, we need at least two more terms. Of the groups of four, only $B'D'$ covers two new 1's; $B'C'$ covers only one. Having chosen the first group, we must use ABC to cover the rest of the function, producing

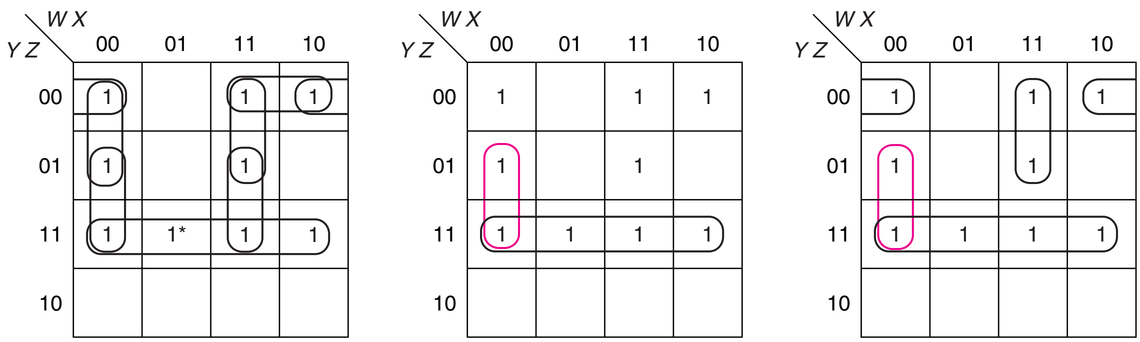
$$F = A'B' + C'D + B'D' + ABC$$

Notice that this is the only set of four prime implicants (regardless of size) that covers the function.

EXAMPLE 3.12

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 7, 8, 11, 12, 13, 15)$$

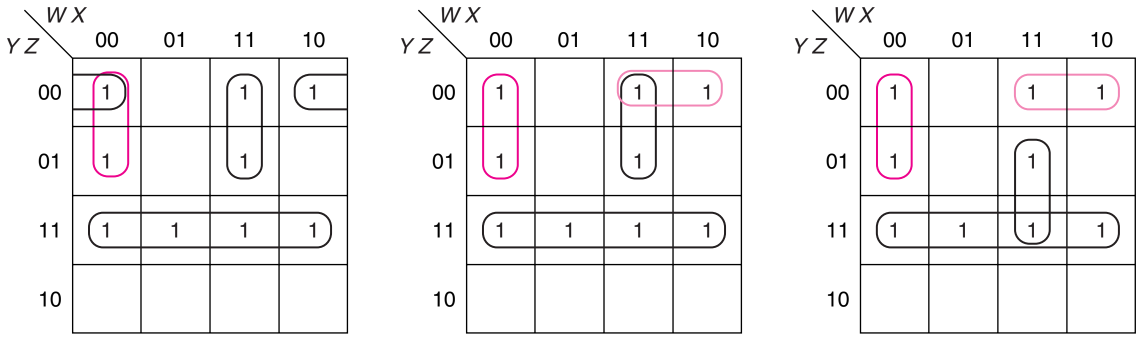
This is a case with more 1's left uncovered after finding the essential prime implicant. The first map shows all the prime implicants circled. The only essential prime implicant is YZ; there are five 1's remaining to be covered. Since all of the other prime implicants are groups of two, we need three more prime implicants. These 1's are organized in a chain, with each prime implicant linked to one on either side. If we are looking for just one solution, we should follow the guidelines from Method 1, choosing two terms that



each cover new 1's and then select a term to cover the remaining 1. One such example is shown on the third map, starting with WXY' and $X'Y'Z'$. If we wish to find all of the minimum solutions, one approach is to start at one end of the chain (as shown in the second map). (We could have started at the other end, with m_{13}, and achieved the same results.) To cover m_1, we must either use $W'X'Z$, as shown in green above, or $W'X'Y'$ (as shown on the maps below). Once we have chosen $W'X'Z$, we have no more freedom, since the terms shown on the third map above are the only way to cover the remaining 1's in two additional terms. Thus, one solution is

$$F = YZ + W'X'Z + X'Y'Z' + WXY'$$

The next three maps show the solutions using $W'X'Y'$ to cover m_0.

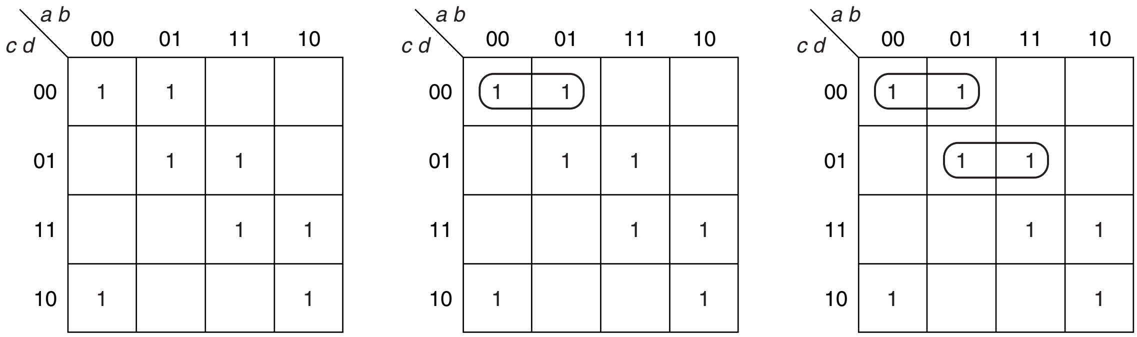


After choosing $W'X'Y'$, there are now three 1's to be covered. We can use the same last two terms as before (left) or use $WY'Z'$ to cover m_8 (right two maps). The other three solutions are thus

$$F = YZ + W'X'Y' + X'Y'Z' + WXY'$$
$$F = YZ + W'X'Y' + WY'Z' + WXY'$$
$$F = YZ + W'X'Y' + WY'Z' + WXZ$$

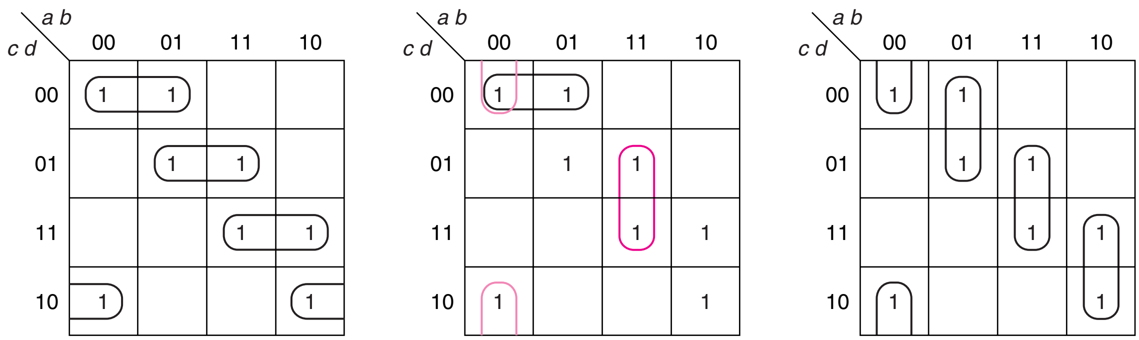
We will now look at some examples with no essential prime implicants. A classic example of such a function is shown in Example 3.13.

EXAMPLE 3.13



There are eight 1's; all prime implicants are groups of two. Thus, we need at least four terms in a minimum solution. There is no obvious place to start; thus, in the second map, we arbitrarily chose one of the terms, $a'c'd'$. Following the guidelines of step 2, we should then choose a second term that covers two new 1's, in such a way as not to leave an isolated uncovered 1. One such term is $bc'd$, as shown on the third map. Another possibility would be $b'cd'$ (the group in the last row). As we will see, that group will also be used. Repeating that procedure, we get the cover on the left map below,

$$f = a'c'd' + bc'd + acd + b'cd'$$



Notice, that if, after starting with $a'c'd'$, we chose one of the prime implicants not included in this solution above, such as abd, shown on the middle map, we leave an isolated uncovered 1 (which would require a third term) plus three more 1's (which would require two more terms). A solution using those two terms would require five terms (obviously not minimum since we found one with four). Another choice would be a term such as $a'b'd'$, which covers only one new 1, leaving five 1's uncovered. That, too, would require at least five terms.

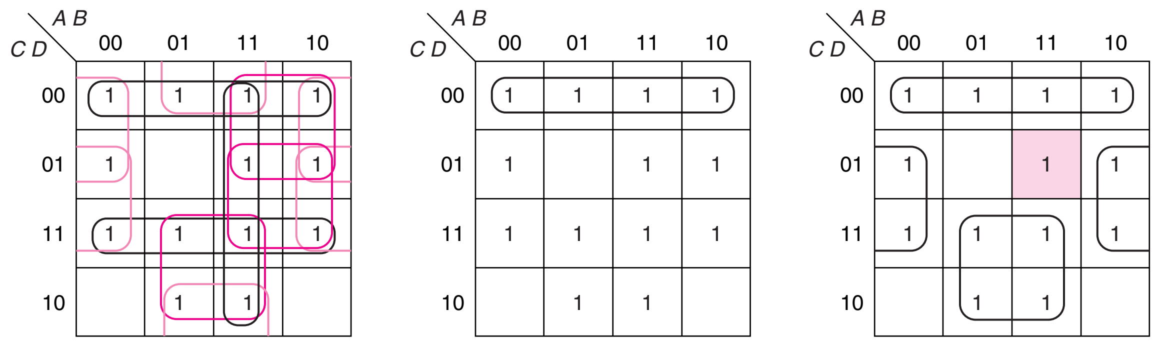
The other solution to this problem starts with $a'b'd'$, the only other prime implicant to cover m_0. Using the same process, we obtain the map on the right and the expression

$$f = a'b'd' + a'bc' + abd + ab'c$$

EXAMPLE 3.14

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

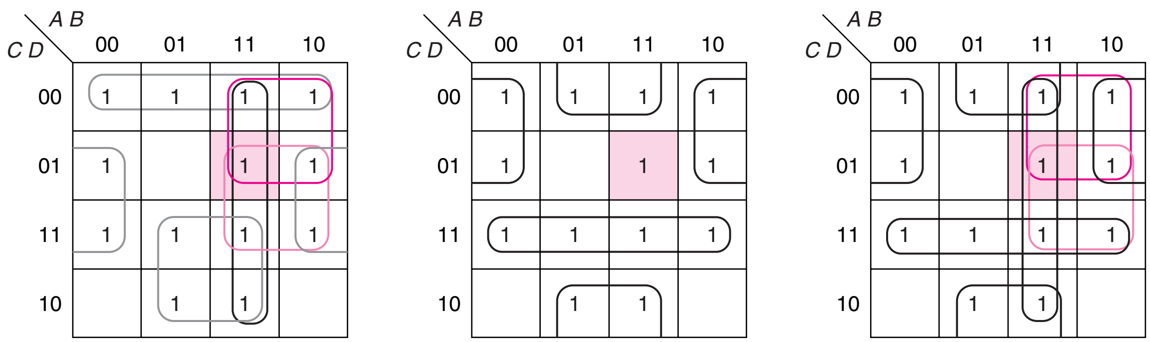
All of the prime implicants are groups of four. Since there are 13 1's, we need at least four terms. The first map shows all of the prime implicants circled; there are nine. There are no 1's circled only once, and thus, there are no essential prime implicants.



As a starting point, we choose one of the minterms covered by only two prime implicants, say m_0. On the second map, we used $C'D'$ to cover it. Next, we found two additional prime implicants that cover four new 1's each, as shown on the third map. That leaves just m_{13} to be covered. As can be seen on the fourth map (shown below), there are three different prime implicants that can be used. Now, we have three of the minimum solutions.

$$F = C'D' + B'D + BC + \{AB \quad \text{or} \quad AC' \quad \text{or} \quad AD\}$$

If, instead of using $C'D'$ to cover m_0, we use $B'C'$ (the only other prime implicant that covers m_0), as shown on the next map, we can find two other groups of four that each cover four new 1's and leave just m_{13} to be covered. Once again, we have three different ways to complete the cover (the same three terms as before).



Thus, there are six equally good solutions

$$F = \begin{Bmatrix} C'D' + B'D + BC \\ B'C' + BD' + CD \end{Bmatrix} + \begin{Bmatrix} AB \\ AC' \\ AD \end{Bmatrix}$$

where one group of terms is chosen from the first bracket and an additional term from the second. We are sure that there are no better solutions, since each uses the minimum number of prime implicants, four. Although it may not be obvious without trying other combinations, there are no additional minimum solutions.

A number of other examples are included in Solved Problems 1 and 2. Example 3.15 is one of the most complex four-variable problems, requiring more terms than we might estimate at first.

EXAMPLE 3.15

cd \ ab	00	01	11	10
00	1*		1	
01		1	1	1
11	1	1		1
10		1	1	1

cd \ ab	00	01	11	10
00	1		1	
01		1	1	1
11	1	1		1
10		1	1	1

cd \ ab	00	01	11	10
00	1		1	
01		1	1	1
11	1	1		1
10		1	1	1

This function has one essential prime implicant (a minterm) and ten other 1's. All of the other prime implicants are groups of two. The second map shows all 13 prime implicants. Note that every 1 (other than m_0) can be covered by two or three different terms.

Since there are ten 1's to be covered by groups of two, we know that we need at least five terms, in addition to $a'b'c'd'$. The third map shows the beginnings of an attempt to cover the function. Each term covers two new 1's without leaving any isolated uncovered 1. (The 1 at the top could be combined with m_{14}.) The four 1's that are left require three additional terms. After trying several other groupings, we can see that it is not possible to cover this function with less than seven terms. There are 32 different minimum solutions to this problem. A few of the solutions are listed below. The remainder are left as an exercise (Ex 1p).

$$
\begin{aligned}
f &= a'b'c'd' + a'cd + bc'd + ab'd + abc' + a'bc + acd' \\
&= a'b'c'd' + a'cd + bc'd + ab'd + abd' + bcd' + ab'c \\
&= a'b'c'd' + b'cd + a'bd + ac'd + abd' + acd' + bcd' \\
&= a'b'c'd' + b'cd + abc' + bcd' + a'bd + ab'c + ab'd
\end{aligned}
$$

[SP 1, 2; EX 1, 2, 3]

3.2 DON'T CARES

Finding minimum solutions for functions with don't cares does not significantly change the methods we developed in the last section. We need to modify slightly the definitions of a prime implicant and clarify the definition of an essential prime implicant.

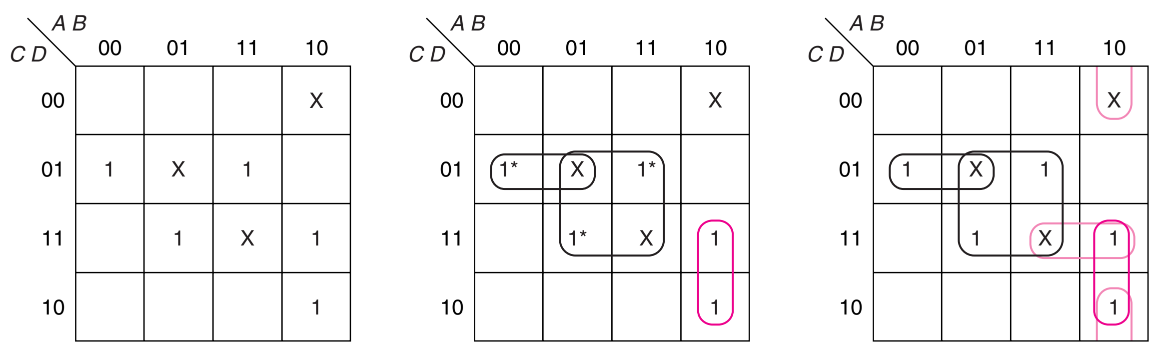
> A *prime implicant* is a rectangle of 1, 2, 4, 8, . . . 1's or X's not included in any one larger rectangle. Thus, from the point of view of finding prime implicants, X's (don't cares) are treated as 1's.

An *essential prime implicant* is a prime implicant that covers at least one 1 not covered by any other prime implicant (as always). Don't cares (X's) do not make a prime implicant essential.

Now, we just apply either of the methods of the last section. When we are done, some of the X's may be included and some may not. But we *don't care* whether or not they are included in the function.

EXAMPLE 3.16

$F(A, B, C, D) = \Sigma m(1, 7, 10, 11, 13) + \Sigma d(5, 8, 15)$



We first mapped the function, entering a 1 for those minterms included in the function and an X for the don't cares. We found two essential prime implicants, as shown on the center map. In each case, the 1's with an asterisk cannot be covered by any other prime implicant. That left the two 1's circled in green to cover the rest of the function. That is not an essential prime implicant, since each of the 1's could be covered by another prime implicant (as shown in light green on the third map). However, if we did not use $AB'C$, we would need two additional terms, instead of one. Thus, the only minimum solution is

$$F = BD + A'C'D + AB'C$$

and terms $AB'D'$ and ACD are prime implicants not used in the minimum solution. Note that if all of the don't cares were made 1's, we would need a fourth term to cover m_8, making

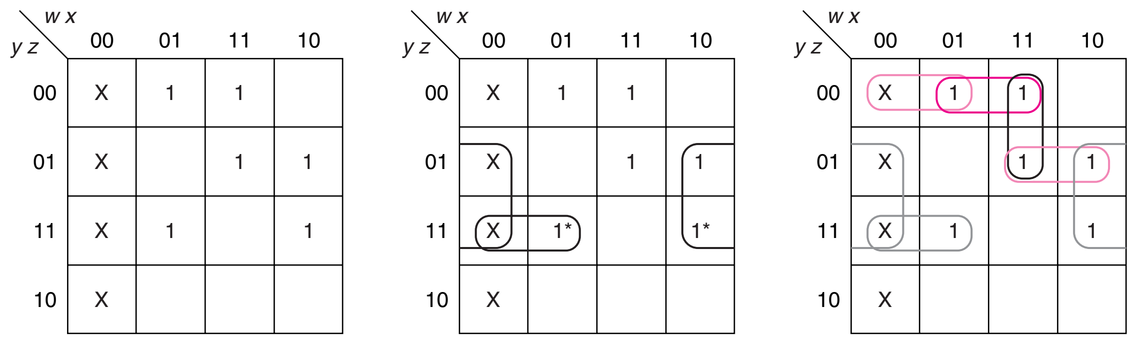
$$F = BD + A'C'D + AB'C + AB'D' \quad \text{or}$$
$$F = BD + A'C'D + ACD + AB'D'$$

and that if all of the don't cares were 0's, the function would become

$$F = A'B'C'D + A'BCD + ABC'D + AB'C$$

In either case, the solution is much more complex then when we treated those terms as don't cares (and made two of them 1's and the other a 0).

EXAMPLE 3.17

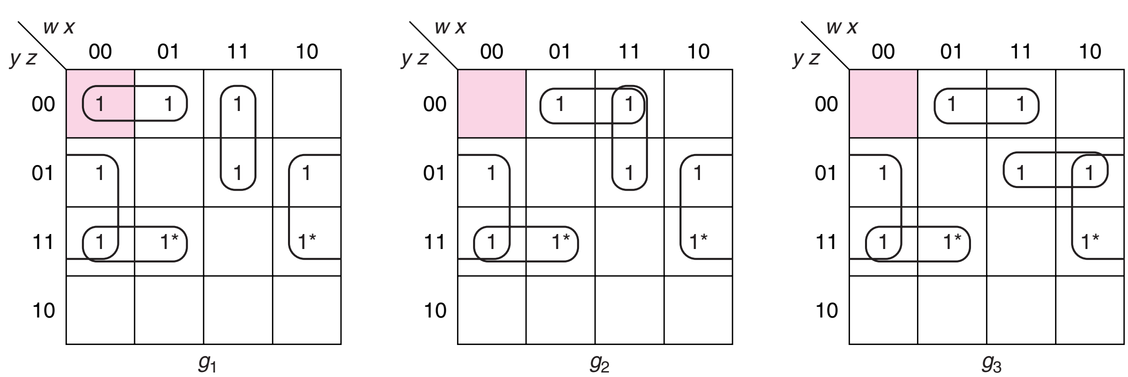


There are two essential prime implicants, as shown on the center map, $x'z$ and $w'yz$. The group of four don't cares, $w'x'$, is a prime implicant (since it is a rectangle of four 1's or X's) but it is not essential (since it does not cover any 1's not covered by some other prime implicant). Surely, a prime implicant made up of all don't cares would never be used, since that would add a term to the sum without covering any additional 1's. The three remaining 1's require two groups of two and thus there are three equally good solutions, each using four terms and 11 literals:

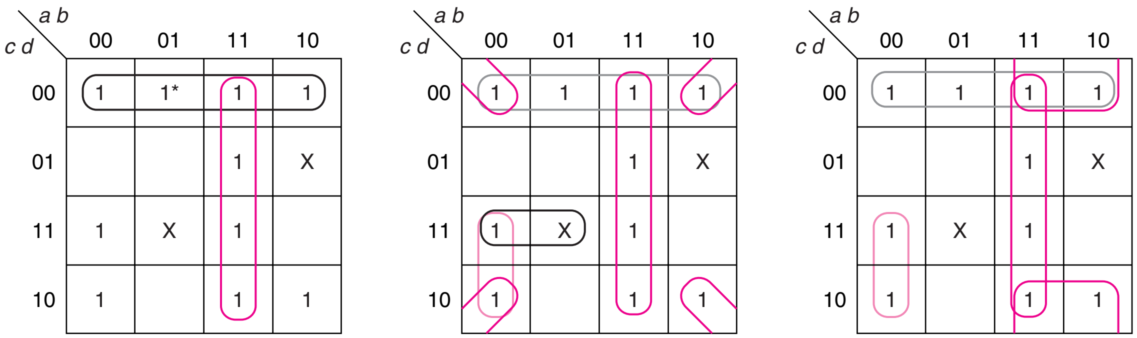
$$g_1 = x'z + w'yz + w'y'z' + wxy'$$
$$g_2 = x'z + w'yz + xy'z' + wxy'$$
$$g_3 = x'z + w'yz + xy'z' + wy'z$$

An important thing to note about Example 3.17 is that the three algebraic expressions are not all equal. The first treats the don't care for m_0 as a 1, whereas the other two (which are equal to each other) treat it as a 0. This will often happen with don't cares. They must treat the specified part of the function (the 1's and the 0's) the same, but the don't cares may take on different values in the various solutions. The maps of Map 3.4 show the three functions.

Map 3.4 The different solutions for Example 3.17.



EXAMPLE 3.18



On the first map, we have shown the only essential prime implicant, $c'd'$, and the other group of four that is used in all three solutions, ab. (This must be used since the only other prime implicant that would cover m_{15} is bcd, which requires one more literal and does not cover any 1's that are not covered by ab.) The three remaining 1's require two terms, one of which must be a group of two (to cover m_3) and the other must be one of the groups of four that cover m_{10}. On the second map, we have shown two of the solutions, those that utilize $b'd'$ as the group of four. On the third map, we have shown the third solution, utilizing ad'. Thus, we have

$$g_1 = c'd' + ab + b'd' + a'cd$$
$$g_2 = c'd' + ab + b'd' + a'b'c$$
$$g_3 = c'd' + ab + ad' + a'b'c$$

We can now ask if these solutions are equal to each other. We can either map all three solutions as we did for Example 3.17 or we can make a table of the behavior of the don't cares—one column for each don't care and one row for each solution.

	m_7	m_9
g_1	1	0
g_2	0	0
g_3	0	0

From the table, it is clear that $g_2 = g_3$, but neither is equal to g_1. A more complex example is found in the solved problems.

Don't cares provide us with another approach to solving map problems for functions with or without don't cares.

Map Method 3

1. Find all essential prime implicants using either Map Method 1 or 2.
2. Replace all 1's covered by the essential prime implicants with X's. This highlights the 1's that remain to be covered.
3. Then choose enough of the other prime implicants (as in Methods 1 and 2).

Step 2 works because the 1's covered by essential prime implicants may be used again (as part of a term covering some new 1's), but need not be. Thus, once we have chosen the essential prime implicants, these minterms are, indeed, don't cares.

EXAMPLE 3.19

$F(A, B, C, D) = \Sigma m(0, 3, 4, 5, 6, 7, 8, 10, 11, 14, 15)$

CD \ AB	00	01	11	10
00	1	1		1
01		1*		
11	1*	1	1	1
10		1	1	1

CD \ AB	00	01	11	10
00	1	X		1
01		X		
11	X	X	X	X
10		X	1	1

CD \ AB	00	01	11	10
00	1	X		1
01		X		
11	X	X	X	X
10		X	1	1

We first found the two essential prime implicants, $A'B$ and CD. On the second map, we converted all of the 1's covered to don't cares. Finally, we can cover the remaining 1's with AC and $B'C'D'$, producing

$$F = A'B + CD + AC + B'C'D'$$

Replacing covered minterms by don't cares accomplishes the same thing as the shading that we did in Examples 3.10 and 3.11; it highlights the 1's that remain to be covered.

[SP 3, 4; EX 4, 5]

3.3 PRODUCT OF SUMS

Finding a minimum product of sums expression requires no new theory. The following approach is the simplest:

1. Map the complement of the function. (If there is already a map for the function, replace all 0's by 1's, all 1's by 0's and leave X's unchanged.)

2. Find the minimum sum of products expression for the complement of the function (using the techniques of the last two sections).
3. Use DeMorgan's theorem (P11) to complement that expression, producing a product of sums expression.

Another approach, which we will not pursue here, is to define the dual of prime implicants (referred to as prime implicates) and develop a new method.

EXAMPLE 3.20

$f(a, b, c, d) = \Sigma m(0, 1, 4, 5, 10, 11, 14)$

Since all minterms must be either minterms of f or of f', then, f' must be the sum of all of the other minterms, that is

$f'(a, b, c, d) = \Sigma m(2, 3, 6, 7, 8, 9, 12, 13, 15)$

Maps of both f and f' are shown below

cd \ ab	00	01	11	10
00	1	1		
01	1	1		
11				1
10			1	1

f

cd \ ab	00	01	11	10
00			1	1
01			1	1
11	1	1	1	
10	1	1		

f'

We did not need to map f, unless we wanted both the sum of products expression and the product of sums expression. Once we mapped f, we did not need to write out all the minterms of f'; we could have just replaced the 1's by 0's and 0's by 1's. Also, instead of mapping f', we could look for rectangles of 0's on the map of f. This function is rather straightforward. The maps for the minimum sum of product expressions for both f and f' are shown below:

cd \ ab	00	01	11	10
00	1	1		
01	1	1		
11				1
10			1	1

f

cd \ ab	00	01	11	10
00			1	1
01			1	1
11	1	1	1	
10	1	1		

f'

There is one minimum solution for f and there are two equally good solutions for the sum of products for f':

$$f = a'c' + ab'c + acd' \qquad f' = ac' + a'c + abd$$
$$f' = ac' + a'c + bcd$$

We can then complement the solutions for f' to get the two minimum product of sums solutions for f:

$$f = (a' + c)(a + c')(a' + b' + d')$$
$$f = (a' + c)(a + c')(b' + c' + d')$$

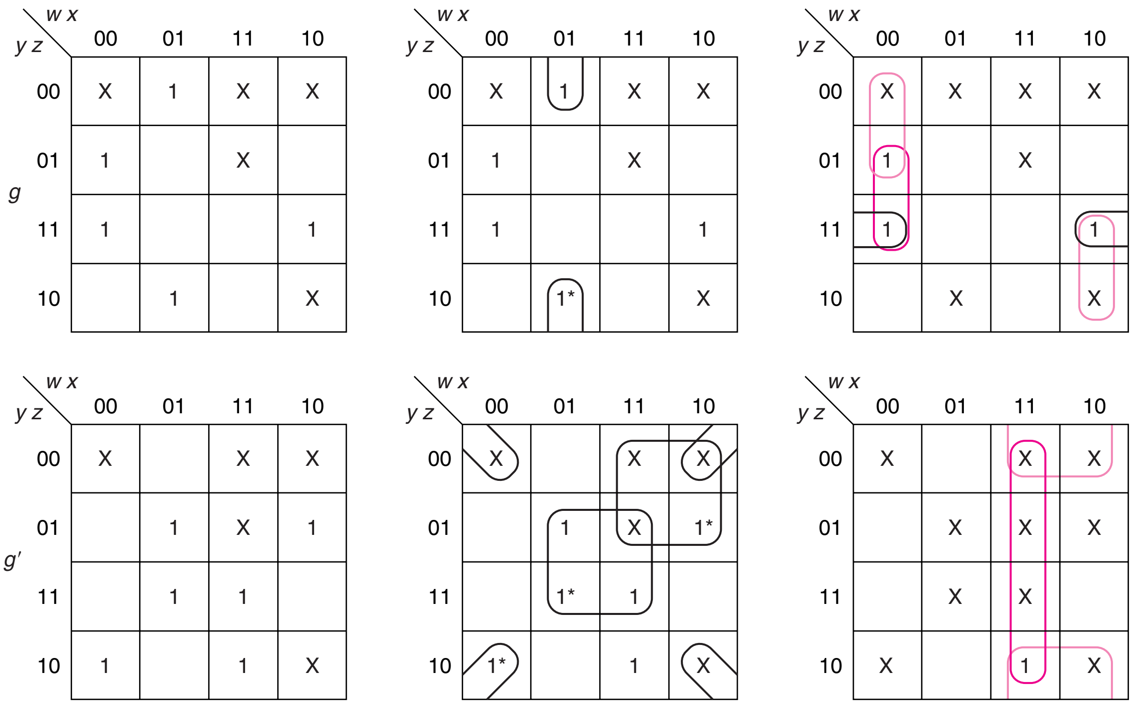
The minimum sum of products solution has three terms and eight literals; the minimum product of sums solutions have three terms and seven literals. (There is no set pattern; sometimes the sum of products solution has fewer terms or literals, sometimes the product of sums does, and sometimes they have the same number of terms and literals.)

EXAMPLE 3.21

Find all of the minimum sum of products and all minimum product of sums solutions for

$$g(w, x, y, z) = \Sigma m(1, 3, 4, 6, 11) + \Sigma d(0, 8, 10, 12, 13)$$

We first find the minimum sum of products expression by mapping g. However, before complicating the map by circling prime implicants, we also map g' (below g). Note that the X's are the same on both maps.



For g, the only essential prime implicant, $w'xz'$ is shown on the center map. The 1's covered by it are made don't cares on the right map and the remaining useful prime implicants are circled. We have seen similar examples before, where we have three 1's to be covered in groups of two. There are three equally good solutions:

$$g = w'xz' + \begin{Bmatrix} w'x'y' + x'yz \\ w'x'z + x'yz \\ w'x'z + wx'y \end{Bmatrix}$$

For g', there are three essential prime implicants, as shown on the center map. Once all of the 1's covered by them have been made don't cares, there is only one 1 left; it can be covered in two ways as shown on the right map:

$$g' = x'z' + xz + wy' + \begin{Bmatrix} wx \\ wz' \end{Bmatrix}$$

$$g = (x + z)(x' + z')(w' + y)\begin{Bmatrix} (w' + x') \\ (w' + z) \end{Bmatrix}$$

Note that in this example, the sum of product solutions each require only three terms (with nine literals), whereas the product of sums solutions each require four terms (with eight literals).

Finally, we want to determine which, if any, of the five solutions are equal. The complication (compared to this same question in the last section) is that when we treat a don't care as a 1 for g', that means that we are treating it as a 0 of g. Labeling the three sum of product solutions as g_1, g_2, and g_3, and the two product of sums solutions as g_4 and g_5, we produce the following table

	0	**8**	**10**	**12**	**13**
g_1	1	0	0	0	0
g_2	0	0	0	0	0
g_3	0	0	1	0	0
g'_4	1	1	1	1	1
g_4	0	0	0	0	0
g'_5	1	1	1	1	1
g_5	0	0	0	0	0

The product of sum solutions treat all of the don't cares as 1's of g' since each is circled by the essential prime implicants of g'. (Thus, they are 0's of g.) We then note that the three solutions that are equal are

$$g_2 = w'xz' + w'x'z + x'yz$$

$$g_4 = (x + z)(x' + z')(w' + y)(w' + x')$$

$$g_5 = (x + z)(x' + z')(w' + y)(w' + z)$$

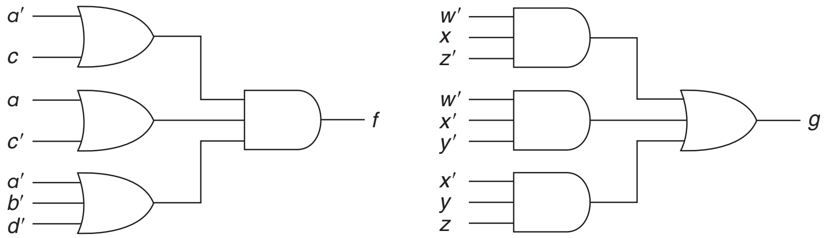
[SP 5, 6; EX 6, 7]

3.4 MINIMUM COST GATE IMPLEMENTATIONS

We are now ready to take another look at implementing functions with various types of gates. In this section, we will limit our discussion to two-level solutions for systems where all inputs are available both uncomplemented and complemented. (In Section 2.10, we examined multilevel circuits.) The minimization criteria is minimum number of gates, and among those with the same number of gates, minimum number of gate inputs. (Other criteria, such as minimum number of integrated circuit packages, were also discussed in Section 2.10 and will be examined further in Chapter 5.) The starting point is almost always to find the minimum sum of products solutions and/or the minimum product of sums solutions. That is because each term (other than single literal ones) corresponds to a gate. Then, unless the function has only one term, there is one output gate. Minimizing the number of literals minimizes the number of inputs to these gates.

First, we will look for solutions using AND and OR gates. We must look at both the minimum sum of products and minimum product of sums solutions. In Examples 3.20 and 3.21 from the last section, the product of sums solutions for f had one less gate input than the sum of products solution and the sum of products solutions for g had one less gate than the product of sums solutions. One of the minimum cost solutions for each is shown in Figure 3.1. (There are three equally good ones for f and two equally good ones for g.)

Figure 3.1 Minimum cost AND/OR implementations.



For a two-level solution using NAND gates, we need to start with a minimum sum of products solution. Thus, for g we can use the solution we obtained for AND and OR, but for f, we must use the sum of products solution, the one with one more gate input, as shown in Figure 3.2.

Similarly, for a two-level solution with NOR gates, we use a minimum product of sums solution, resulting in the circuits of Figure 3.3. Note that the NOR gate solution for g uses one more gate than the NAND gate solution.

If we are not limited to two levels, we have one additional option for implementing NAND gate solutions (or NOR gate solutions) beyond the algebraic manipulation of Section 2.8. We could find a minimum sum of

Figure 3.2 NAND gate implementations.

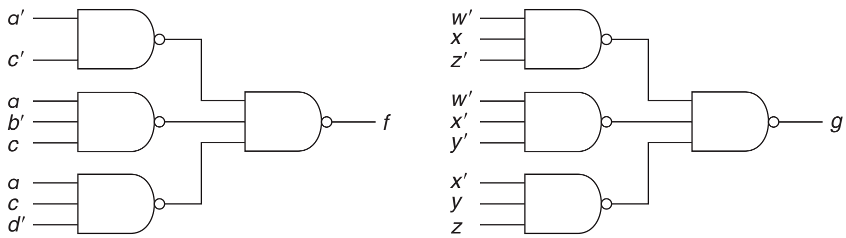
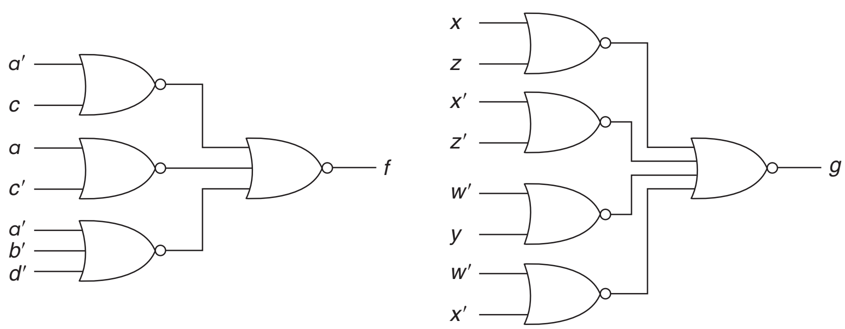


Figure 3.3 NOR gate implementations.



products expression for f' and implement that with NAND gates. We would then place a NOT gate at the output to produce f.

EXAMPLE 3.22

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

In Example 3.14, we found six equally good minimum sum of products solutions, each of which has four terms and eight literals. These solutions would require five gates. One of them is

$$G = C'D' + B'D + BC + AB$$

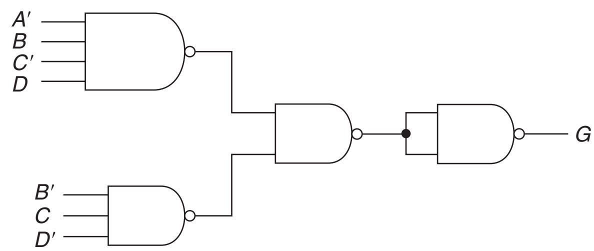
The map for G' is shown below

CD \ AB	00	01	11	10
00				
01		1		
11				
10	1			1

and thus

$$G' = A'BC'D + B'CD'$$

We could implement G' with three NAND gates and then use a NOT gate (or a two-input NAND with the inputs tied together) on the output as shown below:



This requires only four gates compared to the sum of products solution which required five gates. (Either would require two 7400 series packages.)

[SP 7; EX 8]

3.5 FIVE- AND SIX-VARIABLE MAPS

A five-variable map consists of $2^5 = 32$ squares. Although there are several arrangements that have been used, we prefer to look at it as two layers of 16 squares each. The top layer (on the left below) contains the squares for the first 16 minterms (for which the first variable, A, is 0) and the bottom layer contains the remaining 16 squares, as pictured in Map 3.5:

Map 3.5 A five-variable map.

$A = 0$

DE \ BC	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

$A = 1$

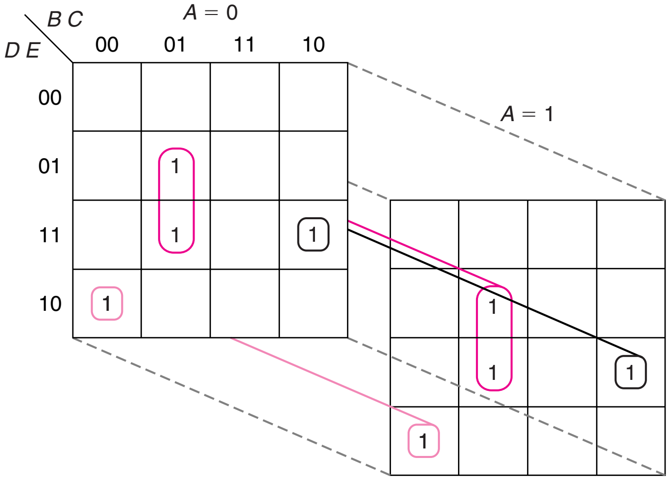
16	20	28	24
17	21	29	25
19	23	31	27
18	22	30	26

Each square in the bottom layer corresponds to the minterm numbered 16 more than the square above it. Product terms appear as rectangular solids of 1, 2, 4, 8, 16, . . . 1's or X's. Squares directly above and below each other are adjacent.

EXAMPLE 3.23

$m_2 + m_5 = A'B'C'DE' + AB'C'DE' = B'C'DE'$
$m_{11} + m_{27} = A'BC'DE + ABC'DE = BC'DE$
$m_5 + m_7 + m_{21} + m_{23} = B'CE$

These terms are circled on the map below.



In a similar manner, six-variable maps are drawn as four layers of 16-square maps, where the first two variables determine the layer and the other variables specify the square within the layer. The layout, with minterm numbers shown, is given in Map 3.6. Note that the layers are ordered in the same way as the rows and the columns, that is 00, 01, 11, 10.

In this section, we will concentrate on five-variable maps, although we will also do an example of six-variable maps at the end. The

Map 3.6 A six-variable map.

$AB = 00$

EF \ CD	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

$AB = 01$

16	20	28	24
17	21	29	25
19	23	31	27
18	22	30	26

$AB = 11$

48	52	60	56
49	53	61	57
51	55	63	59
50	54	62	58

$AB = 10$

32	36	44	40
33	37	45	41
35	39	47	43
34	38	46	42

techniques are the same as for four-variable maps; the only thing new is the need to visualize the rectangular solids. Rather than drawing the maps to look like three dimensions, we will draw them side by side. The function, F, is mapped in Map 3.7.

$$F(A, B, C, D, E) = \Sigma m(4, 5, 6, 7, 9, 11, 13, 15, 16, 18, 27, 28, 31)$$

Map 3.7 A five-variable problem.

A = 0

DE \ BC	00	01	11	10
00		1		
01		1	1	1
11		1	1	1
10		1		

A = 1

DE \ BC	00	01	11	10
00	1		1	
01				
11			1	1
10	1			

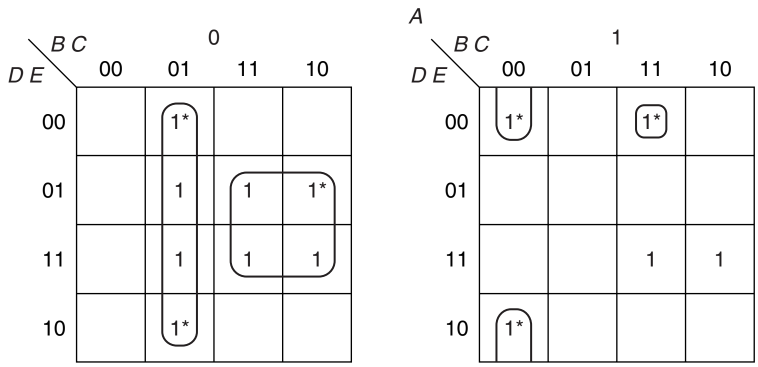
As always, we first look for the essential prime implicants. A good starting point is to find 1's on one layer for which there is a 0 in the corresponding square on an adjoining layer. Prime implicants that cover that 1 are contained completely on that layer (and thus, we really only have a four-variable map problem). In this example, m_4 meets this criteria (since there is a 0 in square 20 below it). Thus, the only prime implicants covering m_4 must be on the first layer. Indeed, $A'B'C$ is an essential prime implicant. (Note that the A' comes from the fact that this group is contained completely on the $A = 0$ layer of the map and the $B'C$ from the fact that this group is in the second column.) Actually, all four 1's in this term have no counterpart on the other layer and m_6 would also make this prime implicant essential. (The other two 1's in that term are part of another prime implicant, as well.) We also note that m_9, m_{16}, m_{18}, and m_{28} have 0's in the corresponding square on the other layer and make a prime implicant essential. Although m_{14} has a 0 beneath it (m_{30}), it does not make a prime implicant on the A' layer essential. Thus Map 3.8 shows each of these circled, highlighting the essential prime implicants that are contained on one layer.

So far, we have

$$F = A'B'C + A'BE + AB'C'E' + ABCD'E' + \cdots$$

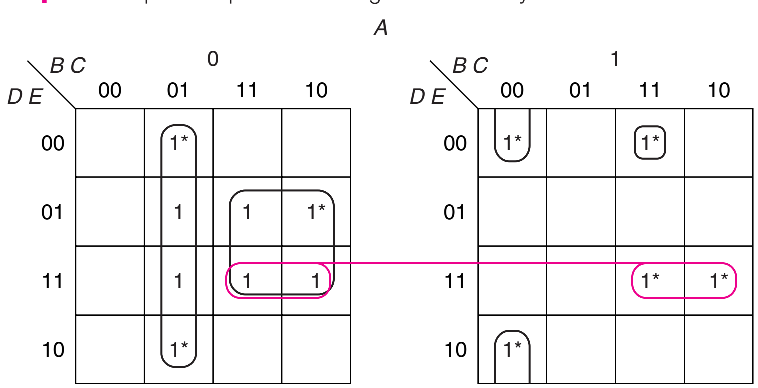
The two 1's remaining uncovered do have counterparts on the other layer. However, the only prime implicant that covers them is *BDE*, as shown on Map 3.9 in green. It, too, is an essential prime implicant. (Note that prime implicants that include 1's from both layers do not have the variable *A* in

Map 3.8 Essential prime implicants on one layer.



them. Such prime implicants must, of course, have the same number of 1's on each layer; otherwise, they would not be rectangular.)

Map 3.9 A prime implicant covering 1's on both layers.



The complete solution is thus

$$F = A'B'C + A'BE + AB'C'E' + ABCD'E' + BDE$$

Groups of eight 1's are not uncommon in five-variable problems, as illustrated in Example 3.24.

EXAMPLE 3.24

$$G(A, B, C, D, E) = \Sigma m(1, 3, 8, 9, 11, 12, 14, 17, 19, 20, 22, 24, 25, 27)$$

The first map shows a plot of that function. On the second map, to the right, we have circled the two essential prime implicants that we found by considering 1's on one layer with 0's in the corresponding square on the other layer. The group of eight 1's, $C'E$ (also an essential prime implicant), is shown in green on the third map (where the essential prime implicants found on the second map are shown as don't cares). Groups of eight have three literals missing (leaving only two). At this point, only two 1's are left uncovered; that requires the essential prime implicant, $BC'D'$, shown on the fourth map in light green.

DE \ BC (A = 0)	00	01	11	10
00			1	1
01	1			1
11	1			1
10			1	

DE \ BC (A = 1)	00	01	11	10
00		1		1
01	1			1
11	1			1
10		1		

DE \ BC (A = 0)	00	01	11	10
00			1	1
01	1			1
11	1			1
10			1*	

DE \ BC (A = 1)	00	01	11	10
00		1*		1
01	1			1
11	1			1
10		1*		

DE \ BC (A = 0)	00	01	11	10
00			X	1
01	1*			1
11	1*			1*
10			X	

DE \ BC (A = 1)	00	01	11	10
00		X		1
01	1*			1
11	1*			1*
10		X		

DE \ BC (A = 0)	00	01	11	10
00			1	1
01	1			1
11	1			1
10			1	

DE \ BC (A = 1)	00	01	11	10
00		1		1*
01	1			1
11	1			1
10		1		

The solution is thus

$$G = A'BCE' + AB'CE' + C'E + BC'D'$$

Note that there is only one other prime implicant in this function, $A'BD'E'$; it covers no 1's not already covered.

EXAMPLE 3.25

The next problem is shown on the maps below. Once again, we start by looking for 1's that are on one layer, with a corresponding 0 on the other layer. Although there are several such 1's on the $A = 0$ layer, only m_{10} makes a prime implicant essential. Similarly, on the $A = 1$ layer, m_{30} is covered by an essential prime implicant. These terms, $A'C'E'$ and $ABCD$, are shown on the second map. The 1's covered are shown as don't cares on the next map.

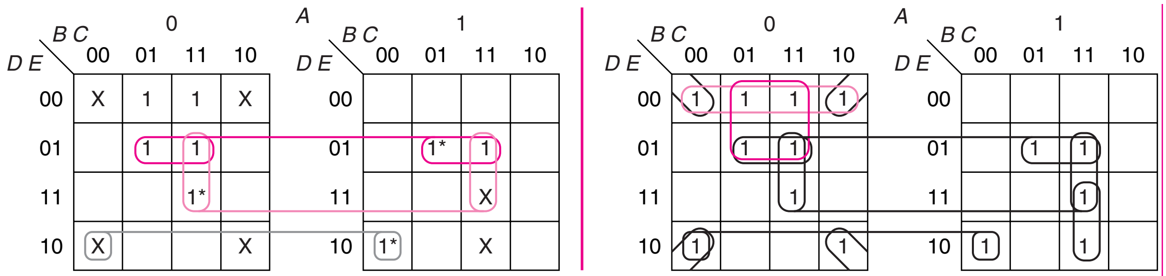
DE \ BC (A = 0)	00	01	11	10
00	1	1	1	1
01		1	1	
11			1	
10	1			1

DE \ BC (A = 1)	00	01	11	10
00				
01		1	1	
11			1	
10	1		1	

DE \ BC (A = 0)	00	01	11	10
00	1	1	1	1
01		1	1	
11			1	
10	1			1*

DE \ BC (A = 1)	00	01	11	10
00				
01		1	1	
11			1	
10	1		1*	

Three other essential prime implicants include 1's from both layers of the map; they are $CD'E$, BCE and $B'C'DE'$, as shown on the left map below. These were found by looking for isolated 1's, such as m_{21}, m_{15}, and m_{18}.



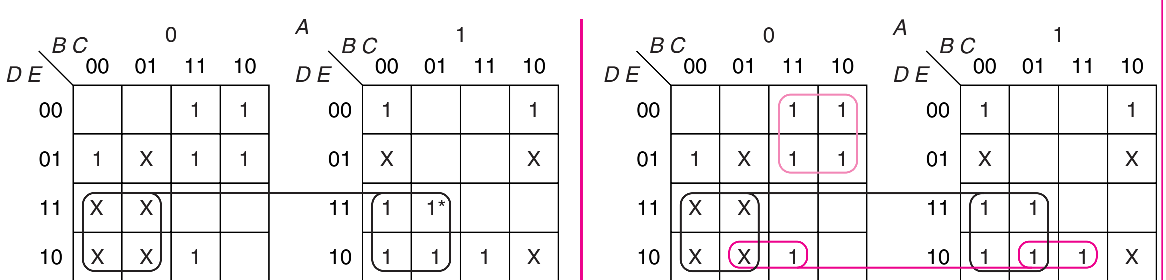
Finally, the remaining two 1's (m_4 and m_{12}) can be covered in two ways, as shown on the right map above, $A'CD'$ and $A'D'E'$. Thus, the two solutions are

$$F = A'C'E' + ABCD + CD'E + BCE + B'C'DE' + A'CD'$$
$$F = A'C'E' + ABCD + CD'E + BCE + B'C'DE' + A'D'E'$$

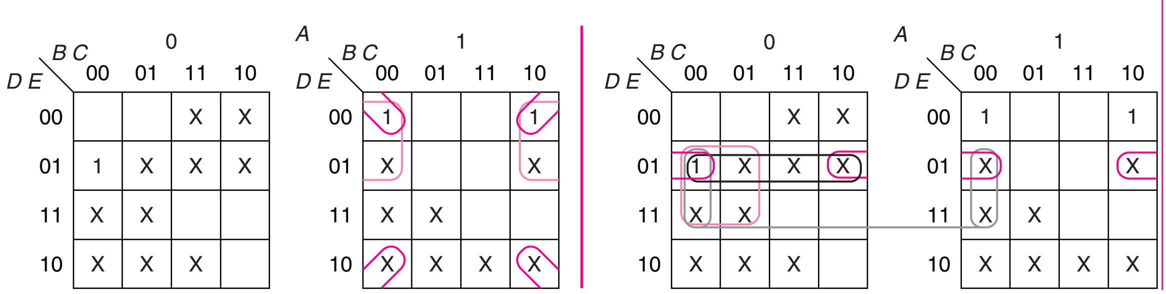
EXAMPLE 3.26

$$H(A, B, C, D, E) = \Sigma m(1, 8, 9, 12, 13, 14, 16, 18, 19, 22, 23, 24, 30) + \Sigma d(2, 3, 5, 6, 7, 17, 25, 26)$$

A map of H is shown below on the left with the only essential prime implicant, $B'D$, (a group of eight, including four 1's and four don't cares) circled.



Next, we choose CDE', since otherwise separate terms would be needed to cover m_{14} and m_{30}. We also chose $A'BD'$ since it covers four new 1's. Furthermore, if that were not used, a group of two ($A'BCE'$) would be needed to cover m_{12}. That leaves us with three 1's (m_1, m_{16}, and m_{24}) to be covered. On the maps below, we have replaced all covered 1's by don't cares (X's) to highlight the remaining 1's. No term that covers m_1 also covers either of the other terms. However, m_{16} and m_{24} can be covered with one term in either of two ways ($AC'E'$ or $AC'D'$) as shown on the first map below, and m_1 can



be covered by four different groups of four, as shown on the second map ($A'D'E$, $A'B'E$, $B'C'E$, or $C'D'E$), yielding the eight solutions shown.

$$H = B'D + CDE' + A'BD' + \begin{Bmatrix} AC'E' \\ AC'D' \end{Bmatrix} + \begin{Bmatrix} A'D'E \\ A'B'E \\ B'C'E \\ C'D'E \end{Bmatrix}$$

Finally, we will look at one example of a six-variable function.

EXAMPLE 3.27

$$G(A, B, C, D, E, F) = \Sigma m(1, 3, 6, 8, 9, 13, 14, 17, 19, 24, 25, 29, 32, 33, 34, 35, 38, 40, 46, 49, 51, 53, 55, 56, 61, 63)$$

The map is drawn horizontally, with the first two variables determining the 16-square layer (numbered, of course 00, 01, 11, 10).

AB = 00

EF \ CD	00	01	11	10
00				1
01	1		1	1
11	1*			
10		1	1	

AB = 01

EF \ CD	00	01	11	10
00				1
01	1		1	1
11	1*			
10				

AB = 11

EF \ CD	00	01	11	10
00				1*
01	1	1	1	
11	1	1	1*	
10				

AB = 10

EF \ CD	00	01	11	10
00	1			1
01	1			
11	1			
10	1	1	1	

The first map shows three of the essential prime implicants. The only one that is confined to one layer is on the third layer, $ABDF$. The 1's in the upper right corner of each layer form another group of four (without the first two variables), $CD'E'F'$. The green squares form a group of eight, $C'D'F$. The next map shows 1's covered by the first three prime implicants as don't cares.

AB = 00

EF \ CD	00	01	11	10
00				X
01	X		1*	1
11	X			
10		1*	1*	

AB = 01

EF \ CD	00	01	11	10
00				X
01	X		1	1
11	X			
10				

AB = 11

EF \ CD	00	01	11	10
00				X
01	X	X	X	
11	X	X	X	
10				

AB = 10

EF \ CD	00	01	11	10
00	1			X
01	X			
11	X			
10	1	1	1*	

The other two essential prime implicants are $A'CE'F$ and $B'DEF'$. (Remember that the top and bottom layers are adjacent.) Finally, m_{32} and m_{34} (on the fourth layer) remain uncovered; they are covered by the term, $AB'C'D'$. (Each of them could have been covered by a group of two; but that would take two terms.) Thus, the minimum expression is

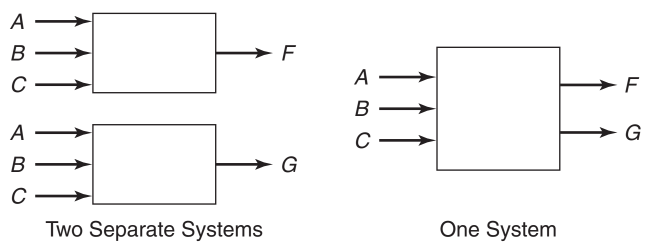
$$G = ABDF + CD'E'F' + C'D'F + A'CE'F + B'DEF' + AB'C'D'$$

[SP 8, 9; EX 9, 10]

3.6 MULTIPLE OUTPUT PROBLEMS

Many real problems involve designing a system with more than one output. If, for example, we had a problem with three inputs, A, B, and C and two outputs, F and G, we could treat this as two separate problems (as shown on the left in Figure 3.4). We would then map each of the functions, and find minimum solutions. However, if we treated this as a single system with three inputs and two outputs (as shown on the right), we may be able to economize by sharing gates.

Figure 3.4 Implementation of two functions.



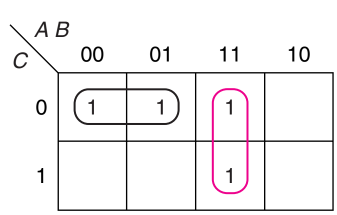
In this section, we will illustrate the process of obtaining two-level solutions using AND and OR gates (sum of products solutions), assuming all variables are available both uncomplemented and complemented. We could convert each of these solutions into NAND gate circuits (using the same number of gates and gate inputs). We could also find product of sums solutions (by minimizing the complement of each of the functions and then using DeMorgan's theorem).

We will illustrate this by first considering three very simple examples.

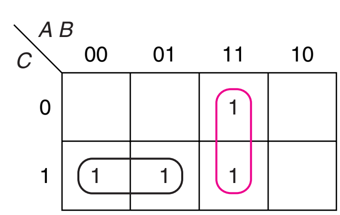
EXAMPLE 3.28

$$F(A, B, C) = \Sigma m(0, 2, 6, 7) \qquad G(A, B, C) = \Sigma m(1, 3, 6, 7)$$

If we map each of these and solve them separately,



F

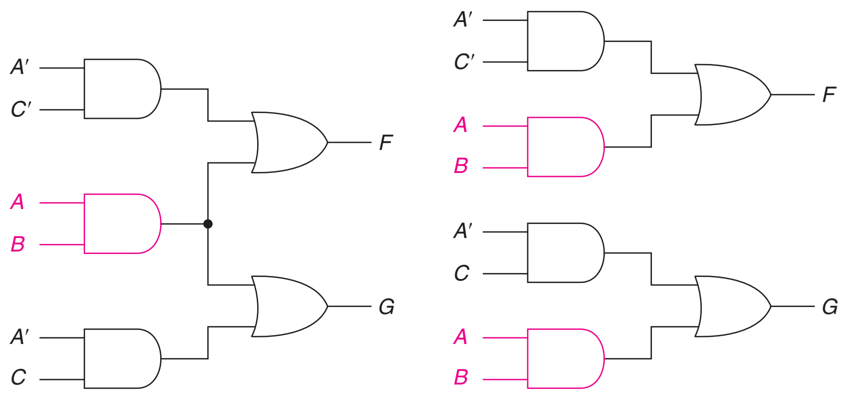


G

we obtain

$$F = A'C' + AB \qquad G = A'C + AB$$

Looking at the maps, we see that the same term (AB) is circled on both. Thus, we can build the circuit on the left, rather than the two circuits on the right.



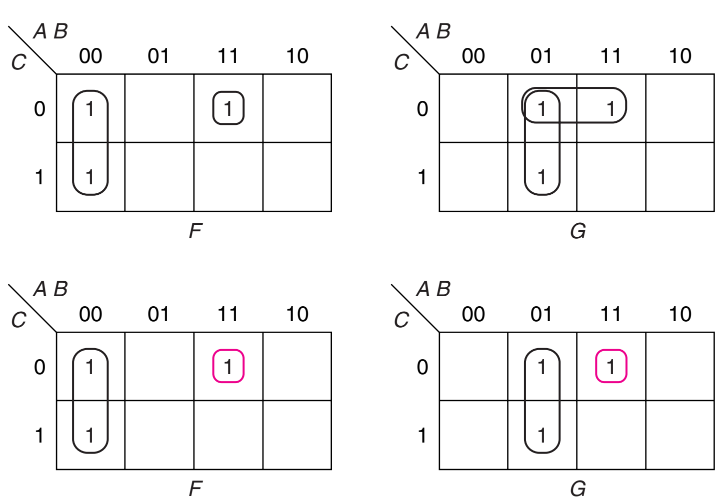
Obviously, the version on the left requires only five gates, whereas the one on the right uses six.

This example is the simplest. Each of the minimum sum of product expressions contains the same term. It would take no special techniques to recognize this and achieve the savings.

Even when the two solutions do not have a common prime implicant, we can share as illustrated in the following example:

EXAMPLE 3.29

$F(A, B, C) = \Sigma m(0, 1, 6) \qquad G(A, B, C) = \Sigma m(2, 3, 6)$



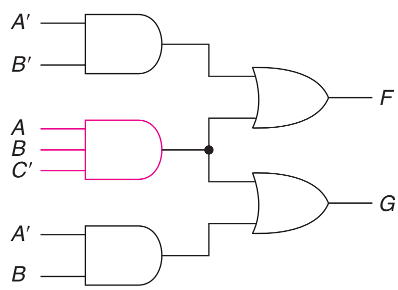
In the top maps, we considered each function separately and obtained

$$F = A'B' + ABC' \qquad G = A'B + BC'$$

This solution requires six gates (four ANDs and two ORs) with 13 inputs. However, as can be seen from the second pair of maps, we can share the term ABC' and obtain

$$F = A'B' + ABC' \qquad G = A'B + ABC'$$

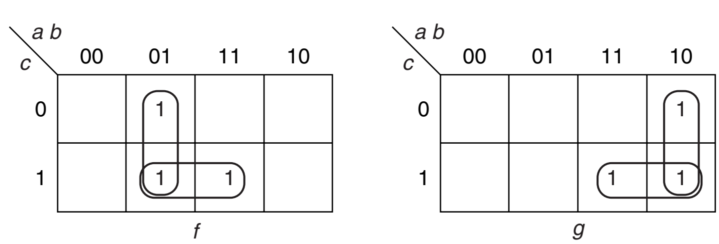
(To emphasize the sharing, we have shown the shared term in green, and will do that in other examples that follow.) As can be seen from the circuit below, this only requires five gates with 11 inputs.

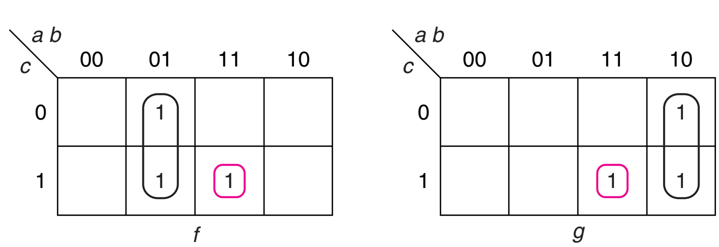


This example illustrates that a shared term in a minimum solution need not be a prime implicant. (In Example 3.29, ABC' is a prime implicant of F but not of G; in Example 3.30, we will use a term that is not a prime implicant of either function.)

EXAMPLE 3.30

$$F(A, B, C) = \Sigma m(2, 3, 7) \qquad G(A, B, C) = \Sigma m(4, 5, 7)$$





In the first pair of maps, we solved this as two problems. Using essential prime implicants of each function, we obtained

$$f = a'b + bc \qquad g = ab' + ac$$

However, as can be seen in the second set of maps, we can share the term *abc*, even though it is not a prime implicant of either function, and once again get a solution that requires only five gates:

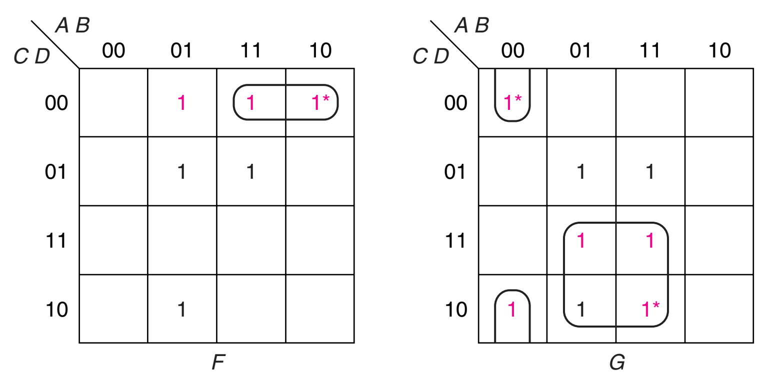
$$f = a'b + abc \qquad g = ab' + abc$$

The method for solving this type of problem is to begin by looking at the 1's of each function that are 0's of the other function. They must be covered by prime implicants of that function. Only the shared terms need not be prime implicants. In this last example, we chose $a'b$ for f since m_2 makes that an essential prime implicant of F and we chose ab' for g since m_4 makes that an essential prime implicant of g. That left just one 1 uncovered in each function—the same 1—which we covered with *abc*. We will now look at some more complex examples.

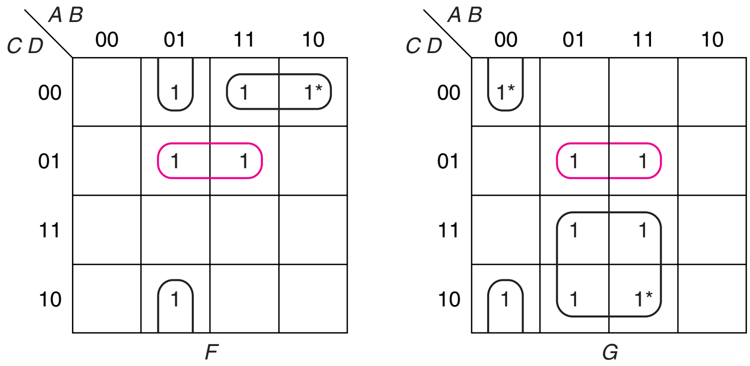
EXAMPLE 3.31

$$F(A, B, C, D) = \Sigma m(4, 5, 6, 8, 12, 13)$$
$$G(A, B, C, D) = \Sigma m(0, 2, 5, 6, 7, 13, 14, 15)$$

The maps of these functions are shown below. In them, we have shown in green the 1's that are included in one function and not the other.



We then circled each of those prime implicants that was made essential by a green 1. The only green 1 that was not circled in F is m_4 because that can be covered by two prime implicants. Even though one of the terms would have fewer literals, we must wait. Next, we will use $A'BD'$ for F. Since m_6 was covered by an essential prime implicant of G, we are no longer looking for a term to share. Thus, m_6 will be covered in F by the prime implicant, $A'BD'$. As shown on the maps below, that leaves m_4 and m_{12} to be covered in both functions, allowing us to share the term $BC'D$, as shown on the following maps circled in green.



leaving

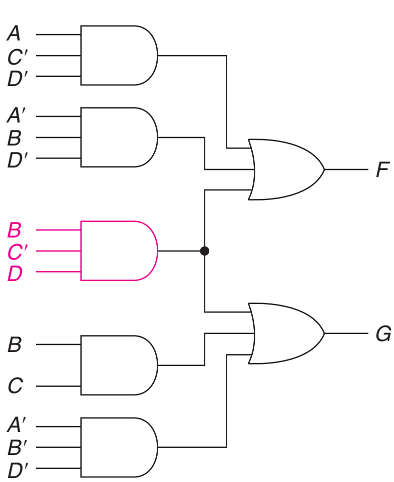
$$F = AC'D' + A'BD' + BC'D$$
$$G = A'B'D' + BC + BC'D$$

for a total of seven gates with 20 gate inputs. Notice that if we had minimized the functions individually, we would have used two separate terms for the third term in each expression, resulting in

$$F = AC'D' + A'BD' + BC'$$
$$G = A'B'D' + BC + BD$$

for a total of eight gates with 21 gate inputs. Clearly, the shared circuit costs less.

The shared version of the circuit is shown below.



EXAMPLE 3.32

$F(A, B, C, D) = \Sigma m(0, 2, 3, 4, 6, 7, 10, 11)$

$G(A, B, C, D) = \Sigma m(0, 4, 8, 9, 10, 11, 12, 13)$

Once again the maps are shown with the unshared 1's in green and the prime implicants made essential by one of those 1's circled.

CD \ AB	00	01	11	10
00	1	1		
01				
11	1	1*		1
10	1	1		1

CD \ AB	00	01	11	10
00	1	1	1	1
01			1*	1
11				1
10				1

Each of the functions can be solved individually with two more groups of four, producing

$$F = A'C + A'D' + B'C \qquad G = AC' + C'D' + AB'$$

That would require eight gates with 18 gate inputs. However, sharing the groups of two as shown on the next set of maps reduces the number of gates to six and the number of gate inputs to 16. If these functions were implemented with NAND gates, the individual solutions would require a total of three packages, whereas the shared solution would require only two.

CD \ AB	00	01	11	10
00	1	1		
01				
11	1	1		1
10	1	1		1

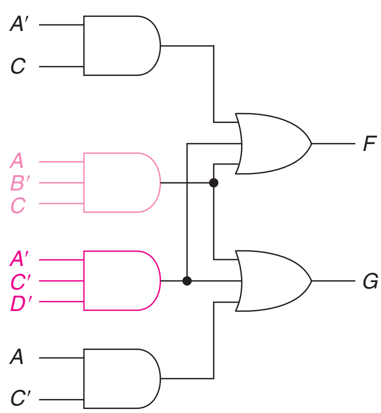
F

CD \ AB	00	01	11	10
00	1	1	1	1
01			1	1
11				1
10				1

G

leaving the equations and the resulting AND/OR circuit.

$$F = A'C + A'C'D' + AB'C \qquad G = AC' + A'C'D' + AB'C$$



EXAMPLE 3.33

$$F(W, X, Y, Z) = \Sigma m(2, 3, 7, 9, 10, 11, 13)$$

$$G(W, X, Y, Z) = \Sigma m(1, 5, 7, 9, 13, 14, 15)$$

On the maps below, the 1's that are not shared are shown in green and the essential prime implicants that cover these 1's are circled.

F:

YZ \ WX	00	01	11	10
00				
01			1	1
11	1	1		1
10	1*			1*

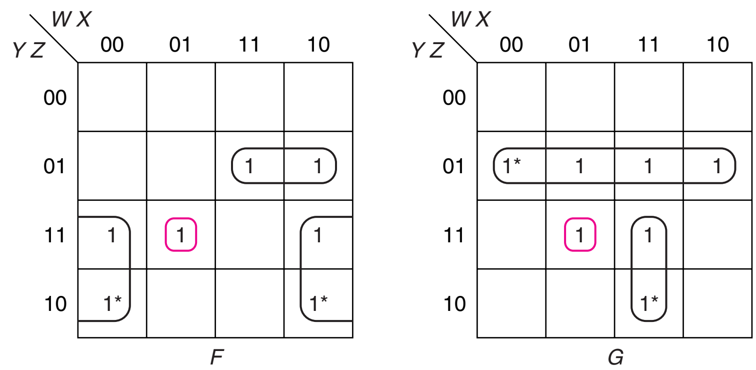
G:

YZ \ WX	00	01	11	10
00				
01	1*	1	1	1
11		1	1	
10			1*	

$$F = X'Y + \cdots$$

$$G = Y'Z + WXY + \cdots$$

Now, there are three 1's left in F. Since m_9 and m_{13} have been covered in G by an essential prime implicant, no sharing is possible for these terms in F. Thus, $WY'Z$, a prime implicant of F, is used in the minimum cover. Finally, there is one uncovered 1 in each function, m_7; it can be covered by a shared term, producing the solution



$F = X'Y + WY'Z + W'XYZ$

$G = Y'Z + WXY + W'XYZ$

This requires seven gates and 20 inputs, compared to the solution we obtain by considering these as separate problems

$F = X'Y + WY'Z + W'YZ$

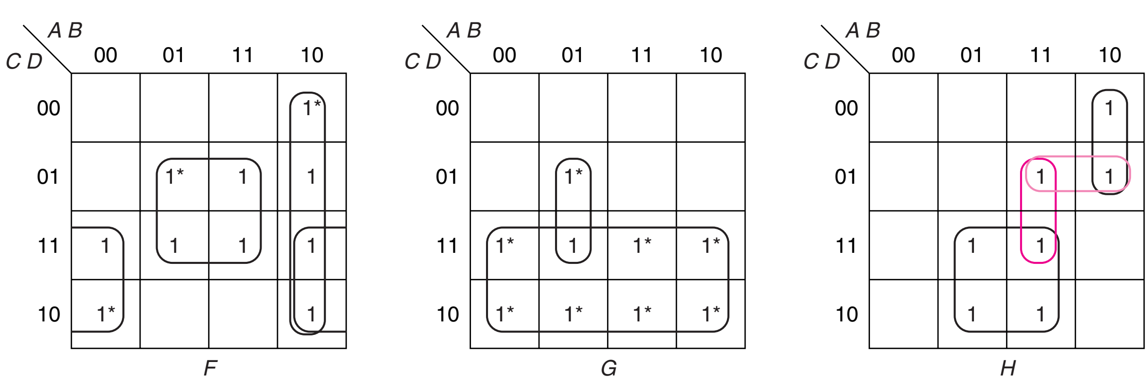
$G = Y'Z + WXY + XZ$

which requires eight gates with 21 inputs.

The same techniques can be applied to problems with three or more outputs.

EXAMPLE 3.34

First, we show the solution obtained if we considered them as three separate problems.



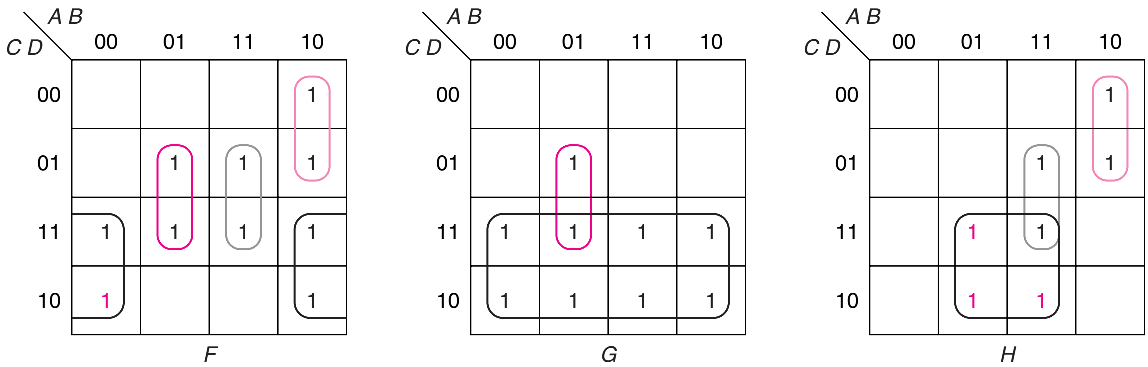
$F = AB' + BD + B'C$

$G = C + A'BD$

$H = BC + AB'C' + (ABD \text{ or } AC'D)$

This solution requires 10 gates and 25 gate inputs. (Note that the term C in function G does not require an AND gate.)

The technique of first finding 1's that are only minterms of one of the functions does not get us started for this example, since each of the 1's is a minterm of at least two of the functions. The starting point, instead, is to choose *C* for function *G*. The product term with only one literal does not require an AND gate and uses only one input to the OR gate. Any other solution, say sharing *B′C* with *F* and *BC* with *H*, requires at least two inputs to the OR gate. Once we have made that choice, however, we must then choose *B′C* for *F* and *BC* for *H*, because of the 1's shown in green on the following maps. There is no longer any sharing possible for those 1's and they make those prime implicants essential in *F* and *H*.



The term *AB′C′* (circled in light green) was chosen next for *H* since it is an essential prime implicant of *H* and it can be shared (that is, all of the 1's in that term are also 1's of *F*, the only place where sharing is possible). *AB′C′* is also used for *F*, since it covers two 1's and we would otherwise require an additional term, *AB′*, to cover m_8. In a similar fashion, the term *A′BD* is used for *G* (it is the only way to cover m_5) and can then be shared with *F*. Finally, we can finish covering *F* and *H* with *ABD* (a prime implicant of *H*, one of the choices for covering *H* when we treated that as a separate problem). It would be used also for *F*, rather than using another AND gate to create the prime implicant *BD*. The solution then becomes

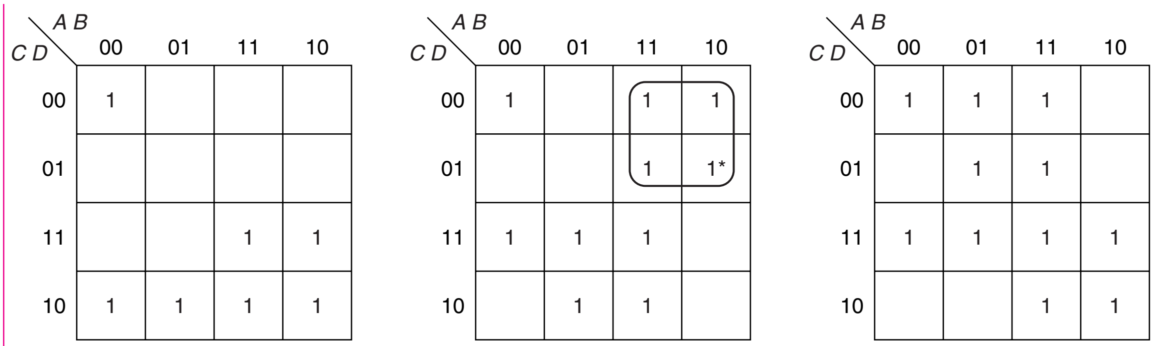
$$F = B'C + AB'C' + A'BD + ABD$$
$$G = C + A'BD$$
$$H = BC + AB'C' + ABD$$

which requires only eight gates and 22 gate inputs (a savings of two gates and three-gate inputs).

EXAMPLE 3.35

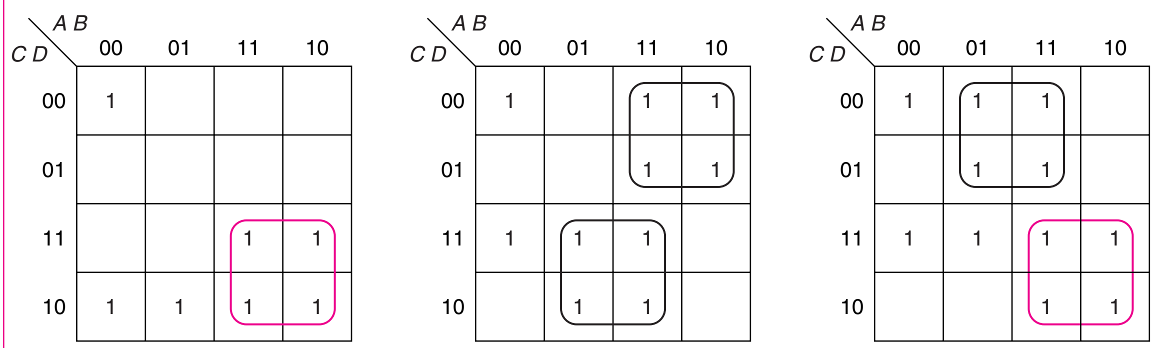
$$F(A, B, C, D) = \Sigma m(0, 2, 6, 10, 11, 14, 15)$$
$$G(A, B, C, D) = \Sigma m(0, 3, 6, 7, 8, 9, 12, 13, 14, 15)$$
$$H(A, B, C, D) = \Sigma m(0, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15)$$

The map on the next page shows these functions; the only 1 that is not shared and makes a prime implicant essential is m_9 in *G*. That prime implicant, *AC′*, is shown circled.



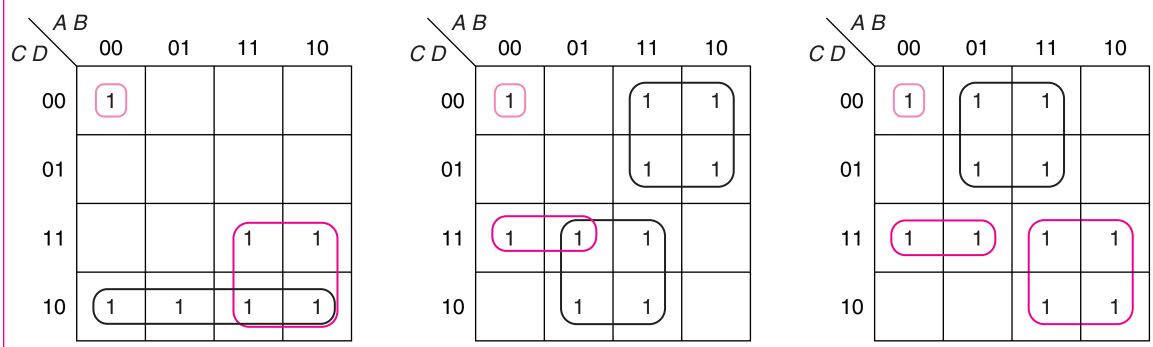
F G H

Next, we note that AC is an essential prime implicant of F (because of m_{11} and m_{15}) and of H (because of m_{10}). Furthermore, neither m_{10} nor m_{11} are 1's of G. Thus, that term is used for both F and H. Next, we chose BC' for H and BC for G; each covers four new 1's, some of which can no longer be shared (since the 1's that correspond to other functions have already been covered).



F G H

At this point, we can see that $A'B'C'D'$ can be used to cover m_0 in all three functions; otherwise, we would need three different three-literal terms. $A'CD$ can be used for G and H, and, finally, CD' is used for F, producing the following map and algebraic functions.



F G H

$$F = AC + A'B'C'D' + CD'$$
$$G = AC' + BC + A'B'C'D' + A'CD$$
$$H = AC + BC' + A'B'C'D' + A'CD$$

This solution requires 10 gates with 28 inputs, compared to 13 gates and 35 inputs if these were implemented separately.

EXAMPLE 3.36

Finally, we will consider an example of a system with don't cares:

$$F(A, B, C, D) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$$
$$G(A, B, C, D) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$$

A map of the functions, with the only prime implicant made essential by a 1 that is not shared circled, $B'D$, is shown below.

CD \ AB	00	01	11	10
00	X	1	1	
01	X			1*
11	1		X	1
10	1	1	X	

F

CD \ AB	00	01	11	10
00	X		1	
01	X			
11			X	1
10	1	1	X	1

G

Since m_{11} has now been covered in F, we must use the essential prime implicant of G, AC, to cover m_{11} there. Also, as shown on the next maps, ABD' is used for G, since that is an essential prime implicant of G and the whole term can be shared. (We will share it in the best solution.)

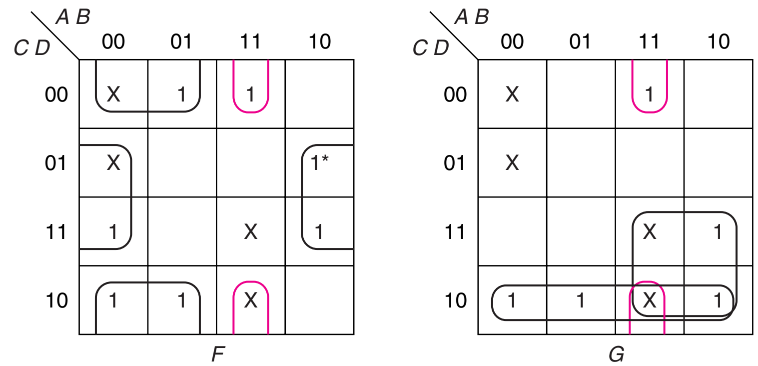
CD \ AB	00	01	11	10
00	X	1	1	
01	X			1*
11	1		X	1
10	1	1	X	

F

CD \ AB	00	01	11	10
00	X		1	
01	X			
11			X	1
10	1	1	X	1

G

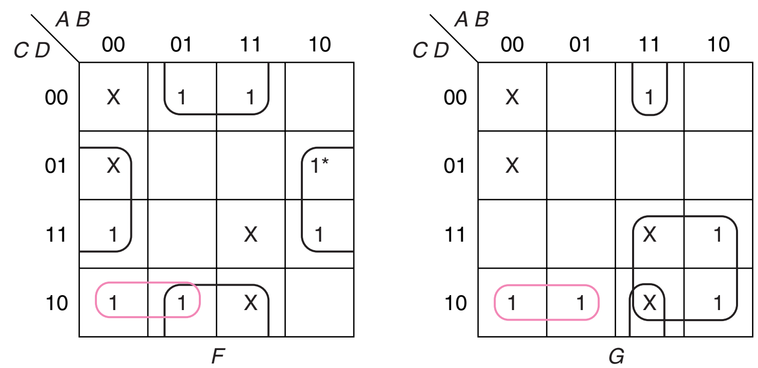
Since we need the term ABD' for G, one approach is to use it for F also. (That only costs a gate input to the OR gate.) If we do that, we could cover the rest of F with $A'D'$ and the rest of G with CD', yielding the map and equations that follow.



$F = B'D + ABD' + A'D'$

$G = AC + ABD' + CD'$

That solution uses seven gates and 17 inputs. Another solution using the same number of gates but one more input shares $A'CD'$. That completes G and then the cover of F is completed with BD'. The maps and equations are thus:



$F = B'D + A'CD' + BD'$

$G = AC + ABD' + A'CD'$

That, too, requires seven gates, but using a three-input AND gate instead of a two-input one, bringing the total number of inputs to 18.

[SP 10; EX 11, 12]

3.7 SOLVED PROBLEMS

1. For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

a. $G(X, Y, Z) = \Sigma m(1, 2, 3, 4, 6, 7)$

b. $f(w, x, y, z) = \Sigma m(2, 5, 7, 8, 10, 12, 13, 15)$